MICROECONOMIC THEORY

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CONTENTS

1. INTRODUCTION AND OVERVIEW

This text provides a succinct treatment of the basic elements of microeconomic theory. It deliberately avoids the use of words and lengthy explanations wherever possible, and for most students should be used as accompaniment to a lecture-based course. It is designed for senior undergraduates with good mathematical skills, and entry-level graduate students.

It begins with a basic treatment of consumer theory (Chapters 2 & 3), and then extends the analysis to consumer welfare measures (Chapter 4), choice under uncertainty (Chapter 5), and intertemporal choice (Chapter 6). An Appendix to Chapter 2 provides a review of optimization techniques.

It then moves to production and the theory of the competitive firm (Chapters 7 – 9). Competitive equilibrium is then examined, first in partial equilibrium (Chapter 10) and then in a simple two-sector general equilibrium model, with an informal presentation of the first welfare theorem (Chapter 11).

It then moves on to sources of market failure, starting with monopoly (Chapter 12) and a game-theoretic treatment of oligopoly (Chapter 13). This is followed by a game-theoretic treatment of externalities and public goods (Chapter 14), and asymmetric information (Chapter 15).

An Appendix contains four problem sets covering the main material. Fully worked solutions are included.

2. FOUNDATIONS OF CONSUMER THEORY

2.1 Preferences

The consumption set, *X*. Preferences over bundles in *X*:

- $x \sim\succ y$ means *x* is *weakly preferred* to (at least as good as) *y*
- if $x \sim\succ y$ and $\sim [y \sim\succ x]$ then $x \succ y$; that is, *x* is *strictly preferred* to *y*
- if $x \sim\succ y$ and $y \sim\succ x$ then $x \sim y$; that is, the agent is *indifferent* between *x* and *y*

<u>Assumptions on Preferences</u>

1. **completeness**: $\forall x \in X$ and $\forall y \in X$, either $x \sim\succ y$ or $y \sim\succ x$; that is, all bundles can be compared.
2. **reflexivity**: $\forall x \in X$, $x \sim\succ x$
3. **transitivity**: $\forall \{x, y, z\} \in X$, if $x \sim\succ y$ and $y \sim\succ z$ then $x \sim\succ z$. This ensures no ambiguity in ranking.
4. **continuity**: $\forall x \in X$, the sets $B = \{x : x \sim\succ y\}$ and $A = \{x : y \sim\succ x\}$ are closed; that is, these sets contain their boundaries. This allows for the existence of indifference curves, along which points belong to both *A* and *B*.
5. **strong monotonicity**: if $x \geq y$ then $x \succ y$; that is, more is strictly better.
6. **strict convexity**: $\forall \{x, y, z\} \in X$ and $x \neq y$, if $x \sim\succ z$ and $y \sim\succ z$, then $tx + (1-t)y \succ z \;\; \forall t \in (0,1)$; that is, indifference curves are bowed towards the origin. See Figure 2.1. Strict convexity implies a preference for variety.

<u>Representation Theorem (without proof)</u>

If preferences satisfy assumptions (1) – (5) then there exists a *utility function* $u(x)$ that represents those preferences, such that

$$x \sim\succ y \;\Leftrightarrow\; u(x) \geq u(y)$$

$$x \succ y \;\Leftrightarrow\; u(x) > u(y)$$

$$x \sim y \;\Leftrightarrow\; u(x) = u(y)$$

The utility function representing a particular preference ordering is *not* unique. If $u(x)$ represents a preference ordering, then any monotonic transformation of that function also represents those preferences.

In particular, the *marginal rate of substitution* (MRS) is invariant to a monotonic transformation of the utility function. To obtain the MRS between two goods x_i and x_j, totally differentiate the utility function to obtain

$$du = \frac{\partial u}{\partial x_i} dx_i + \frac{\partial u}{\partial x_j} dx_j$$

By definition, along an indifference curve, $du = 0$. Thus, the slope of the indifference curve (in absolute value) is

$$\left| \frac{dx_i}{dx_j} \right| = \frac{\partial u}{\partial x_j} \Big/ \frac{\partial u}{\partial x_i} \equiv MRS_{ij}$$

Now let $v(u)$ be a monotonic transformation of *u*. Then

$$dv = v'(u) \frac{\partial u}{\partial x_i} dx_i + v'(u) \frac{\partial u}{\partial x_j} dx_j$$

Setting $dv = 0$:

$$\left| \frac{dx_i}{dx_j} \right| = v'(u) \frac{\partial u}{\partial x_j} \Big/ v'(u) \frac{\partial u}{\partial x_i} = \frac{\partial u}{\partial x_j} \Big/ \frac{\partial u}{\partial x_i}$$

Why must $v(u)$ be monotonic? This ensures that $v'(u)$ does not change sign; otherwise rankings could be reversed.

The non-uniqueness of a utility representation of preferences means that $u(x)$ has no cardinality; that is, a cardinal value for utility, such as $u = 25$, has no meaning. The utility function only has *ordinal* interpretation: it orders (or ranks) consumption bundles as better or worse, but says nothing about how much better or worse.

We now wish to relate convexity of preferences to the utility function. To do so, we first need to define *quasi-concavity*.

Technical Note. A function $f(x)$ is (strictly) quasi-concave iff

$$f(tx' + (1-t)x'') \geq (>) \min\{f(x'), f(x'')\} \quad \forall x', x'' \quad \forall t \in (0,1)$$

The critical feature of a quasi-concave function is that it has no local minima. See Figure 2.2 for an example of a quasi-concave function, and Figure 2.3 for an example of a function that is not quasi-concave. ♠

Theorem

If preferences are (strictly) convex, then the utility function representing those preferences is (strictly) quasi-concave.

Proof. If preferences are strictly convex then $tx + (1-t)y \succ z$ for $x \sim z$ and $y \sim z$. If $u(x)$ represents those preferences then:

$$u(x) = u(z),\ u(y) = u(z) \text{ and } u(tx + (1-t)y) > u(z)$$

Therefore

$$u(tx + (1-t)y) > \min\{u(x), u(y)\}$$

That is, $u(x)$ is quasi-concave. ♣

2.2 Utility Maximization

The consumer's problem is:

$$\max_x u(x) \quad \text{subject to } px \leq m$$

Since $u(x)$ is strictly increasing – by assumption (5) on preferences – the income constraint will bind at the optimum, so we can write

$$\max_x u(x) \quad \text{subject to } px = m$$

Let $x(p,m)$ be the solution vector. The elements of that vector, $x_i(p,m)$, are the *Marshallian (or ordinary) demand functions*.

If $u(x)$ is differentiable (and it need not be), we can derive $x(p,m)$ using calculus.[1]

Form the Lagrangean:

$$L = u(x) + \lambda(m - px)$$

and differentiate with respect to *x* and λ to obtain the first order conditions:

$$\partial u / \partial x_i - \lambda p_i = 0 \quad \forall i$$

$$m - px = 0$$

To interpret these conditions, take the ratio of a pair of conditions *i* and *j*:

$$\frac{\partial u / \partial x_i}{\partial u / \partial x_j} = \frac{p_i}{p_j}$$

That is, $MRS_{ij} =$ price ratio. Geometrically, this is a *tangency condition*: the slope of the indifference curve is equal to the slope of the budget constraint. The intercepts of the budget constraint tie down that tangency point – and the associated Marshallian demands – to the particular income level faced by the consumer. See Figure 2.4.

<u>Second-Order Conditions and Optimizing Behaviour</u>

Technical Note. It is straightforward to show that in any constrained optimization problem, if the constraint is linear (as the budget constraint is) and the objective function is quasi-concave and continuously differentiable then the first-order conditions are sufficient for a global maximum.♠

Implication for consumer theory: quasi-concavity of $u(x)$ (convexity of preferences) is sufficient for a maximum. Weak convexity allows the possibility of a non-unique maximum; strict convexity (strict quasi-concavity of $u(x)$) ensures a unique maximum.

It is essential to understand that the consumer always acts to maximize utility, by assumption. If the consumer has convex preferences then the first-order conditions describe maximizing behavior. We can then make inferences about behaviour from these

[1] See the Appendix to this Chapter for a brief review of optimization methods.

conditions by imposing the SOCs. That is, we use the SOC to inform us about what maximizing behaviour implies for consumer demand.

2.3 Examples

1. Cobb-Douglas

$$u(x) = x_1^a x_2^b$$

$$L = x_1^a x_2^b + \lambda(m - p_1 x_1 - p_2 x_2)$$

First-order conditions:

$$a x_1^{a-1} x_2^b - \lambda p_1 = 0$$

$$b x_1^a x_2^{b-1} - \lambda p_2 = 0$$

Take ratio:

$$\frac{a x_2}{b x_1} = \frac{p_1}{p_2}$$

$$\Rightarrow x_2 = \frac{b p_1 x_1}{a p_2}$$

Substitute into budget constraint:

$$m = p_1 x_1 + p_2 \left(\frac{b p_1 x_1}{a p_2} \right)$$

$$\Rightarrow x_1(p, m) = \frac{am}{(a+b) p_1}$$

$$\Rightarrow x_2(p, m) = \frac{bm}{(a+b) p_2}$$

2. Log-Linear

$$u(x) = a \log x_1 + b \log x_2$$

$$L = a \log x_1 + b \log x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

First-order conditions:

$$\frac{a}{x_1} - \lambda p_1 = 0$$

$$\frac{b}{x_2} - \lambda p_2 = 0$$

Take ratio:

$$\frac{ax_2}{bx_1} = \frac{p_1}{p_2}$$

Same as C-D result above. Hence,

$$x_1(p,m) = \frac{am}{(a+b)p_1}$$

$$x_2(p,m) = \frac{bm}{(a+b)p_2}$$

Why the same as C-D? Case (2) is a logarithmic (and therefore monotonic) transformation of (1).

3. Generalized Log-Linear

$$u(x) = \sum_{i=1}^{n} a_i \log x_i$$

or equivalently,

$$u(x) = \prod_{i=1}^{n} x_i^{a_i}$$

First-order conditions:

$$\frac{a_i}{x_i} - \lambda p_i = 0$$

(*) $\Rightarrow a_i = \lambda p_i x_i$

Sum both sides over *i*:

$$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} p_i x_i$$

Substitute for m:

$$\sum_{i=1}^{n} a_i = \lambda m$$

$$\Rightarrow \lambda = \sum_{i=1}^{n} a_i \Big/ m$$

Substitute into (*):

$$x_i(p,m) = \frac{a_i m}{p_i \sum_{i=1}^{n} a_i}$$

4. Quasi-Linear

$$u(x) = \log x_1 + x_2$$

$$L = \log x_1 + x_2 + \lambda(m - p_1 x_1 - p_2 x_2)$$

First-order conditions:

$$\frac{1}{x_1} - \lambda p_1 = 0$$

$$1 - \lambda p_2 = 0$$

$$\Rightarrow \lambda = \frac{1}{p_2}$$

$$\Rightarrow x_1(p,m) = \frac{p_2}{p_1} \quad \text{if} \quad p_1\left(\frac{p_2}{p_1}\right) \le m$$

Then from the budget constraint:

If $p_2 \le m$:

$$x_1(p,m) = \frac{p_2}{p_1}$$

$$x_2(p,m) = \frac{m - p_1 x_1}{p_2} = \frac{m - p_2}{p_2}$$

If $p_2 > m$:

$$x_2(p,m) = 0$$

$$x_1(p,m) = \frac{m}{p_1}$$

That is, if $p_2 > m$ we have a corner solution. See Figure 2-5.

5. Leontief (Fixed Proportions)

This is a non-differentiable example:

$$u(x) = \min\{ax_1, bx_2\}$$

If $ax_1 = bx_2$ then any increase in either x_1 or x_2 (but not both) has no effect on $u(x)$. Thus, the consumer will never choose any bundle $\{x_1, x_2\}$ such that $ax_1 \neq bx_2$. So the "tangency condition" is:

$$ax_1 = bx_2$$

See Figure 2.6. Then using the budget constraint:

$$m = p_1 x_1 + p_2 \left(\frac{ax_1}{b} \right)$$

$$\Rightarrow x_1(p,m) = \frac{bm}{bp_1 + ap_2}$$

$$\Rightarrow x_2(p,m) = \frac{am}{bp_1 + ap_2}$$

2.4 The Indirect Utility Function

The maximum value function for the consumer problem is the *indirect utility function*:

$$v(p,m) = u\big(x(p,m)\big)$$

Example: Cobb-Douglas

$$u(x) = x_1^a x_2^b$$

$$x_1(p,m) = \frac{am}{(a+b)p_1}$$

$$x_2(p,m) = \frac{bm}{(a+b)p_2}$$

Upon substitution:

$$v(p,m) = \left(\frac{am}{(a+b)p_1} \right)^a \left(\frac{bm}{(a+b)p_2} \right)^b = \left(\frac{m}{a+b} \right)^{a+b} \left(\frac{a}{p_1} \right)^a \left(\frac{b}{p_2} \right)^b$$

Technical Note: The Envelope Theorem

Consider the following general optimization problem:

$$\max_{x} \; f(x,a) \text{ subject to } g(x,a)=0$$

where x and a are vectors. Utility maximization is a special case where $f(x,a)=u(x)$ (and hence $f'(a)=0$) and $g(x,a)=m-px$ (and hence $a=\{m,p\}$). Let

$$L = f(x,a)+\lambda g(x,a)$$

be the associated Lagrangean, and let $x^*(a)$ denote the solution. Furthermore, let $v(a)\equiv f(x^*(a),a)$ denote the associated maximum value function. Then the *envelope theorem* states that

$$v'(a)=\frac{\partial f}{\partial a}+\lambda\frac{\partial g}{\partial a}$$

That is, in calculating the effect of a change in a parameter on the maximum value function, we need only consider the direct effect through *f* and *g*; the indirect effect, through $x^*(a)$, can be ignored.

Proof. By differentiation of $v(a)$:

(1) $$v'(a)=\sum_{i=1}^{n}\left(\frac{\partial f(x^*(a),a)}{\partial x_i}\right)\left(\frac{\partial x_i^*}{\partial a}\right)+\frac{\partial f(x^*(a),a)}{\partial a}$$

But since $x^*(a)$ is optimal for *a*, it must satisfy the FOCs. Thus,

(2) $$\frac{\partial f(x^*(a),a)}{\partial x_i}=-\lambda\frac{\partial g(x^*(a),a)}{\partial x_i} \quad \forall i$$

(3) $$g(x^*(a),a)=0$$

Differentiating equation (3) – the constraint – yields

(4) $$\sum_{i=1}^{n}\left(\frac{\partial g(x^*(a),a)}{\partial x_i}\right)\left(\frac{\partial x_i^*}{\partial a}\right)+\frac{\partial g(x^*(a),a)}{\partial a}=0$$

Substituting (2) and (4) into (1) yields

$$v'(a)=\frac{\partial f}{\partial a}+\lambda\frac{\partial g}{\partial a}$$

which proves the result.♣

Interpretation. A change in the parameter does cause a change in the value of $x^*(a)$ but at the margin that change in x^* has no effect on *L* since x^* is a turning point of *L*.

The Marginal Utility of Income

The envelope theorem implies an important relationship between the indirect utility function and the Lagrange multiplier. In the utility-maximization problem, the constraint is $g(x,m) = m - \sum_{i=1}^{n} p_i x_i$, where m takes the role of *a*. This is linear in m, so $\partial g / \partial a = 1$ for this problem. Moreover, m does not enter the utility function directly in this problem, so $\partial f / \partial a = 0$. Thus, the envelope theorem tells us that $v'(m) = \lambda$; the marginal utility of income is equal to the Lagrange multiplier. That is, the value of the Lagrange multiplier at the optimum measures the amount by which utility rises if income rises by one dollar. It is for this reason that the Lagrange multiplier is called the *shadow price* of the constraint.

2.5 Expenditure Minimization and the Hicksian Demands

An equivalent mathematical representation of utility maximization is *expenditure minimization*:

$$\min_x \; px \text{ subject to } u(x) = u$$

Construct the Lagrangean:

$$L = px + \phi\big(u - u(x)\big)$$

First-order conditions:

$$p_i = \phi \frac{\partial u}{\partial x_i} \quad \forall i$$

$$u(x) = u$$

Take ratio of a pair *i* and *j*:

$$\frac{p_i}{p_j} = \frac{\partial u}{\partial x_i} \Big/ \frac{\partial u}{\partial x_j}$$

That is, price ratio $= MRS_{ij}$. Thus, utility maximization $\Leftrightarrow$ expenditure minimization. In Section 2.7 below we will see more clearly how the expenditure minimization problem and the utility maximization problem are related.

Solution of the FOCs to the expenditure minimization problem yields the *Hicksian* (or compensated) demand functions: $h_i(p,u)$.

Example: Cobb-Douglas

$$L = p_1 x_1 + p_2 x_2 + \phi(u - x_1^a x_2^b)$$

First-order conditions:

(1) $p_1 - \phi a x_1^{a-1} x_2^b = 0$

(2) $p_2 - \phi b x_1^a x_2^{b-1} = 0$

(3) $u - x_1^a x_2^b = 0$

From (1) and (2) we obtain the tangency condition:

(4) $\dfrac{p_1}{p_2} = \dfrac{a x_2}{b x_1}$

Substitute (4) into (3) and solve for x_2 :

$$h_2(p,u) = \left(u \left(\frac{b p_1}{a p_2} \right)^a \right)^{\frac{1}{a+b}}$$

Then use (4) to solve for x_1 :

$$h_1(p,u) = \left(u \left(\frac{a p_2}{b p_1} \right)^b \right)^{\frac{1}{a+b}}$$

Relationship between the Hicksian and Marshallian Demands

Consider the demand response to $\Delta p_1 : p_1^o \rightarrow p_1'$, as illustrated in Figure 2.7 for $p_1' < p_1^0$. The demand response comprises a *substitution effect* (SE) and an *income effect* (IE).

SE measures the demand response associated with the price change given that the individual is compensated with just enough income to restore her utility to its initial level (at the new prices), at the point labeled $\tilde{x}$ in the upper frame of Figure 2.7.

IE measures the demand response due to the change in real income associated with the price change. (Figure 2.7 is drawn for the case of a normal good: the income effect and substitution effect work in the same direction. For an inferior good the income effect for a price fall is negative. We will return to this issue in Topic 3).

The Hicksian (or compensated) demand function measures only the SE associated with a price change; see the lower frame of Figure 2.7. That is, it represents the relationship between price and quantity demanded when utility is held constant via an income compensation. The Marshallian (or ordinary) demand function measures the combined SE and IE.

In the case of a normal good (as illustrated in Figure 2.7), the Marshallian demand is flatter than the Hicksian demand because the SE and IE act in the same direction. For an inferior good, the Marshallian demand is steeper than the Hicksian demand.

2.6 The Expenditure Function

The minimum value function for the expenditure minimization problem is the *expenditure function*:

$$e(p,u) = p \cdot h(p,u)$$

It measures the minimum expenditure that is necessary to achieve *u* at given *p*.

<u>Example:</u> Cobb-Douglas

$$e(p,u) = p_1 h_1(p,u) + p_2 h_2(p,u)$$

$$= p_1 \left(u \left(\frac{ap_2}{bp_1} \right)^b \right)^{\frac{1}{a+b}} + p_2 \left(u \left(\frac{bp_1}{ap_2} \right)^a \right)^{\frac{1}{a+b}}$$

Properties of the Expenditure Function

The following properties are implied directly by the definition of $e(p,u)$ as a minimum value function; they do *not* rely on convexity of preferences.

1. $e(p,u)$ is non-decreasing in *p* and increasing in *u*.
2. $e(p,u)$ is homogeneous of degree 1 in *p*.
3. $e(p,u)$ is concave in *p*.

Technical Note. A function $f(x)$ is homogeneous of degree α iff

$$f(\theta x) = \theta^{\alpha} f(x) \quad \forall \theta$$

Technical Note. A function $f(x)$ is (strictly) concave iff

$$f(tx' + (1-t)x'') \geq (>) tf(x') + (1-t) f(x'') \quad \forall x', x'' \quad \forall t \in (0,1)$$

See Figure 2.8 for an example of a strictly concave function, and Figure 2.9 for an example of a function that is not concave (though it is quasi-concave). Note that all concave functions are quasi-concave but not all quasi-concave functions are concave.♠

Proof of Property 3

Let $p'' = tp^o + (1-t)p'$. Then

$$\begin{aligned} e(p'',u) &= p'' \cdot h(p'',u) \\ &= [tp^o + (1-t)p']h(p'',u) \\ &= tp^o h(p'',u) + (1-t)p'h(p'',u) \end{aligned}$$

By definition of $e(p,u)$ as the minimum value function:

$$\begin{bmatrix} p^o h(p'',u) \geq e(p^o,u) \\ p'h(p'',u) \geq e(p',u) \end{bmatrix}$$

Therefore

$$tp^o h(p'',u) + (1-t)p'h(p'',u) \geq te(p^o,u) + (1-t)e(p',u)$$

which in turn implies

$$e(p'', u) \geq te(p^o, u) + (1-t)e(p', u)$$

That is, $e(p,u)$ is concave in p.♣

In Topic 3 we will use the concavity of $e(p,u)$ to derive the properties of the Hicksian demands, and by extension, of the Marshallian demands. Thus, the concavity of $e(p,u)$ is a key result in consumer theory.

2.7 Some Important Identities

1. The minimum expenditure needed to achieve utility $v(p,m)$ is m:

$$e(p, v(p,m)) \equiv m$$

2. The maximum utility attainable from income $e(p,u)$ is u:

$$v(p, e(p,u)) \equiv u$$

3. Identities (1) and (2) together imply that $v(p,m)$ and $e(p,u)$ are inverses of each other:

$$e(p,u) \equiv v^{-1}(p,u)$$

$$v(p,m) \equiv e^{-1}(p,m)$$

4. The value of the Marshallian demand at income m is the same as the value of the Hicksian demand at utility $v(p,m)$:

$$x(p,m) \equiv h(p, v(p,m))$$

5. The value of the Hicksian demand at utility u is the same as the value of the Marshallian demand at income $e(p,u)$:

$$h(p,u) \equiv x(p, e(p,u))$$

6. Roy's Identity

$$-\frac{\partial v / \partial p_i}{\partial v / \partial m} = x_i(p,m)$$

if $m > 0$, $p_i > 0$ and the LHS derivative exists.

Proof. By the envelope theorem:

$$\frac{\partial v}{\partial p_i} = -\lambda x_i(p,m) \quad \text{and} \quad \frac{\partial v}{\partial m} = \lambda$$

Take the ratio to obtain the result.♣

7. Shephard's Lemma

$$\frac{\partial e(p,u)}{\partial p_i} = h_i(p,u)$$

if $p_i > 0$ and the LHS derivative exists.

Proof. By the envelope theorem.♣

For a graphical summary of these results, see Figure 2.10

2.8 Example: Cobb-Douglas Utility

Identity 1

$$e(p,v(p,m)) = p_1\left(v(p,m)\left(\frac{ap_2}{bp_1}\right)^b\right)^{\frac{1}{a+b}} + p_2\left(v(p,m)\left(\frac{bp_1}{ap_2}\right)^a\right)^{\frac{1}{a+b}}$$

Substituting for

$$v(p,m) = \left(\frac{m}{a+b}\right)^{a+b}\left(\frac{a}{p_1}\right)^a\left(\frac{b}{p_2}\right)^b$$

yields $e(p,v(p,m)) = m$.

Identity 2

$$v(p,e(p,u)) = \left(\frac{e(p,u)}{a+b}\right)^{a+b}\left(\frac{a}{p_1}\right)^{a}\left(\frac{b}{p_2}\right)^{b}$$

Substituting for

$$e(p,u) = p_1\left(u\left(\frac{ap_2}{bp_1}\right)^{b}\right)^{\frac{1}{a+b}} + p_2\left(u\left(\frac{bp_1}{ap_2}\right)^{a}\right)^{\frac{1}{a+b}}$$

yields $v(p,e(p,u)) \equiv u$.

Identity 3

Set $v(p,m) = u$:

$$u = \left(\frac{m}{a+b}\right)^{a+b}\left(\frac{a}{p_1}\right)^{a}\left(\frac{b}{p_2}\right)^{b}$$

Solve for *m* and set $e(p,u) = m$ to obtain

$$e(p,u) = p_1\left(u\left(\frac{ap_2}{bp_1}\right)^{b}\right)^{\frac{1}{a+b}} + p_2\left(u\left(\frac{bp_1}{ap_2}\right)^{a}\right)^{\frac{1}{a+b}}$$

Similarly, set $e(p,u) = m$:

$$m = p_1\left(u\left(\frac{ap_2}{bp_1}\right)^{b}\right)^{\frac{1}{a+b}} + p_2\left(u\left(\frac{bp_1}{ap_2}\right)^{a}\right)^{\frac{1}{a+b}}$$

Solve for *u* and set $v(p,m) = u$ to obtain

$$v(p,m) = \left(\frac{m}{a+b}\right)^{a+b}\left(\frac{a}{p_1}\right)^{a}\left(\frac{b}{p_2}\right)^{b}$$

Identity 4

Substitute

$$v(p,m) = \left(\frac{m}{a+b}\right)^{a+b}\left(\frac{a}{p_1}\right)^{a}\left(\frac{b}{p_2}\right)^{b}$$

for *u* in

$$h_1(p,u) = \left(u \left(\frac{ap_2}{bp_1} \right)^b \right)^{\frac{1}{a+b}}$$

to obtain

$$x_1(p,m) = \frac{am}{(a+b)p_1}$$

Identity 5

Substitute

$$e(p,u) = p_1 \left(u \left(\frac{ap_2}{bp_1} \right)^b \right)^{\frac{1}{a+b}} + p_2 \left(u \left(\frac{bp_1}{ap_2} \right)^a \right)^{\frac{1}{a+b}}$$

for *m* in

$$x_1(p,m) = \frac{am}{(a+b)p_1}$$

to obtain

$$h_1(p,u) = \left(u \left(\frac{ap_2}{bp_1} \right)^b \right)^{\frac{1}{a+b}}$$

Roy's Identity

Differentiate

$$v(p,m) = \left(\frac{m}{a+b} \right)^{a+b} \left(\frac{a}{p_1} \right)^a \left(\frac{b}{p_2} \right)^b$$

with respect to p_1 to obtain

$$\frac{\partial v}{\partial p_1} = -a \left(\frac{m}{a+b} \right)^{a+b} \left(\frac{a}{p_1} \right)^{a-1} \left(\frac{b}{p_2} \right)^b \left(\frac{a}{p_1^2} \right)$$

and with respect to *m* to obtain

$$\frac{\partial v}{\partial m} = \left(\frac{m}{a+b} \right)^{a+b-1} \left(\frac{a}{p_1} \right)^a \left(\frac{b}{p_2} \right)^b$$

Then take the (negative of) the ratio to obtain

$$x_1(p,m) = \frac{am}{(a+b)p_1}$$

Shephard's Lemma

Differentiate

$$e(p,u) = p_1\left(u\left(\frac{ap_2}{bp_1}\right)^b\right)^{\frac{1}{a+b}} + p_2\left(u\left(\frac{bp_1}{ap_2}\right)^a\right)^{\frac{1}{a+b}}$$

with respect to p_1 to obtain

$$h_1(p,u) = \left(u\left(\frac{ap_2}{bp_1}\right)^b\right)^{\frac{1}{a+b}}$$

APPENDIX A2: CONSTRAINED OPTIMIZATION

A2.1 The Constrained Optimization Problem

We begin with a constrained optimization problem of the type

$$\max_{x} \quad f(x_1,...,x_n) \quad \text{subject to} \quad g(x_1,...,x_n) = b$$

The function $f(x_1,...,x_n)$ is called the *objective function* or *maximand*; the equation $g(x_1,...,x_n) = b$ is called the *constraint*.

Remarks

1. We are restricting attention here to equality-constrained problems. An inequality-constrained problem would arise where the constraint is $g(x_1,...,x_n) \leq b$. The techniques we develop here can be extended easily to that case.
2. A minimization problem with objective function $f(x)$ can be set up as a maximization problem with objective function $-f(x)$.

A2.2 Characteristics of the Optimum

At the maximum of the objective function subject to the constraint, infinitesimal changes in the variables $x_1, x_2,..., x_n$ which satisfy the constraint must have no effect on the value of the objective function. Otherwise, we could not be at a maximum.

Thus, a necessary condition for the maximum is that $df = 0$ whenever $dg = 0$. That is, we require

(A2.1) $$\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + ... + \frac{\partial f}{\partial x_n}dx_n = 0$$

for all dx_1, dx_2, …, dx_n satisfying

(A2.2) $$\frac{\partial g}{\partial x_1}dx_1 + \frac{\partial g}{\partial x_2}dx_2 + ... + \frac{\partial g}{\partial x_n}dx_n = 0$$

where $\partial f / \partial x_i$ is the partial derivative of *f* with respect to x_i, and $\partial g / \partial x_i$ is the partial derivative of *g* with respect to x_i.

These conditions tells us that at the optimum there must be no way in which we can change the x_i's such that we can change the value of the function (and in particular, increase the value of the function) and still satisfy the constraint.

A2.3 The Unconstrained Optimum

Note that in the absence of the constraint we would be seeking conditions under which (A2.1) holds for all dx_1, dx_2, …, dx_n, rather than only those changes in *x* that satisfy (A2.2). That is, we would require that the dx_i have zero coefficients in (A2.1):

(A2.3) $$\frac{\partial f}{\partial x_i} = 0 \quad \forall i$$

Note that this is a set of n equations in *n* unknowns.

Example

Profit maximization for a "competitive" firm with Cobb-Douglas technology, given by

(A2.4) $$h(x) = x_1^a x_2^b$$

The profit maximization problem is

(A2.5) $$\max_x \quad p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

with first-order conditions

(A2.6) $$a p x_1^{a-1} x_2^b = w_1$$

and

(A2.7) $$b p x_1^a x_2^{b-1} = w_2$$

We can solve these equations by first taking the ratio of (A2.6) and (A2.7) to obtain

(A2.8) $$\frac{a x_2}{b x_1} = \frac{w_1}{w_2}$$

Now rearrange (A2.8) to obtain:

(A2.9) $$x_2 = \frac{bw_1x_1}{aw_2}$$

Substitute (A2.9) into (A2.6) or (A2.7) and solve for x_1:

(A2.10) $$x_1(p,w) = \left(\frac{bp}{w_2}\right)^{\frac{1}{1-a-b}} \left(\frac{aw_2}{bw_1}\right)^{\frac{1-b}{1-a-b}}$$

Then substitute (A2.10) into (A2.9) to obtain $x_2(p,w)$. These are the *factor demands* or *input demands*. We can then construct the *supply function* by substituting these factor demands into the production function:

(A2.11) $$y(p,w) = x_1(p,w)^a\, x_2(p,w)^b$$

A2.4 The Constrained Optimum: Solution by Substitution

Rewrite (A2.2) to isolate dx_1:

(A2.12) $$dx_1 = -\frac{\sum_{i\neq 1}(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}$$

Now substitute this expression for dx_1 into (A2.1) to obtain

(A2.13) $$-\frac{\partial f}{\partial x_1}\left(\frac{\sum_{i\neq 1}(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}\right) + \sum_{i\neq 1}\frac{\partial f}{\partial x_i}dx_i = 0$$

This can be written as

(A2.14) $$\sum_{i\neq 1}\left(-\frac{(\partial f/\partial x_1)(\partial g/\partial x_i)dx_i}{\partial g/\partial x_1}\right) + \sum_{i\neq 1}\frac{\partial f}{\partial x_i}dx_i = 0$$

Then by collecting terms under the summation operator, we have

(A2.16) $$\sum_{i\neq 1}\left(\frac{\partial f}{\partial x_i} - \frac{(\partial f/\partial x_1)(\partial g/\partial x_i)}{\partial g/\partial x_1}\right)dx_i = 0$$

The only solution to this equation is to set all of the coefficients on the dx_i 's equal to zero since the equation must hold for <u>all</u> possible values of the dx_i 's. That is, (A2.16) $$\frac{\partial f}{\partial x_i} = \frac{(\partial f / \partial x_1)(\partial g / \partial x_i)}{\partial g / \partial x_1} \quad \forall i \neq 1$$

This can in turn be written as

(A2.17) $$\frac{\partial f / \partial x_i}{\partial g / \partial x_i} = \frac{\partial f / \partial x_1}{\partial g / \partial x_1} \quad \forall i \neq 1$$

Note that (A2.17) comprises $n-1$ equations. Together with the constraint itself we therefore have *n* equations which can be solved for the *n* unknowns (the x_i 's).

A2.5 Example: Utility Maximization

Recall the utility maximization problem for $n=2$. For that example, equation (A2.17) – which is a single equation in the $n=2$ case – becomes

(A2.18) $$\frac{\partial f / \partial x_2}{\partial g / \partial x_2} = \frac{\partial f / \partial x_1}{\partial g / \partial x_1}$$

This in turn can be rearranged as

(A2.19) $$\frac{\partial f / \partial x_1}{\partial f / \partial x_2} = \frac{\partial g / \partial x_1}{\partial g / \partial x_2}$$

In the utility maximization problem we have $\partial f / \partial x_i \equiv \partial u / \partial x_i$ and $\partial g / \partial x_i \equiv p_i$. Thus, (A2.19) becomes

(A2.20) $$\frac{\partial u / \partial x_1}{\partial u / \partial x_2} = \frac{p_1}{p_2}$$

This is the familiar tangency condition, stating that the slope of the indifference curve (the marginal rate of substitution) is equal to the slope of the budget constraint at the optimum.

In the specific case of Cobb-Douglas (CD) utility, (A2.20) becomes

(A2.21) $$\frac{a x_1^{a-1} x_2^{b}}{b x_1^{a} x_2^{b-1}} \equiv \frac{a x_2}{b x_1} = \frac{p_1}{p_2}$$

Thus, in the case of CD utility, the consumption ratio is inversely proportional to the price ratio.

Note that (A2.21) is one equation in two unknowns. It tells us the relationship between x_1 and x_2 at the optimum but cannot be solved for unique values of x_1 and x_2. In the geometric interpretation, it tells us that we must have a tangency but it does not tells us *where* that tangency must be. For that we need additional information: the position of the budget constraint (as opposed to its slope). That information is contained in the budget constraint itself, which in the $n = 2$ case is

(A2.22) $$p_1 x_1 + p_2 x_2 = m$$

Combining equations (A2.21) and (A2.22), we have two equations in two unknowns, which can be solved by simple substitution. In particular, express (A2.21) as

(A2.23) $$p_1 x_1 = \frac{a p_2 x_2}{b}$$

and substitute into the budget constraint to obtain

(A2.24) $$\frac{a p_2 x_2}{b} + p_2 x_2 = m$$

Solving for x_2 yields

(A2.25) $$x_2(p, m) = \frac{bm}{(a+b)p_2}$$

Substituting this solution for x_2 into (A2.25) then yields the solution for x_1:

(A2.26) $$x_1(p, m) = \frac{am}{(a+b)p_1}$$

Equations (A2.25) and (A2.26) are called the *Marshallian demands*; they relate the demand for each good to the prices, and to income.

A2.6 The Lagrange Multiplier Approach

The Lagrange multiplier approach to the constrained maximization problem is a useful mathematical algorithm that allows us to reconstruct the constrained problem as an unconstrained problem which yields (A2.17) as its solution.

Consider the problem

(A2.27) $$\max_{x} \ f(x) \ \text{ subject to } \ g(x,b)=0$$

where $x=\{x_1,...,x_n\}$ and b is a parameter in the constraint, identified explicitly here because we have some particular interest in it.

To solve this problem, we define the *Lagrangean* by introducing a new variable λ called the *Lagrange multiplier* (LM):

(A2.28) $$L(x,\lambda)=f(x)+\lambda g(x)$$

We then solve the unconstrained maximization problem

(A2.29) $$\max_{x,\lambda} \ L(x,\lambda)$$

The necessary (first-order) conditions for a maximum are

(A2.30) $$\frac{\partial L}{\partial x_i}=0 \quad \forall i$$

and

(A2.31) $$\frac{\partial L}{\partial \lambda}=0$$

Note that (A2.30) comprises n equations, so together with (A2.31) we have $n+1$ equations in $n+1$ unknowns (including λ). Take these derivatives of $L(x,\lambda)$ to yield

(A2.32) $$\frac{\partial L}{\partial x_i}=\frac{\partial f}{\partial x_i}-\lambda\frac{\partial g}{\partial x_i}=0 \quad \forall i$$

and

(A2.33) $$\frac{\partial L}{\partial \lambda}=g(x,b)=0$$

Now take the ratio of any two equations from (A2.32), say for $i=1$ and $i=j$:

(A2.34) $$\frac{\partial f/\partial x_j}{\partial f/\partial x_1}=\frac{\partial g/\partial x_j}{\partial g/\partial x_1}\quad \forall j\neq 1$$

Note that the LM has now been eliminated. This expression can be rearranged to yield

(A2.35) $$\frac{\partial f/\partial x_j}{\partial g/\partial x_j}=\frac{\partial f/\partial x_1}{\partial g/\partial x_1}\quad \forall j\neq 1$$

These $n-1$ equations are the same as conditions (A2.17). Thus, the LM method yields the solution to the constrained optimization problem. The additional information we need to complete the solution is the constraint itself, and this is given by (A2.33). Thus, conditions (A2.33) and (A2.35) describe a complete solution to the constrained problem.

Example 1: CD Utility Maximization Revisited

Recall the constrained optimization problem for CD utility:

(A2.36) $$\max_x\ x_1^a x_2^b \quad \text{subject to} \quad p_1x_1+p_2x_2=m$$

Construct the Lagrangean:

(A2.37) $$L=x_1^a x_2^b+\lambda(m-p_1x_1-p_2x_2)$$

Derive the first-order conditions:

(A2.38) $$ax_1^{a-1}x_2^b-\lambda p_1=0$$

(A2.39) $$bx_1^a x_2^{b-1}-\lambda p_2=0$$

(A2.40) $$m-p_1x_1-p_2x_2=0$$

Take the ratio of (A2.40) and (A2.39) to obtain

(A2.41) $$\frac{ax_2}{bx_1}=\frac{p_1}{p_2}$$

and rearrange this to yield

(A2.42) $$x_2 = \frac{bp_1x_1}{ap_2}$$

Now substitute (A2.42) into the budget constraint (A2.40):

(A2.43) $$m = p_1x_1 + p_2\left(\frac{bp_1x_1}{ap_2}\right)$$

and solve for x_1:

(A2.44) $$x_1(p,m) = \frac{am}{(a+b)p_1}$$

Then substitute (A2.44) into (A2.42) to yield

(A2.45) $$x_2(p,m) = \frac{bm}{(a+b)p_2}$$

Example 2: Generalized Log-Linear Utility

(A2.46) $$\max_x \sum_{i=1}^{n} a_i \log x_i \quad \text{subject to} \quad \sum_{i=1}^{n} p_1 x_i = m$$

Construct the Lagrangean:

(A2.47) $$L = \sum_{i=1}^{n} a_i \log x_i + \lambda\left(m - \sum_{i=1}^{n} p_i x_i\right)$$

Derive the first-order conditions:

(A2.48) $$\frac{a_i}{x_i} - \lambda p_i = 0 \quad \forall i$$

(A2.49) $$m - \sum_{i=1}^{n} p_i x_i = 0$$

In this example, taking the ratio of any pair of equations from (A2.48) will yield the usual tangency condition, but it is not the most efficient way to solve the problem. Instead we will use an alternative solution method that usually performs better when we have $n > 2$ variables.

Rearrange (A2.48) to yield

(A2.50) $$a_i = \lambda p_i x_i \quad \forall i$$

Now take the <u>sum</u> over *i* on both sides of (A2.50) to yield

(A2.51) $$\sum_{i=1}^{n} a_i = \lambda \sum_{i=1}^{n} p_i x_i$$

Substitute *m* for expenditure in the RHS term to yield:

(A2.52) $$\sum_{i=1}^{n} a_i = \lambda m$$

and rearrange to solve for λ:

(A2.53) $$\lambda = \frac{\sum_{i=1}^{n} a_i}{m}$$

Now substitute (A2.53) for λ in (A2.50) and solve for x_i:

(A2.54) $$x_i(p,m) = \frac{a_i m}{p_i \sum_{i=1}^{n} a_i}$$

Note that in the special case where $n = 2$, these solutions become

(A2.55) $$x_1(p,m) = \frac{a_1 m}{(a_1 + a_2) p_1}$$

(A2.56) $$x_2(p,m) = \frac{a_2 m}{(a_1 + a_2) p_2}$$

Compare these with the CD Marshallian demands from (A2.44) and (A2.45). They are the same solutions (with $a_1 = a$ and $a_2 = b$). Why? The CD utility function and the log-linear utility function represent exactly the same preferences; one function is a monotonic transform of the other. A monotonic transform does not change the underlying preferences because preferences have no cardinal interpretation; they are an ordinal notion only.

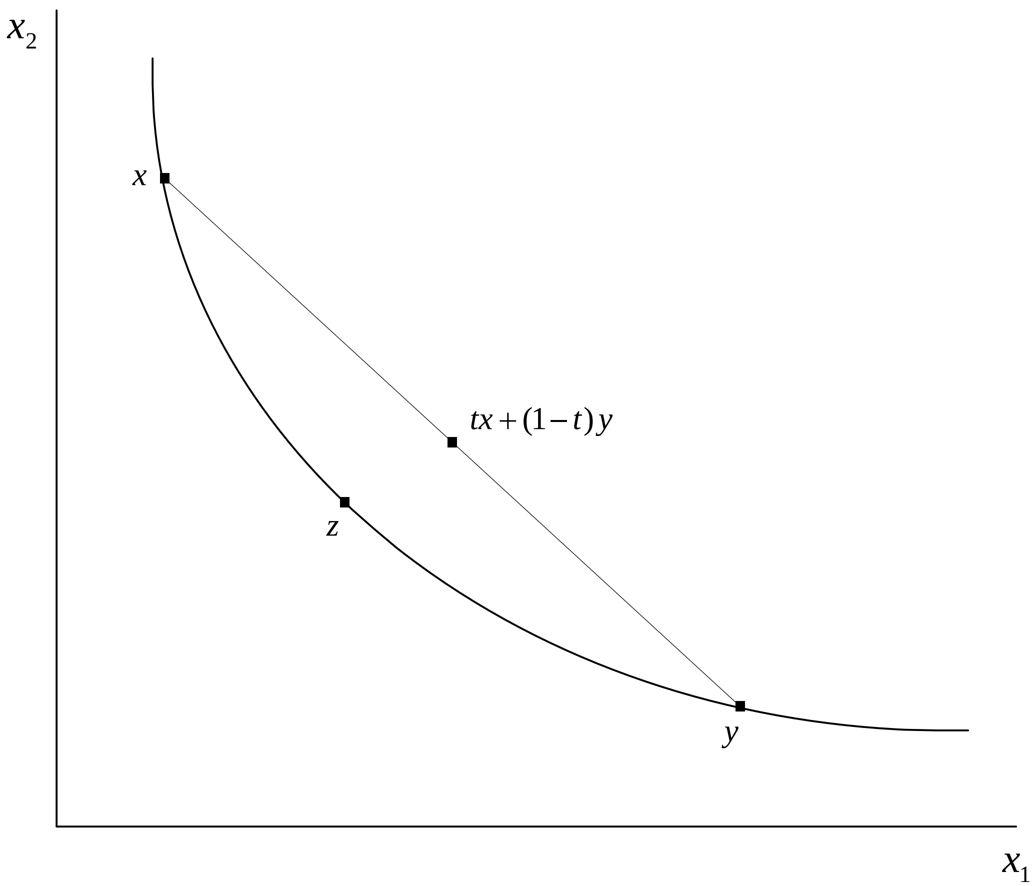


FIGURE 2.1

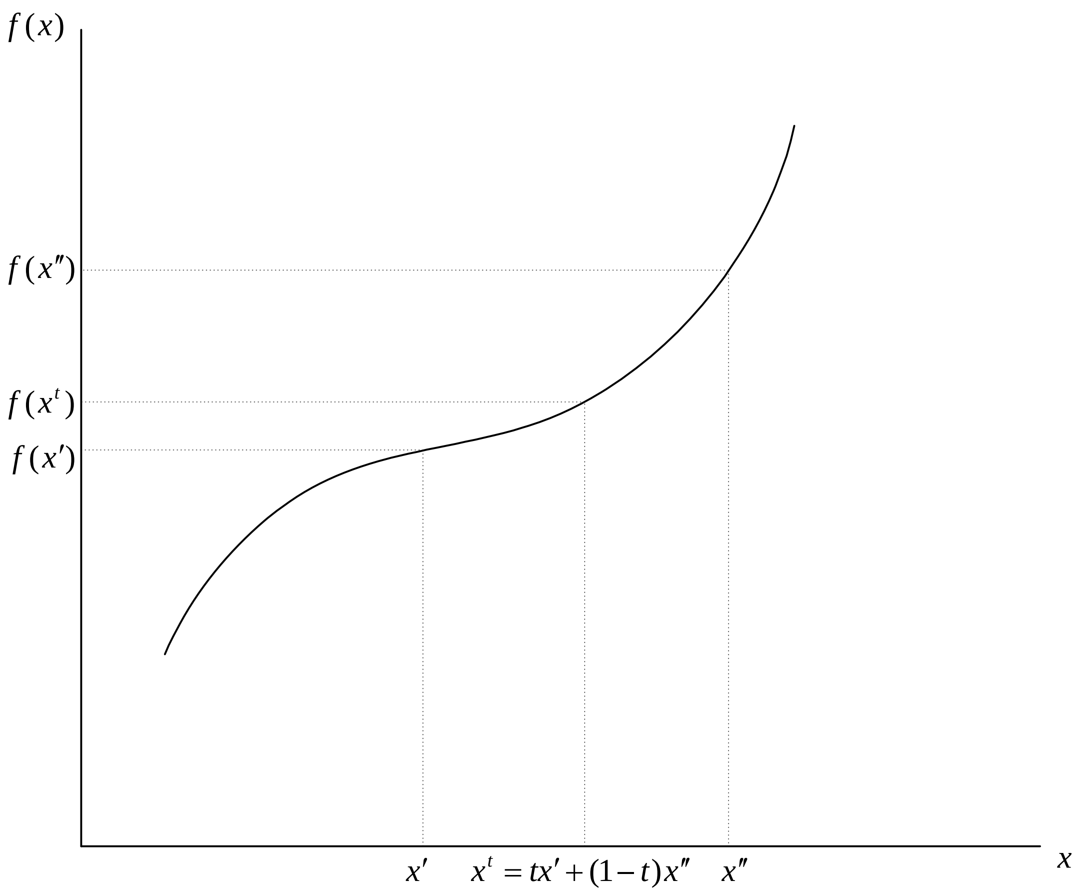


FIGURE 2.2

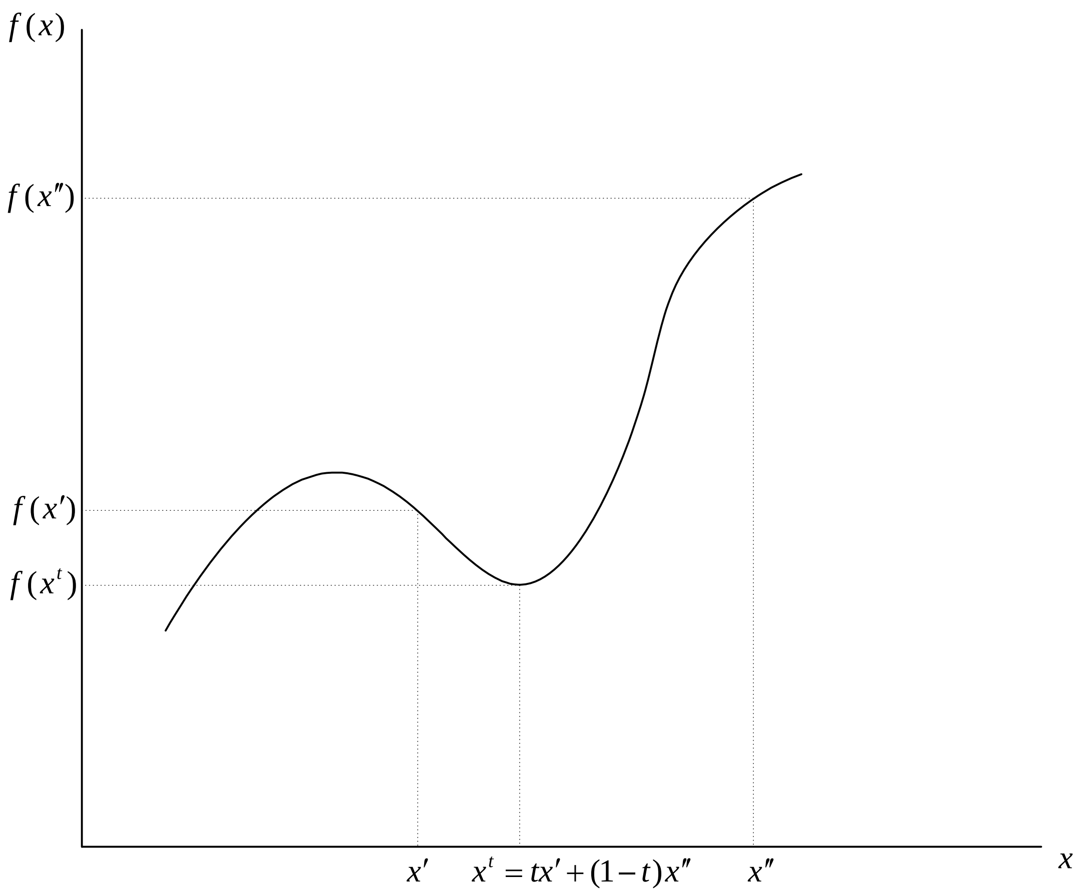


FIGURE 2.3

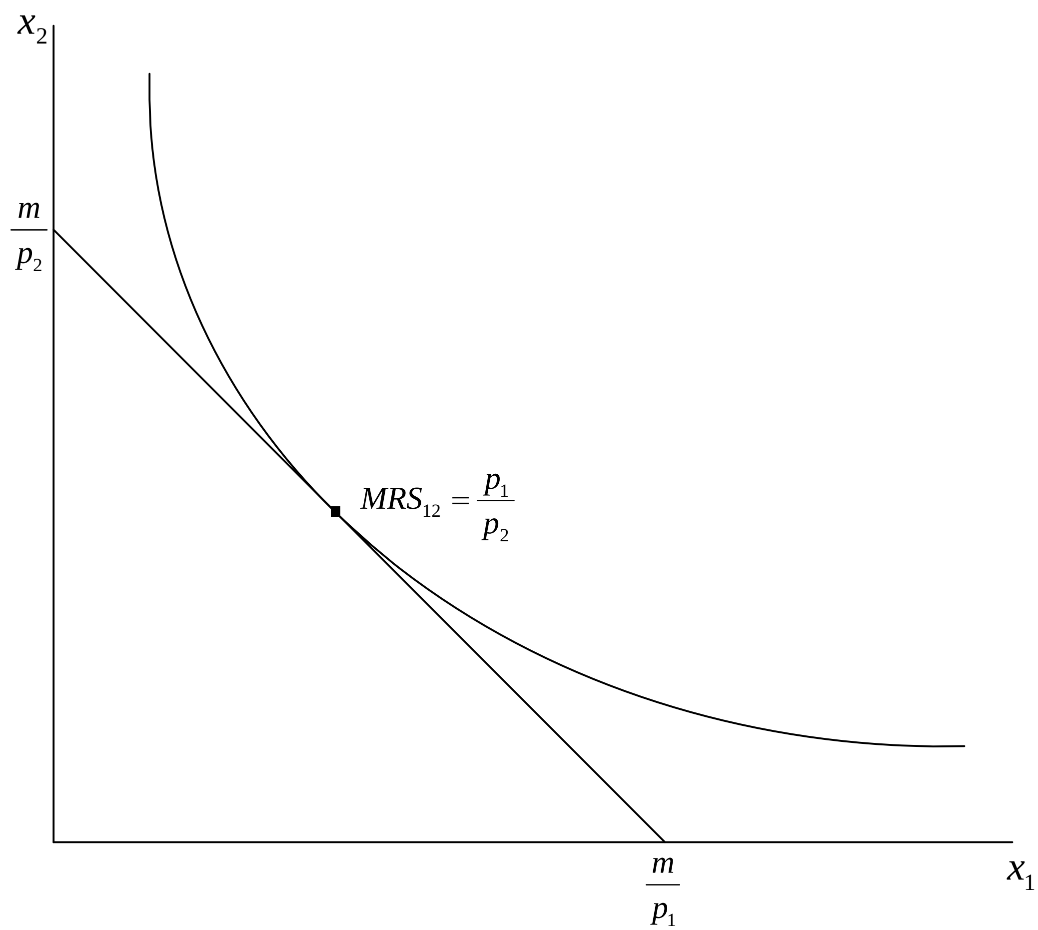


FIGURE 2.4

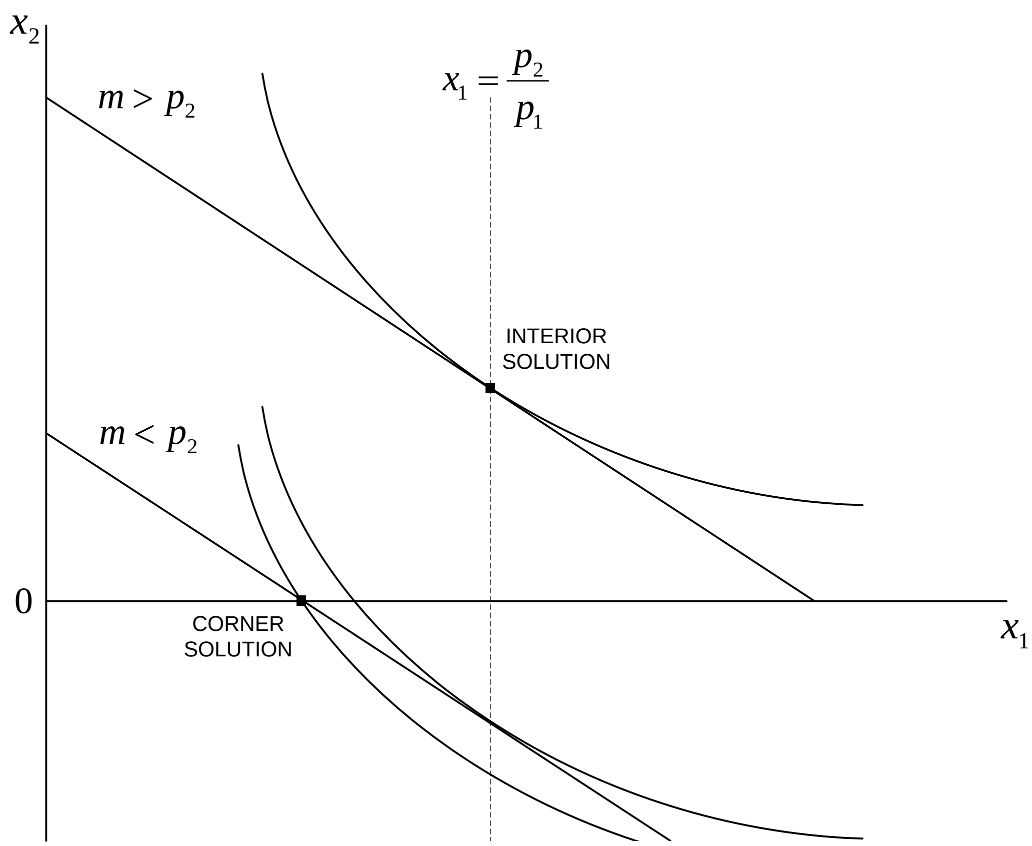

x_2
$m > p_2$
$x_1 = \frac{p_2}{p_1}$
INTERIOR SOLUTION
$m < p_2$
0
CORNER SOLUTION
x_1

FIGURE 2.5

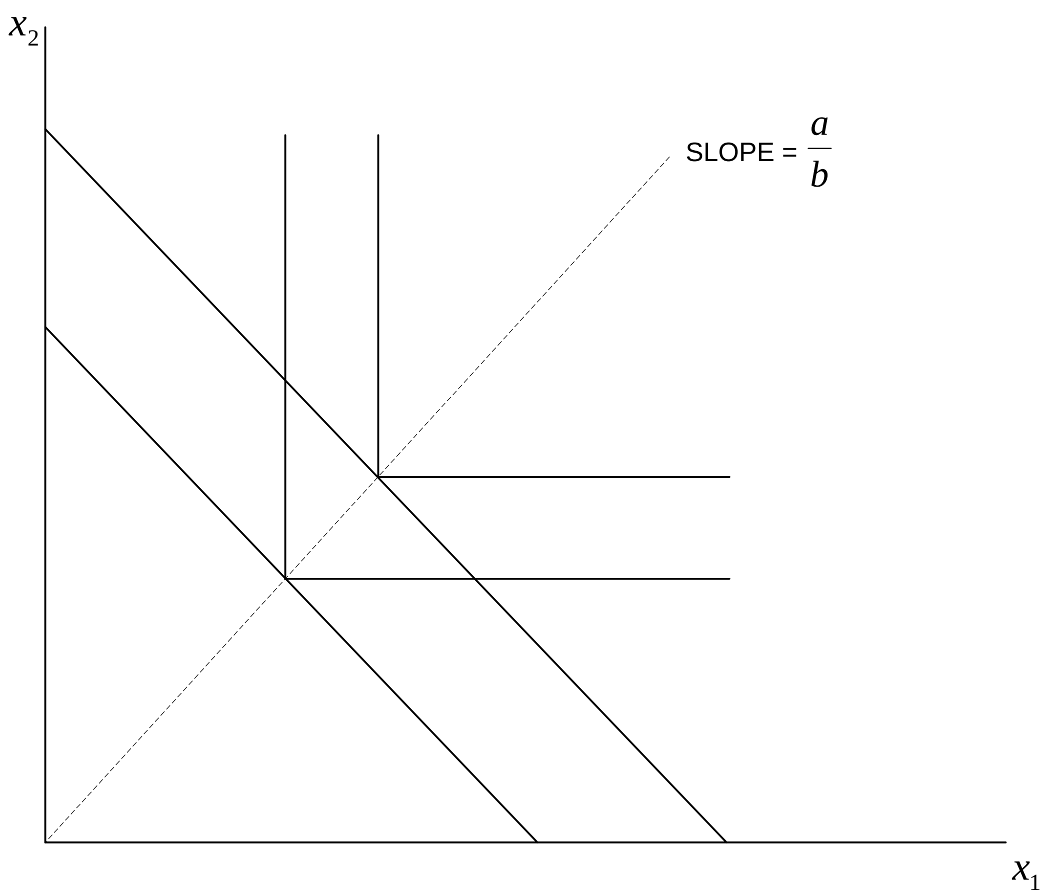


FIGURE 2.6

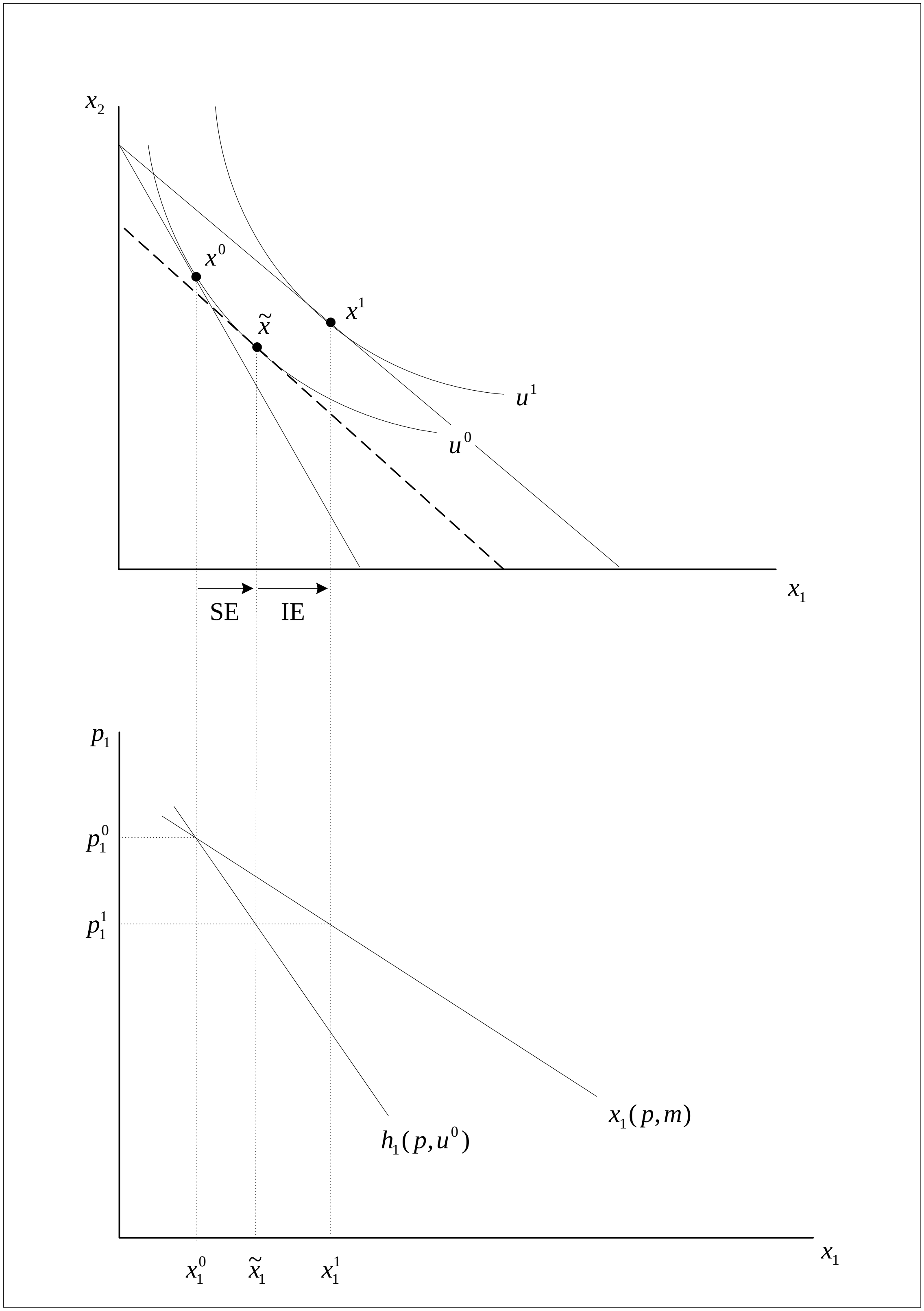

x_2
x^0
$\tilde{x}$
x^1
u^1
u^0
x_1
SE
IE
p_1
p_1^0
p_1^1
$x_1(p,m)$
$h_1(p,u^0)$
x_1
x_1^0
$\tilde{x}_1$
x_1^1

FIGURE 2.7

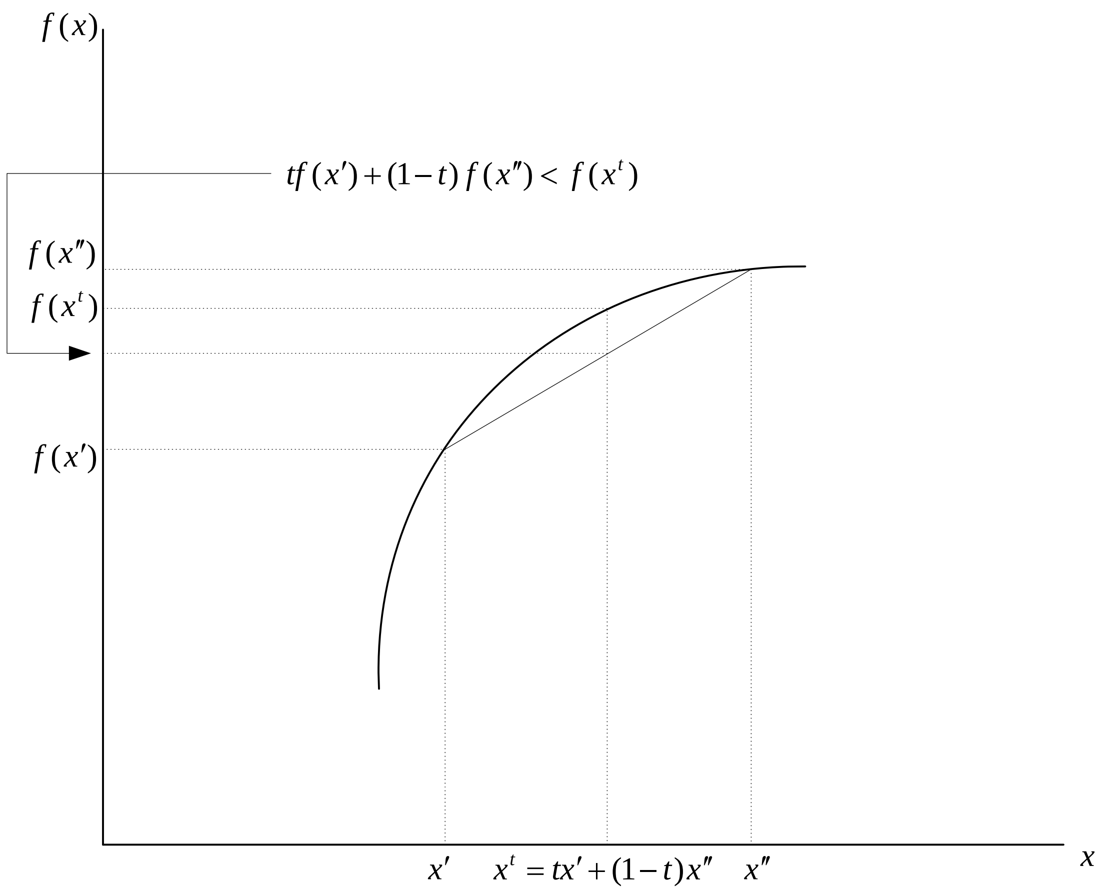


FIGURE 2.8

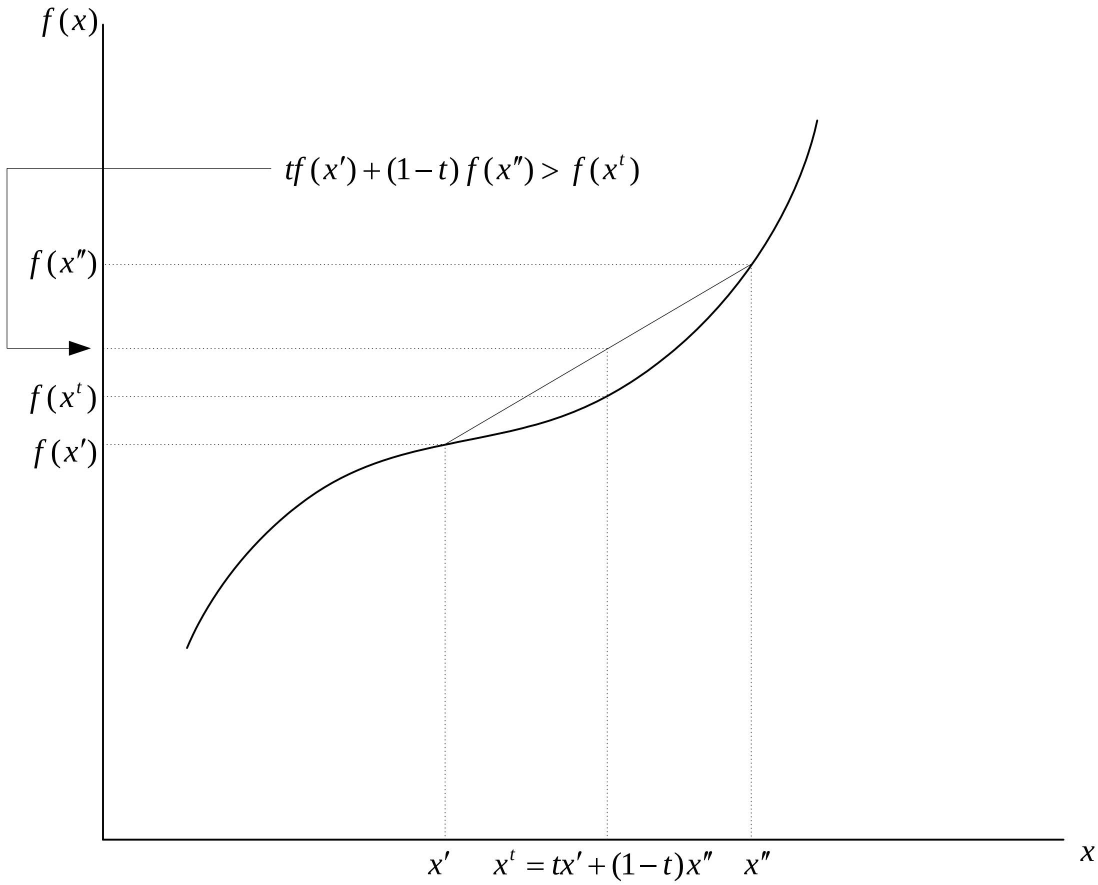


FIGURE 2.9

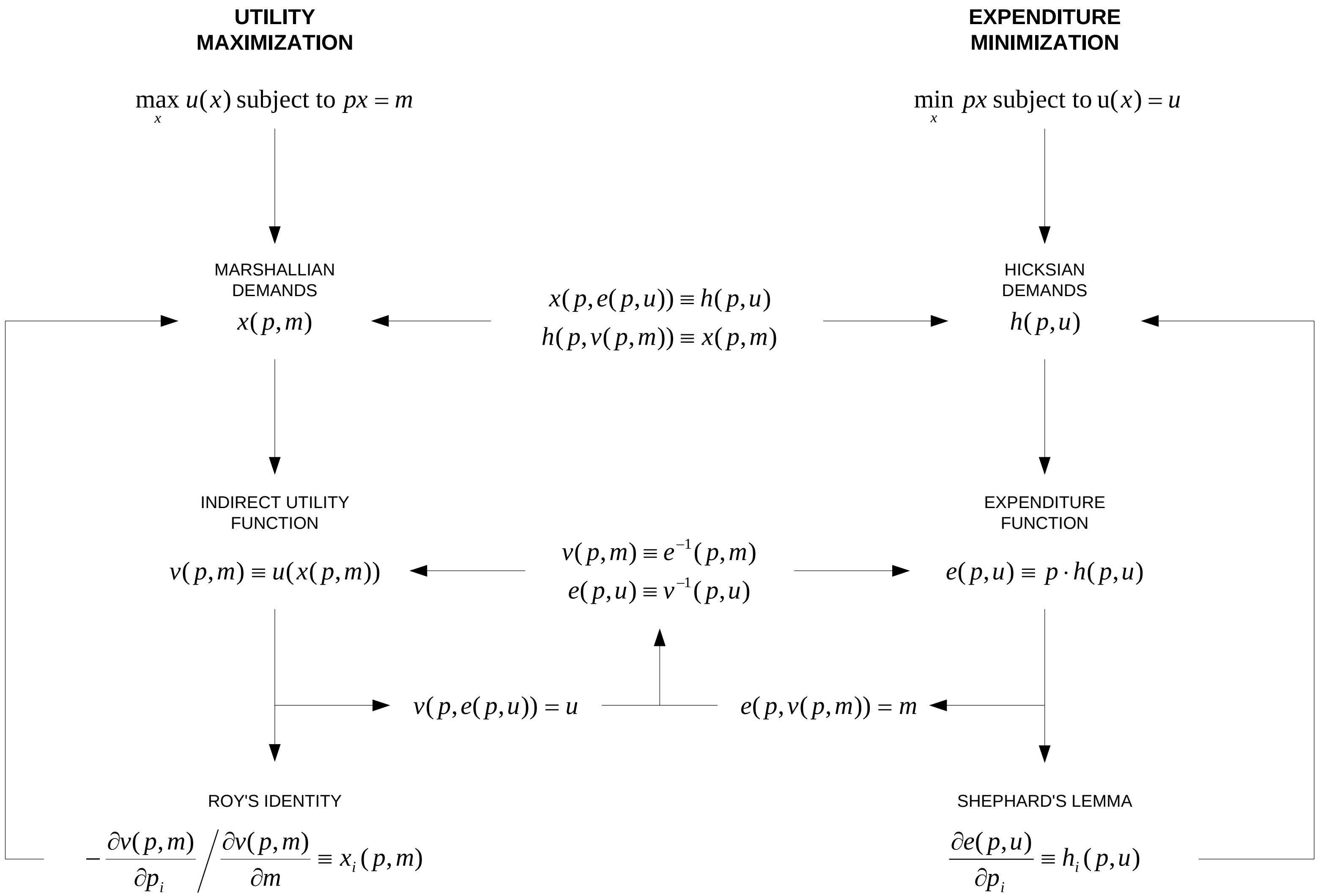

UTILITY MAXIMIZATION
$\max_x u(x)$ subject to $px = m$
EXPENDITURE MINIMIZATION
$\min_x px$ subject to $u(x) = u$
MARSHALLIAN DEMANDS
$x(p,m)$
$x(p,e(p,u)) \equiv h(p,u)$
$h(p,v(p,m)) \equiv x(p,m)$
HICKSIAN DEMANDS
$h(p,u)$
INDIRECT UTILITY FUNCTION
$v(p,m) \equiv u(x(p,m))$
$v(p,m) \equiv e^{-1}(p,m)$
$e(p,u) \equiv v^{-1}(p,u)$
EXPENDITURE FUNCTION
$e(p,u) \equiv p \cdot h(p,u)$
$v(p,e(p,u)) = u$
$e(p,v(p,m)) = m$
ROY'S IDENTITY
$-\frac{\partial v(p,m)}{\partial p_i} \Big/ \frac{\partial v(p,m)}{\partial m} \equiv x_i(p,m)$
SHEPHARD'S LEMMA
$\frac{\partial e(p,u)}{\partial p_i} \equiv h_i(p,u)$

FIGURE 2.10

3. PROPERTIES OF CONSUMER DEMAND

3.1 Properties of the Hicksian Demands

1. Negativity

$$\frac{\partial h_i}{\partial p_i} \leq 0$$

That is, the compensated own-price effect – the *substitution effect* – is non-positive. (Hence, the usual term, "negativity", is somewhat misleading).

Proof. By Shephard's lemma:

$$h_i = \frac{\partial e}{\partial p_i}$$

and by concavity of $e(p,u)$:

$$\frac{\partial h_i}{\partial p_i} = \frac{\partial^2 e}{\partial p_i^2} \leq 0 \clubsuit$$

2. Symmetry

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial h_j}{\partial p_i}$$

That is, compensated cross-price effects are *symmetric*.

Proof. By Shephard's lemma:

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial^2 e}{\partial p_i \partial p_j} \quad \textit{and} \quad \frac{\partial h_j}{\partial p_i} = \frac{\partial^2 e}{\partial p_j \partial p_i}$$

The result then follows by Young's theorem.♣

3. Homogeneity

$h(p,u)$ is homogeneous of degree 0 in *p*.

Proof. $e(p,u)$ is homogeneous of degree 1 in *p*. By Shephard's lemma:

$$h_i = \frac{\partial e}{\partial p_i}$$

and so the result follows from Euler's theorem.♣

Note that these three properties follow from the definition of $e(p,u)$ as a minimum value function. Thus, if the consumer is maximizing utility – in which case she must also be minimizing expenditure for a given utility – then the above properties hold if $e(p,u)$ is differentiable.

A sufficient condition for differentiability of $e(p,u)$ is convexity of preferences. Why? Suppose preferences are non-convex, as illustrated if Figure 3.1.[1] As p_1 rises from p_1^0 to p_1^1, there is a discrete jump in the Hicksian demand. This renders $h_1(p,u)$ discontinuous at some price $\tilde{p}_1 \in (p_1^0, p_1^1)$, as in Figure 3.2. In this case, the expenditure function is kinked at $\tilde{p}_1$, as in Figure 3.3, and therefore not everywhere differentiable.

Suppose instead preferences are weakly convex, as in Figure 3.4. At the kink in the IC, a small change in price induces no response in $h_1(p,u)$, as in Figure 3.5. In this case, the SE is only weakly negative ($SE \leq 0$).

In contrast, if preferences are strictly convex then $e(p,u)$ is strictly concave, $h_1(p,u)$ is smooth and $SE < 0$.

3.2 The Slutsky Equation and the Marshallian Demands

Recall from section 2.5:

$$h_i(p,u) \equiv x_i(p,e(p,u))$$

Differentiate both sides with respect to p_j:

$$\frac{\partial h_i}{\partial p_j} = \frac{\partial x_i}{\partial p_j} + \frac{\partial x_i}{\partial m} \cdot \frac{\partial e}{\partial p_j}$$

We know from Shephard's lemma that

$$\frac{\partial e}{\partial p_j} = x_j$$

[1] Note that the indifference curve is nonetheless negatively sloped; monotonicity is not violated.

Making this substitution and rearranging, we have the **Slutsky equation**:

$$\frac{\partial x_i}{\partial p_j} = \frac{\partial h_i}{\partial p_j} - x_j \frac{\partial x_i}{\partial m}$$

This holds for both $i = j$ and $i \neq j$. In particular, for $i = j$:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - x_i \frac{\partial x_i}{\partial m}$$

<u>Interpretation</u>

Marshallian demand response = substitution effect + income effect:

$$\frac{\partial h_i}{\partial p_i} \equiv \text{ substitution effect}$$

$$-x_i \frac{\partial x_i}{\partial m} \equiv \text{ income effect}$$

We have seen that concavity of $e(p,u)$ allows us to sign the SE≤0, but theory says nothing about the IE. There are three possibilities:

$$\frac{\partial x_i}{\partial m} > 0 : \; x_i \text{ is a normal good}$$

$$\frac{\partial x_i}{\partial m} = 0 : \; x_i \text{ is an income-neutral good}$$

$$\frac{\partial x_i}{\partial m} < 0 : \; x_i \text{ is an inferior good}$$

These are illustrated in Figures 3.6 – 3.8 respectively.

For normal and neutral goods, we know that :

$$\frac{\partial x_i}{\partial p_i} \leq 0$$

That is, the Marshallian demand is non-positively sloped.

For inferior goods, it is possible that over some range, the positive income effect more than offsets the substitution effect, such that the Marshallian demand is positively sloped;

such a good is called a Giffen good. However, for sufficiently high p_i, it must be true that $\partial x_i / \partial p_i$ is negative, or else the budget constraint would be violated.

3.3 Additional Properties of Demand Systems

The budget constraint implies two important restrictions on Marshallian demands as a *system* of equations.

1. Engel Aggregation

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial m} = 1$$

Proof. From the budget constraint:

$$\sum_{i=1}^{n} p_i x_i(p, m) = m$$

Differentiate both sides with respect to *m*:

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial m} = 1 \clubsuit$$

2. Cournot Aggregation

$$\sum_{i=1}^{n} p_i \frac{\partial x_i}{\partial p_j} + x_j = 0$$

Proof. Differentiate the budget constraint with respect to p_j and rearrange.♣

3.4 Demand Elasticities

An elasticity measures the $\%\Delta$ in one variable in response to a $\%\Delta$ in another.

Own-Price Elasticity

$$\varepsilon_{ii} \equiv \frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i}$$

It is often convenient to express this in logarithmic terms. Note that

$$\frac{d\log x}{dx} = \frac{1}{x}$$

$$\frac{d\log p}{dp} = \frac{1}{p}$$

Thus,

$$\frac{d\log x}{d\log p} = \frac{dx}{dp} \cdot \frac{p}{x}$$

and hence

$$\varepsilon_{ii} \equiv \frac{\partial \log x_i}{\partial \log p_i}$$

Some definitions:

- Inelastic demand: $|\varepsilon_{ii}| < 1$
- Elastic demand: $|\varepsilon_{ii}| > 1$

<u>Cross-Price Elasticity</u>

$$\varepsilon_{ij} \equiv \frac{\partial x_i}{\partial p_j} \cdot \frac{p_j}{x_i} = \frac{\partial \log x_i}{\partial \log p_j}$$

<u>Income Elasticity</u>

$$\eta_i \equiv \frac{\partial x_i}{\partial m} \cdot \frac{m}{x_i} = \frac{\partial \log x_i}{\partial \log m}$$

Some definitions:

- A normal good: $\eta_i > 0$
- An inferior good: $\eta_i < 0$
- A luxury: $\eta_i > 1$
- A necessity: $0 < \eta_i < 1$

Example: Cobb-Douglas

$$x_i = \frac{\alpha_i m}{p_i \sum \alpha_i}$$

$$\varepsilon_{ii} = \frac{\partial x_i}{\partial p_i} \cdot \frac{p_i}{x_i} = -1$$

$$\varepsilon_{ij} = 0$$

$$\eta_i = 1$$

These properties make the C-D specification very restrictive.

3.5 Labor Supply

Suppose an agent has one unit of time to split between leisure l and labor supply L, and that utility is increasing in consumption c and leisure l:

$$u(c,l)$$

Labor can be sold at wage w; consumption has price p. Suppose the agent also has non-labor income m.

Utility maximization:

$$\max_{c,l}\ u(c,l) \text{ subject to } pc = wL + m \text{ and } l + L = 1$$

Merge the two constraints to obtain:

$$pc + wl = w + m$$

where we can think of $w+m$ as total income and w as the price of buying back leisure. Thus, the Lagrangean is :

$$J = u(c,l) + \lambda(w + m - pc - wl)$$

and the FOCs are

$$\frac{\partial u}{\partial c} - \lambda p = 0$$

$$\frac{\partial u}{\partial l} - \lambda w = 0$$

Taking the ratio yields a standard tangency condition:

$$MRS_{cl} = \frac{p}{w}$$

Solution of the FOCs yields the demand for leisure: $l(p, w, m)$. Labour supply is then given by the residual, $1 - l(p, w, m)$.

3.6 Homothetic Preferences

A homothetic function is one that can be expressed as a monotonic transform of a homogeneous function. If preferences can be represented by a homothetic utility function then those preferences are said to be homothetic.

If $u(x)$ is homothetic then a monotonic transformation can yield a utility function that is homogeneous of degree 1 that represents the same preferences. Thus, we can represent homothetic preferences with a utility function $u(x)$ such that

$$u(tx) = tu(x)$$

Consider the MRS for such a function, evaluated at some bundle $\tilde{x}$:

$$MRS(\tilde{x}) = -\frac{\partial u(\tilde{x})}{\partial x_1} \Big/ \frac{\partial u(\tilde{x})}{\partial x_2}$$

and at some other bundle proportional to $\tilde{x}$:

$$MRS(t\tilde{x}) = -\frac{\partial u(t\tilde{x})}{\partial x_1} \Big/ \frac{\partial u(t\tilde{x})}{\partial x_2} = -\frac{t\partial u(\tilde{x})}{\partial x_1} \Big/ \frac{t\partial u(\tilde{x})}{\partial x_2} = MRS(\tilde{x})$$

Thus, MRS is constant along a ray; see Figure 3.9. This means that a change in income does not change the *proportion* in which goods are consumed. Equivalently, the expenditure function is linear in u:

$$e(p, u) = a(p)u$$

Why? To double u we must double x (with no change in consumption proportions) since $u(tx) = tu(x)$, and to double x (at given prices) we must double expenditure.

Now consider $v(p, m)$. Set $e(p, u) = m$ and $v(p, m) = u$ to obtain

$$m = a(p)v(p, m)$$

Thus, we can write

$$v(p,m) = \frac{m}{a(p)} \equiv b(p)m$$

Then by Roy's Identity:

$$x_i(p,m) = -\frac{\partial v(p,m)}{\partial p_i} \bigg/ \frac{\partial v(p,m)}{\partial m} = \frac{m}{b(p)}\frac{\partial b}{\partial p_i}$$

Thus, Marshallian demands associated with homothetic preferences are linear in income.

Note that the C-D specification is homothetic.

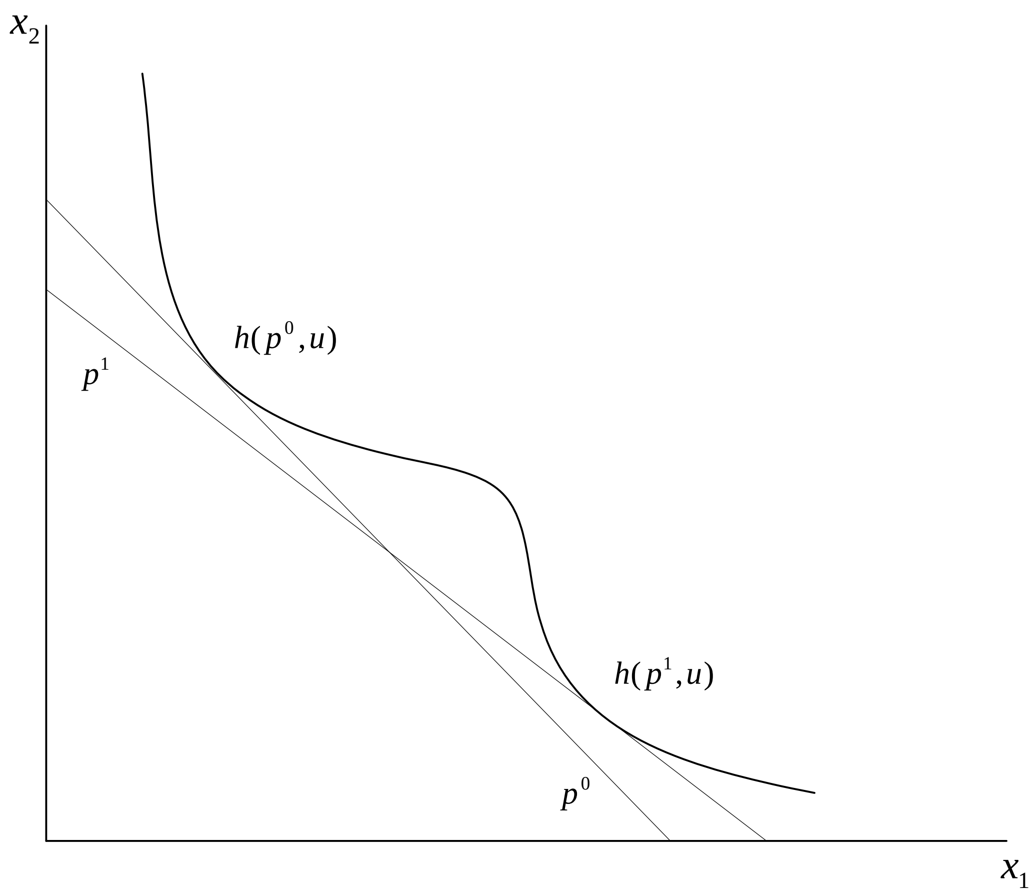


FIGURE 3.1

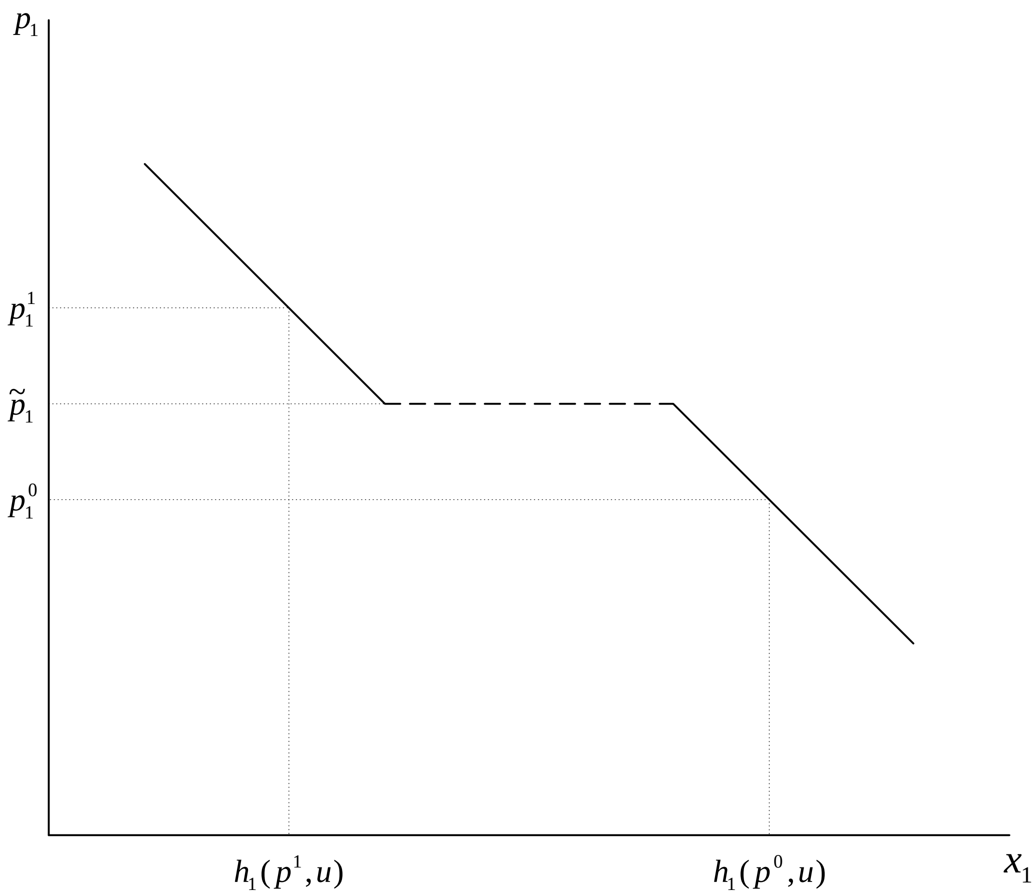


FIGURE 3.2

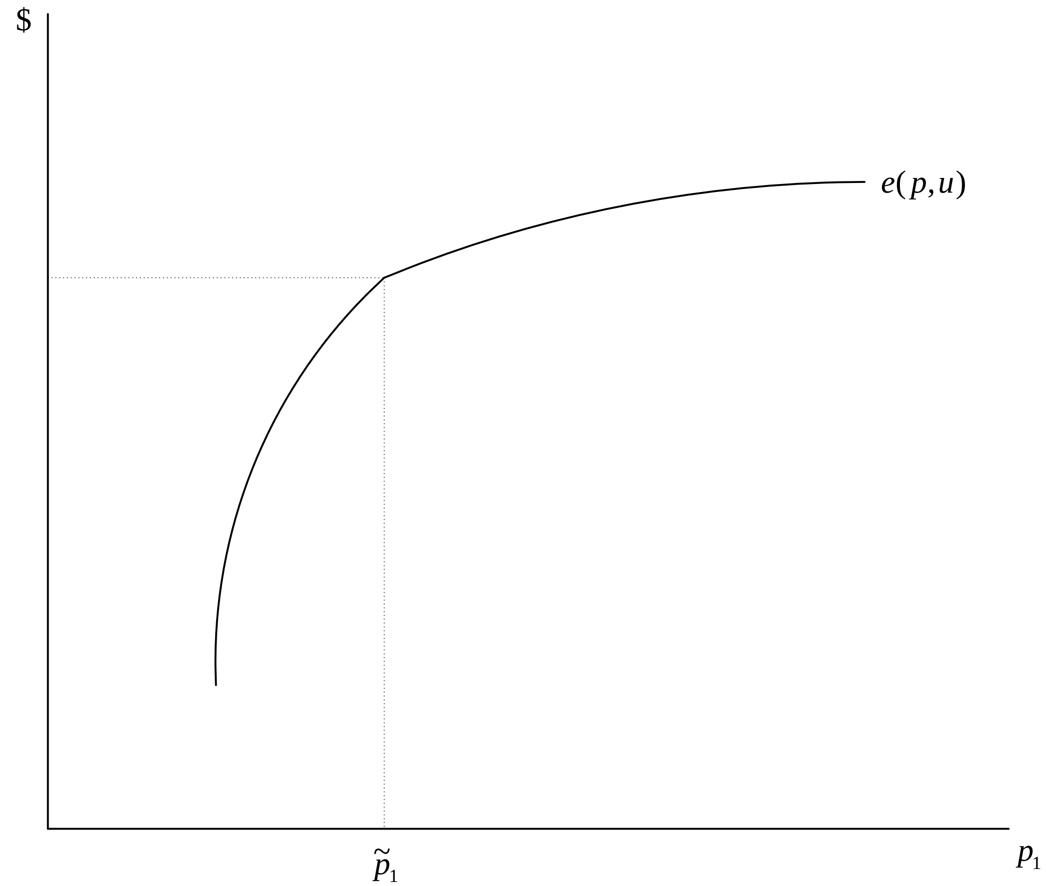


FIGURE 3.3

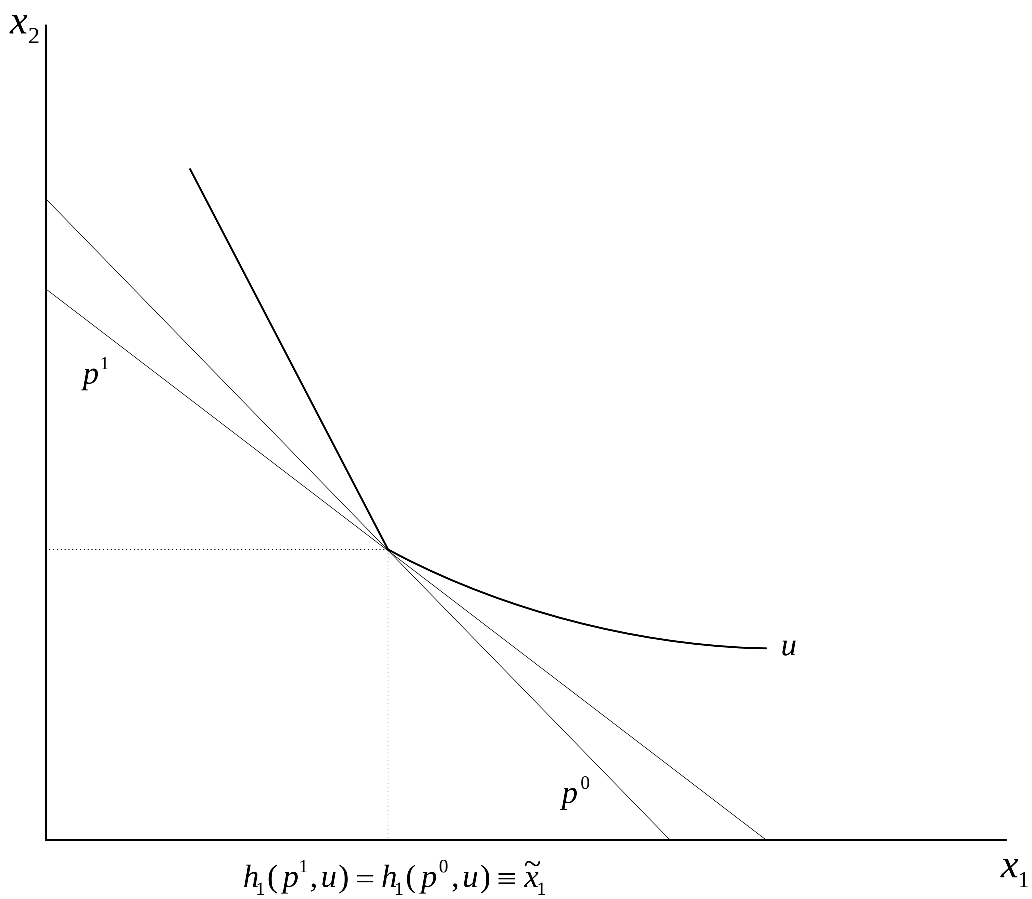


FIGURE 3.4

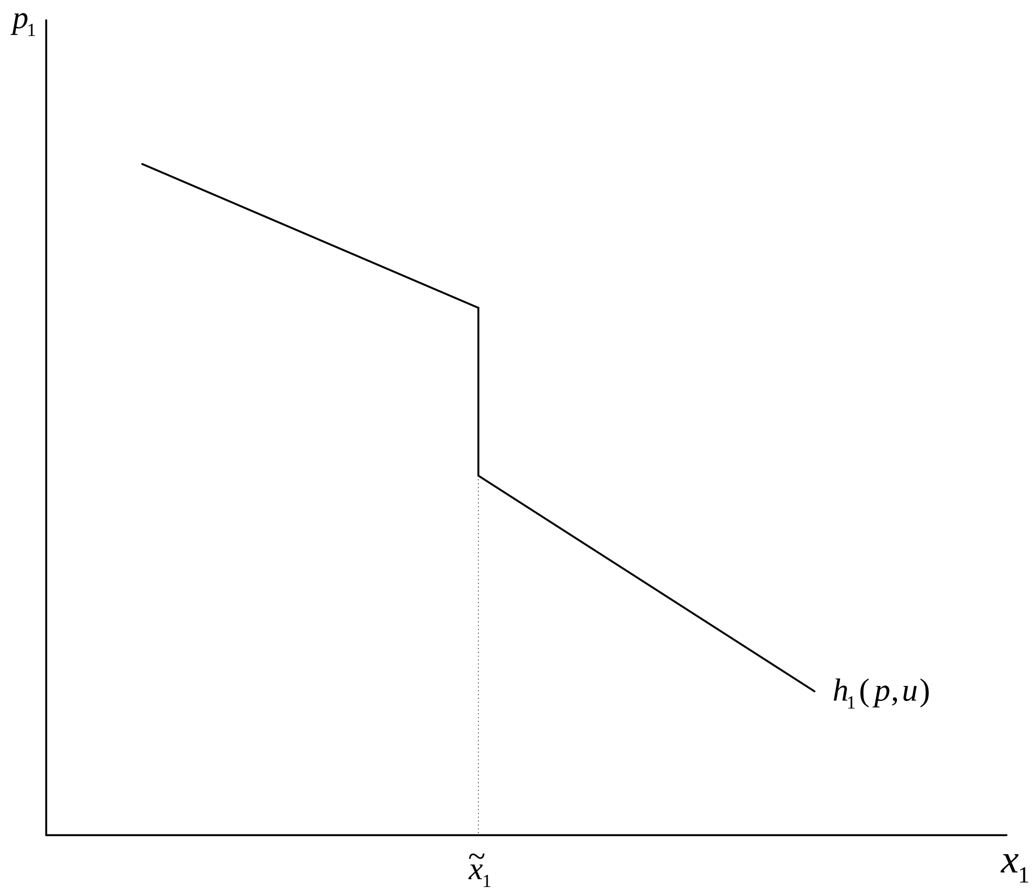


FIGURE 3.5

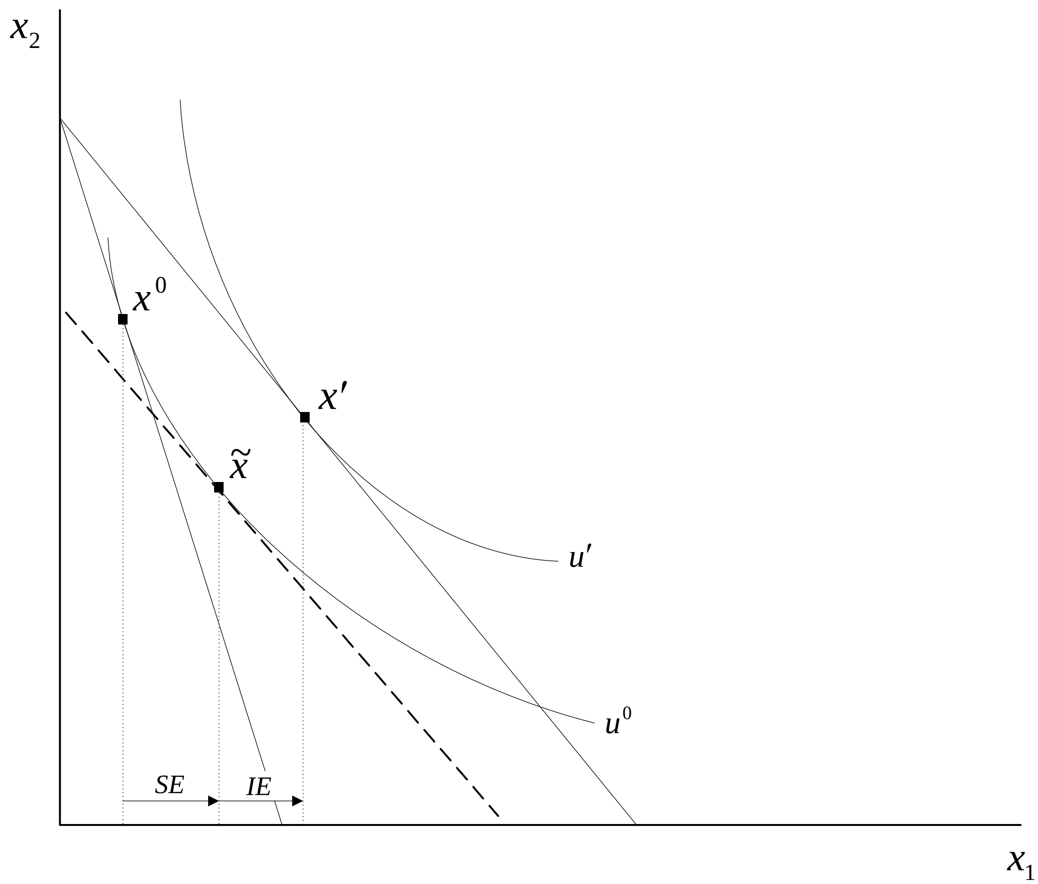


FIGURE 3.6

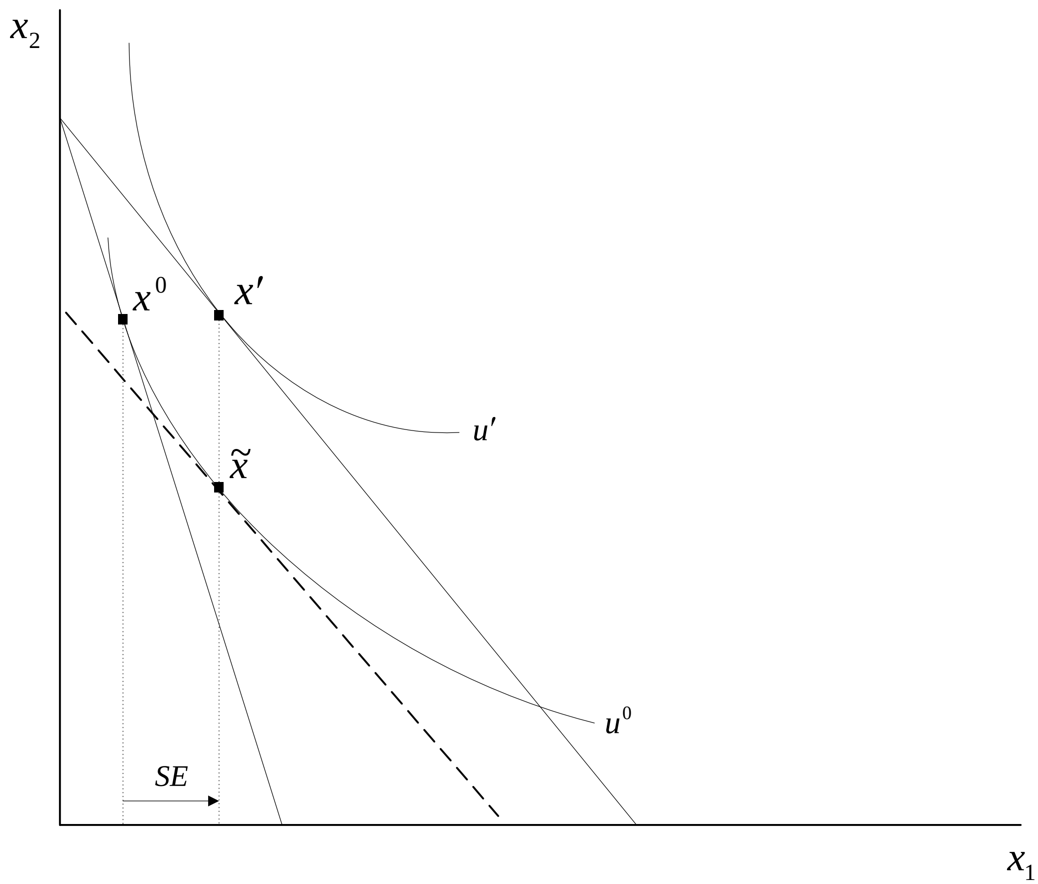


FIGURE 3.7

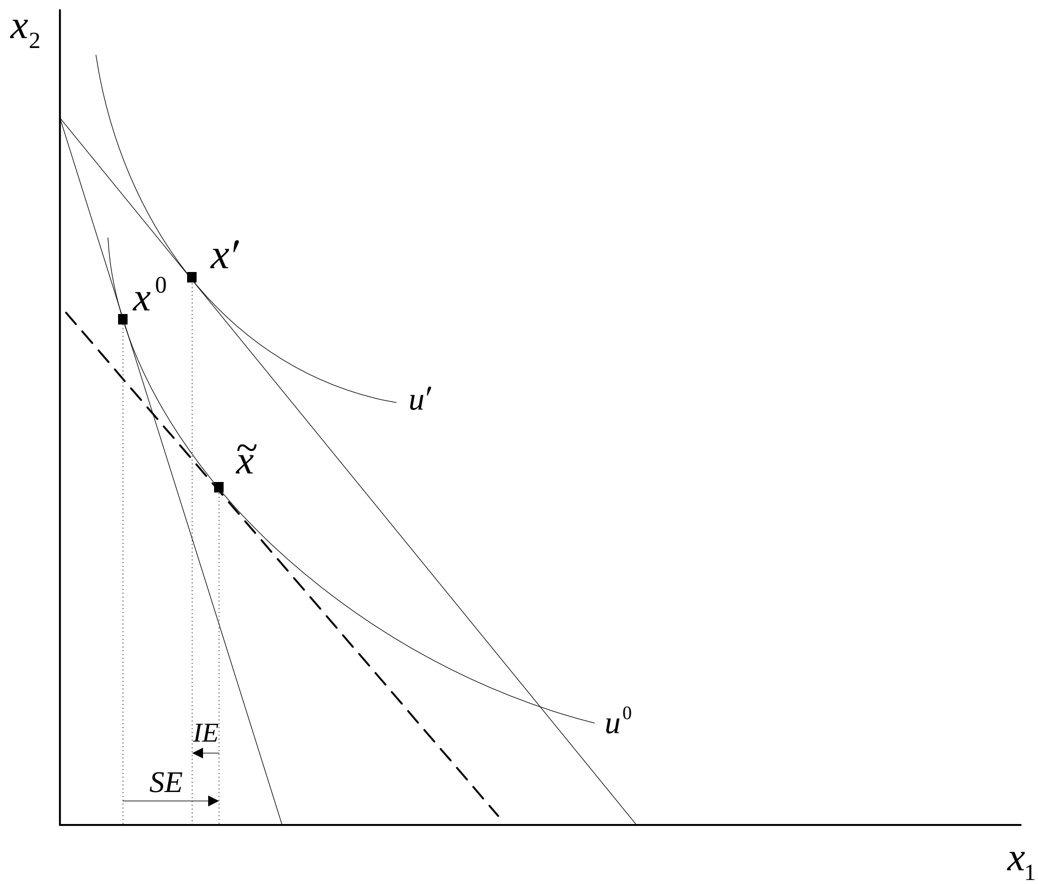


FIGURE 3.8

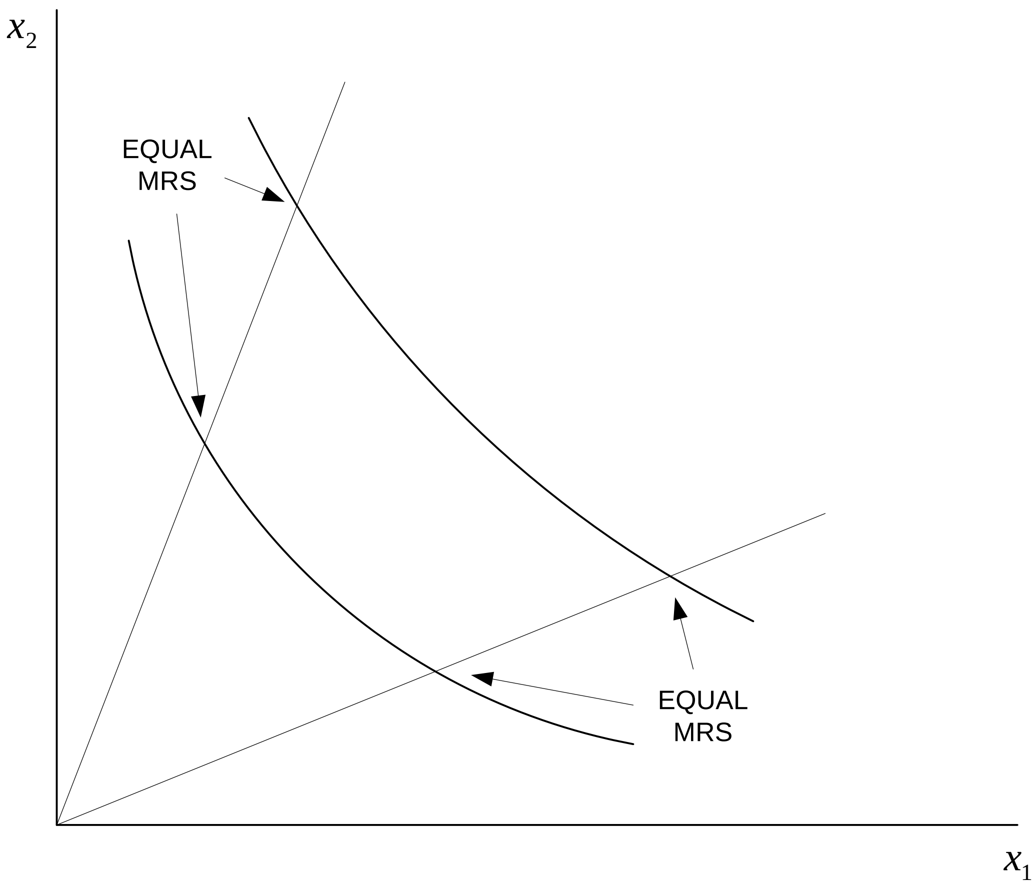


FIGURE 3.9

4. MEASURES OF CONSUMER WELFARE

4.1 Compensating Variation and Equivalent Variation

Suppose prices change from p^0 to p^1. What is the effect on consumer welfare?

We cannot put a cardinal number on the utility change; the best we can do is determine whether utility rises or falls. Consider the indirect utility function. If

$v(p^1, m) - v(p^0, m) > 0$ then the consumer is better-off

$v(p^1, m) - v(p^0, m) < 0$ then the consumer is worse-off

This is not a very useful measure since we often want to compare welfare changes for different policy options, or compare welfare changes across individuals, and for this we need a cardinal measure. For this purpose we use money-metric welfare change measures: *compensating variation* (CV) and *equivalent variation* (EV).

<u>Compensating Variation</u>

CV measures the (negative of the) amount of money necessary to fully *compensate* an individual for a price change. In other words, given the new prices, how much extra income does the consumer need in order to be restored to the original level of utility? In terms of the expenditure function:

$$CV \equiv m - e(p^1, u^0)$$

Note that if $CV > 0$ then the consumer is made better-off by the price changes (the actual compensation needed would be negative) and if $CV < 0$ then the consumer is made worse-off by the price changes.

When representing CV graphically in $\{x_1, x_2\}$ space, it is customary to measure CV in units of x_2 rather than dollars. See Figure 4.1.

It is useful to interpret CV in terms of willingness-to-pay (WTP) and willingness-to-accept (WTA):

- if the individual is made worse-off by the change in conditions (as from a price rise) then $|CV|$ measures her WTA for that change in conditions.
- if the individual is made better-off by the change in conditions (as from a price fall) then $|CV|$ measures her WTP to obtain that change in conditions.

Equivalent Variation

EV measures the (negative of the) amount of money that would have to be taken away from the consumer, at the original prices, to yield a change in utility *equivalent* to the change created by the price changes. In terms of the expenditure function:

$$EV \equiv e(p^0, u^1) - m$$

Note that if $EV > 0$ then the consumer is made better-off by the price changes and if $EV < 0$ then the consumer is made worse by the price changes.

See Figure 4.2 for a graphical representation, where EV is measured in units of x_2.

It is useful to also interpret EV in terms of WTP and WTA:

- if the individual is made worse-off by the change in conditions (as from a price rise) then $|EV|$ measures her WTP to avoid the change in conditions.
- if the individual is made better-off by the change in conditions (as from a price fall) then $|EV|$ measures her WTA to forego that change in conditions.

Which Measure Should We Use?

EV and CV always have the same sign but their magnitudes will generally be different because they use different reference points: CV uses u^0 as the reference point; EV uses u^1 as the reference point.

So which measure should we use? The conventional answer that it depends on the assignment of **property rights** implicit in the analysis:

- if the individual is deemed to have a right to the benefit of the change in conditions, or a right not to be harmed by the change in conditions, then we should use WTA
  - $\Rightarrow$ use EV if she gains from the change, and CV if she loses from the change.
- if the individual is deemed to have no right to the benefits of the change in conditions, or no right not to be harmed by the change in conditions, then we should use WTP
  - $\Rightarrow$ use CV if she gains from the change, and EV if she loses from the change.

This conventional answer is not very satisfactory because property rights are often not defined in the context of many changes induced by policy or by the behaviour of other agents. For example, do you have a right to less polluted air, or does a car driver have a right to drive her car, and pollute the air as a consequence?

A potentially better approach is to first ask what purpose we have in mind for the measurement of the welfare impact. If our purpose is to calculate the payment that will <u>actually</u> be made to compensate a damaged individual, then we should use CV because it is based on WTA in that setting. Similarly, if our purpose is to calculate the payment that a beneficiary will <u>actually</u> make in return for a change in conditions, then we should use CV because it is based on WTP in that setting. This ensures that the actual property-rights assignment implied by the payments is consistent with the welfare measure used.

Conversely, if our purpose is to calculate the loss that a change in conditions will impose on an individual who will <u>not</u> actually be compensated for the change, then we should use EV because it is based on her WTP in that context. Similarly, if our purpose is to calculate the gain that a beneficiary will receive without having to actually pay for that gain, then we use EV because it measures WTA in that context. Again, this ensures that the actual property-rights assignment implied by the <u>absence</u> of payments is consistent with the welfare measure used.

To summarize, if actual payments will be made then we should use CV to calculate those payments. If no actual payments will be made, then we should use EV to measure the gains and losses that will arise precisely because compensating payments were not actually made.

Relationship to the Hicksian Demands

The EV and CV can be represented as areas under Hicksian demands. In particular,

$$\begin{aligned} CV &\equiv m - e(p^1, u^0) \\ &= e(p^0, u^0) - e(p^1, u^0) \\ &= \int_{p^1}^{p^0} h(p, u^0)\,dp \end{aligned}$$

where the final step follows from Shephard's lemma:

$$h_i(p, u^0) = \frac{\partial e(p, u^0)}{\partial p_i}$$

This is illustrated in Figure 4.3 for the case of a price fall for a normal good.

In the case of EV,

$$\begin{aligned} EV &\equiv e(p^0, u^1) - m \\ &= e(p^0, u^1) - e(p^1, u^1) \\ &= \int_{p^1}^{p^0} h(p, u^1)\,dp \end{aligned}$$

This is illustrated in Figure 4.4 for the case of a price fall for a normal good.

4.2 Consumer Surplus

Consumer is defined as an area under the Marshallian demand:

$$\Delta CS = \int_{p_i^1}^{p_i^0} x_i(p)\,dp$$

This is illustrated in Figure 4.5.

<u>Relationship to EV and CV</u>

Recall the Slutsky equation:

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - x_i \frac{\partial x_i}{\partial m}$$

This means

- for a normal good: $h_i(p,u)$ is steeper than $x_i(p,m)$
- for an inferior good: $h_i(p,u)$ is flatter than $x_i(p,m)$

The normal good case is illustrated (for a price fall) in Figures 4.6 – 4.8; the inferior good case is illustrated (for a price fall) in Figures 4.9 – 4.11.

These relationships between the Hicksian and Marshallian demands imply that for a

- price fall for a normal good: $|EV| > |\Delta CS| > |CV|$
- price rise for a normal good: $|CV| > |\Delta CS| > |EV|$
- price fall for an inferior good: $|CV| > |\Delta CS| > |EV|$
- price rise for an inferior good: $|EV| > |\Delta CS| > |CV|$
- price change for a neutral good: $|EV| = |\Delta CS| = |CV|$

If more than one price changes then the CV and EV can be calculated as the sum of the CVs and EVs associated with the individual price changes taken sequentially. The value of the total CV or EV is invariant to the order of this calculation. That is, the sequence of price changes assumed for the calculation is irrelevant. This property of the CV and EV is called *path independency*.

In contrast, ΔCS is *path dependent;* that is, it is not invariant to the sequence of changes assumed for the calculation when more than one price changes or if prices and income change. This reflects the presence of income effects in ordinary demand curves, which means that cross-price effects are generally not symmetric (unless preferences are homothetic; see Question 11 in Problem Set 1). Thus, it matters which demand curve is allowed to shift first for the purposes of measuring areas. This is an important theoretical shortcoming but empirically it is not likely to be of serious concern.

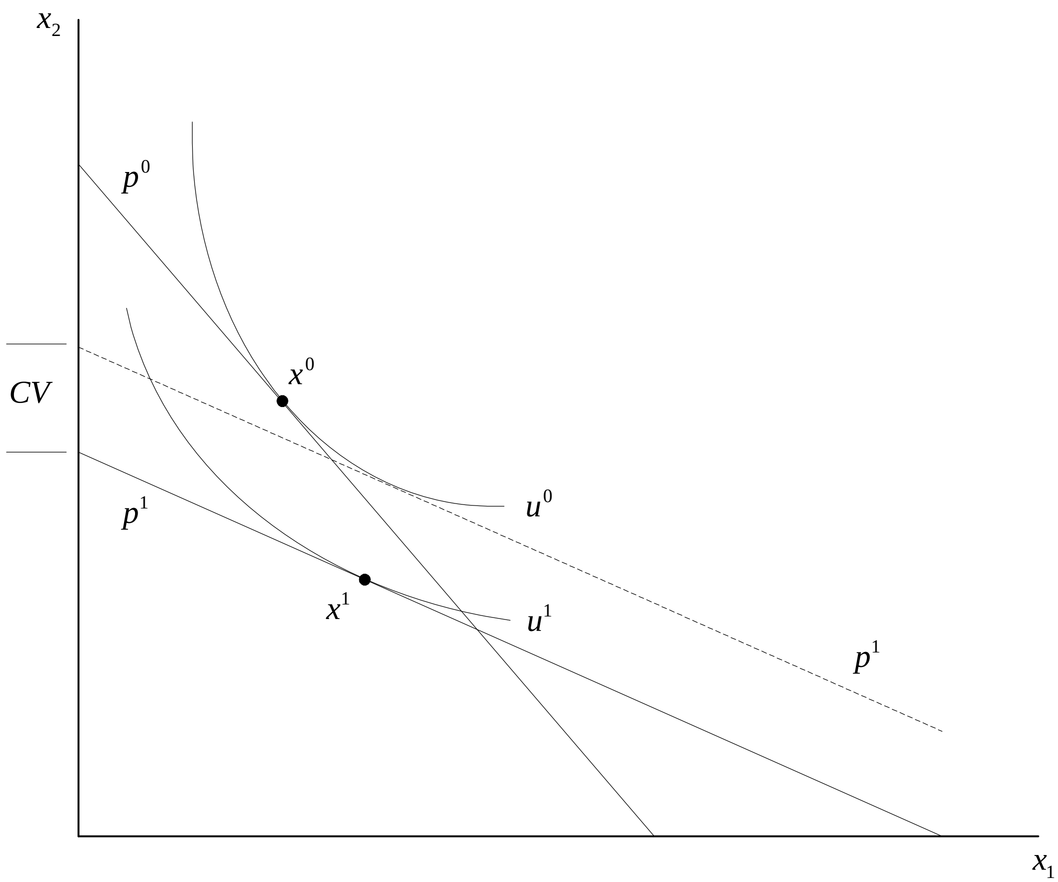


FIGURE 4.1

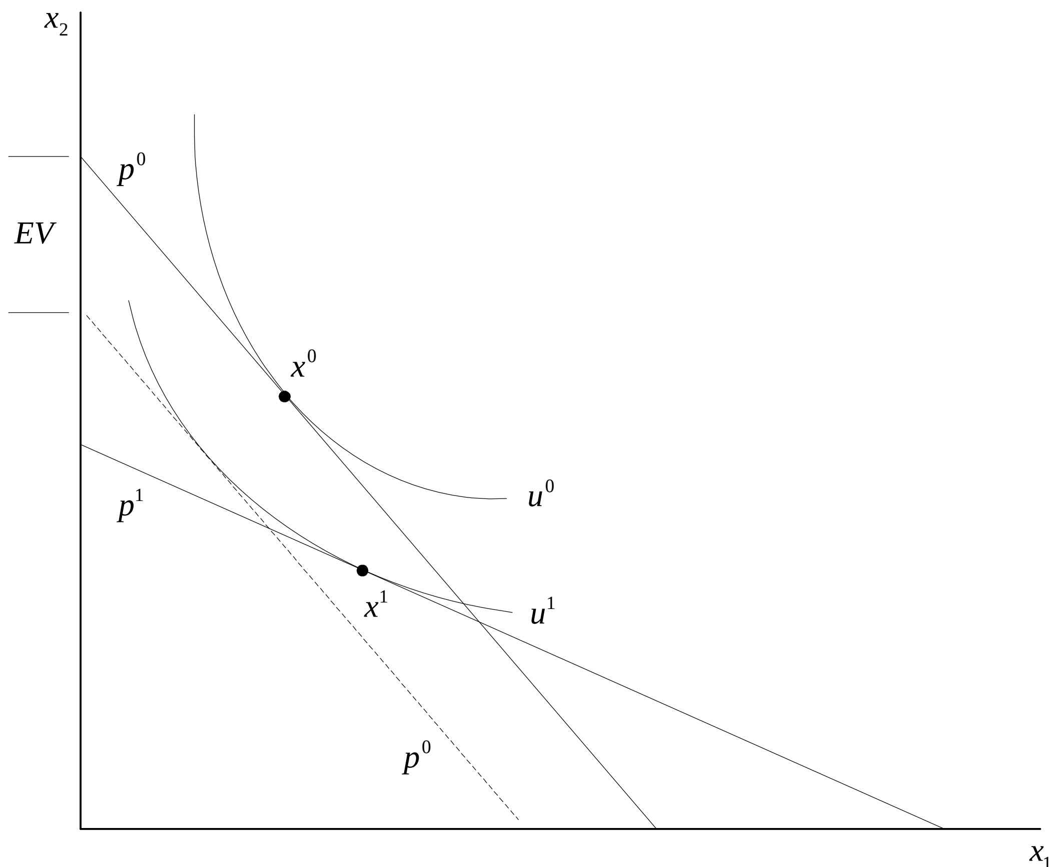


FIGURE 4.2

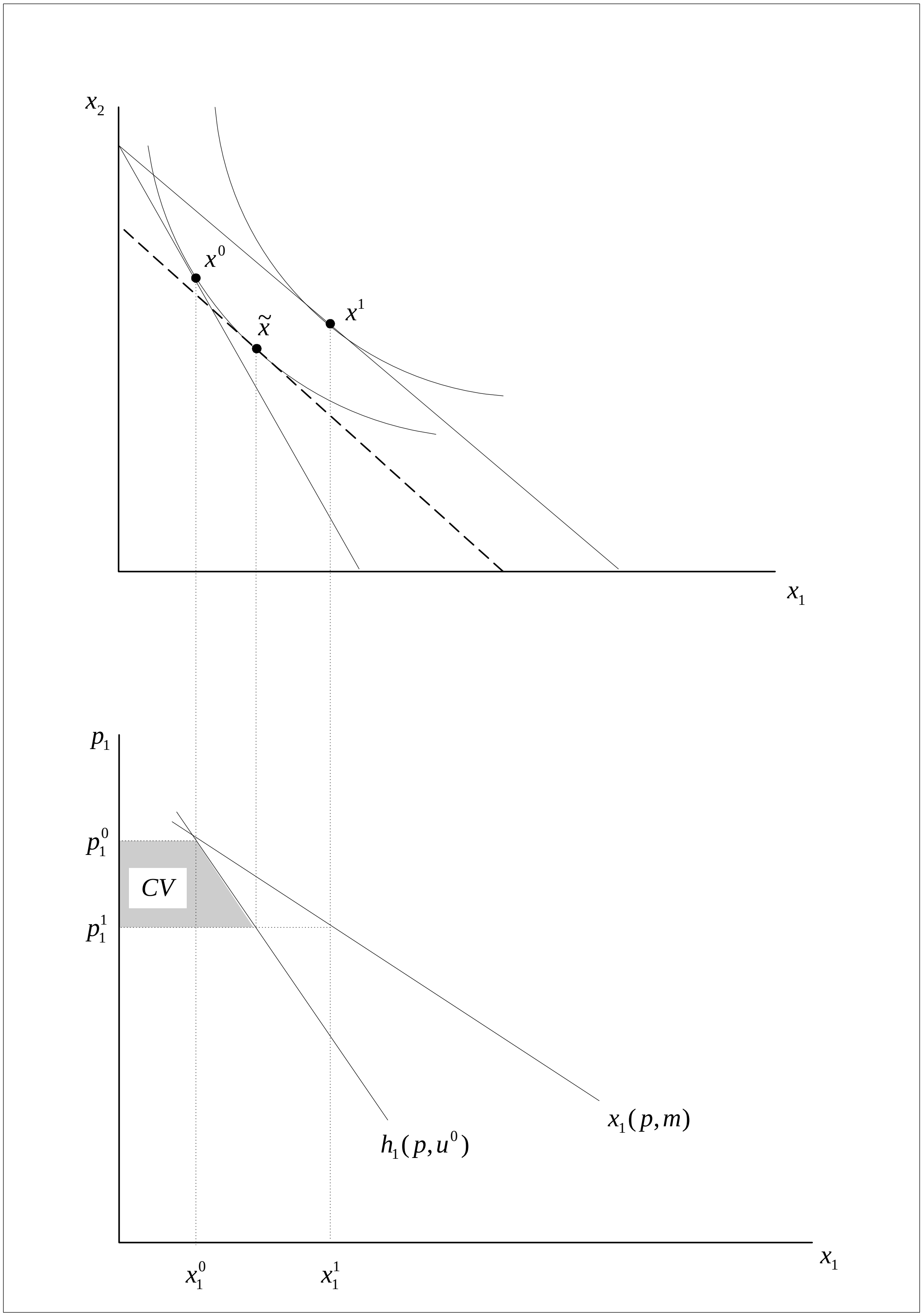


FIGURE 4.3

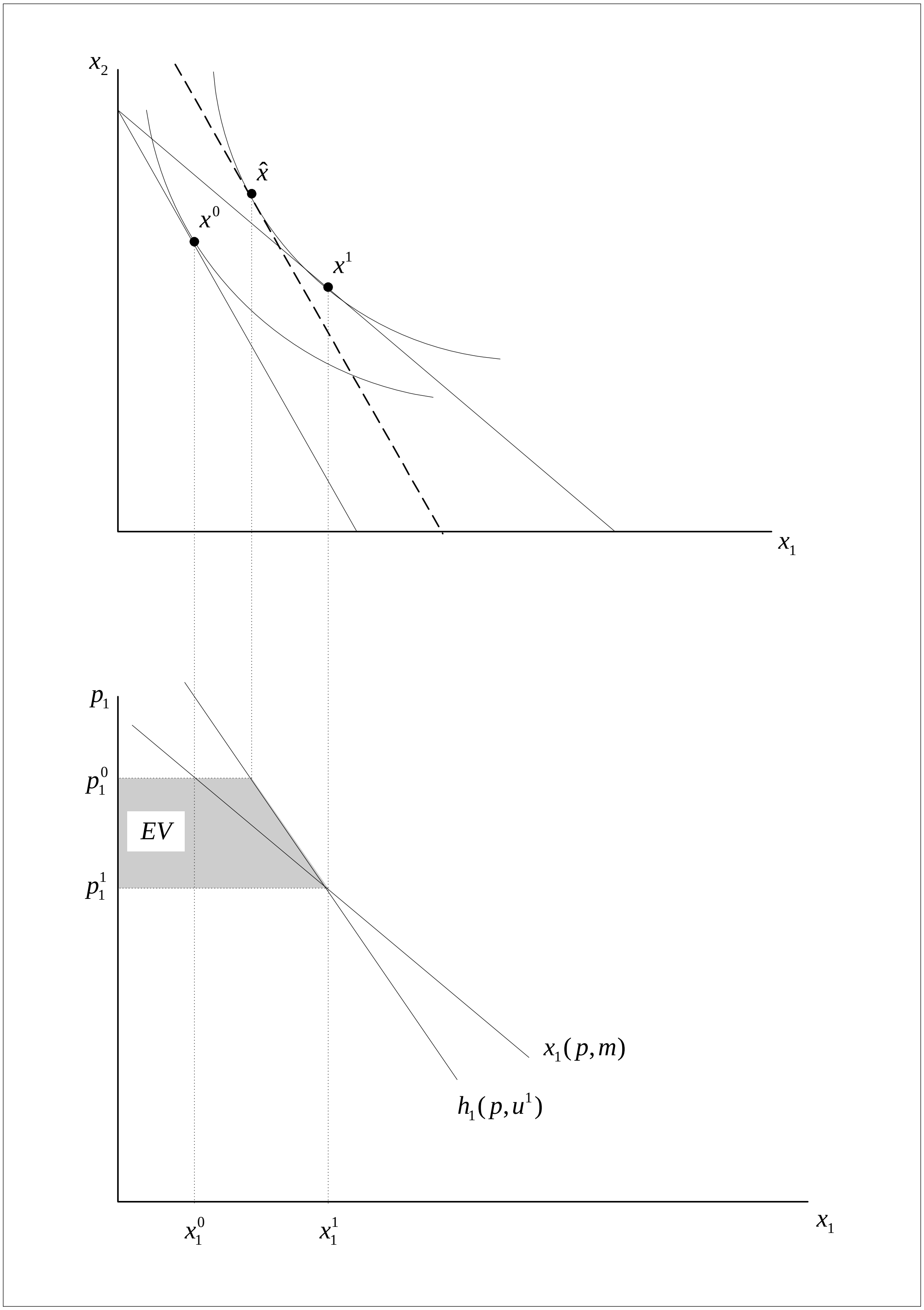


FIGURE 4.4

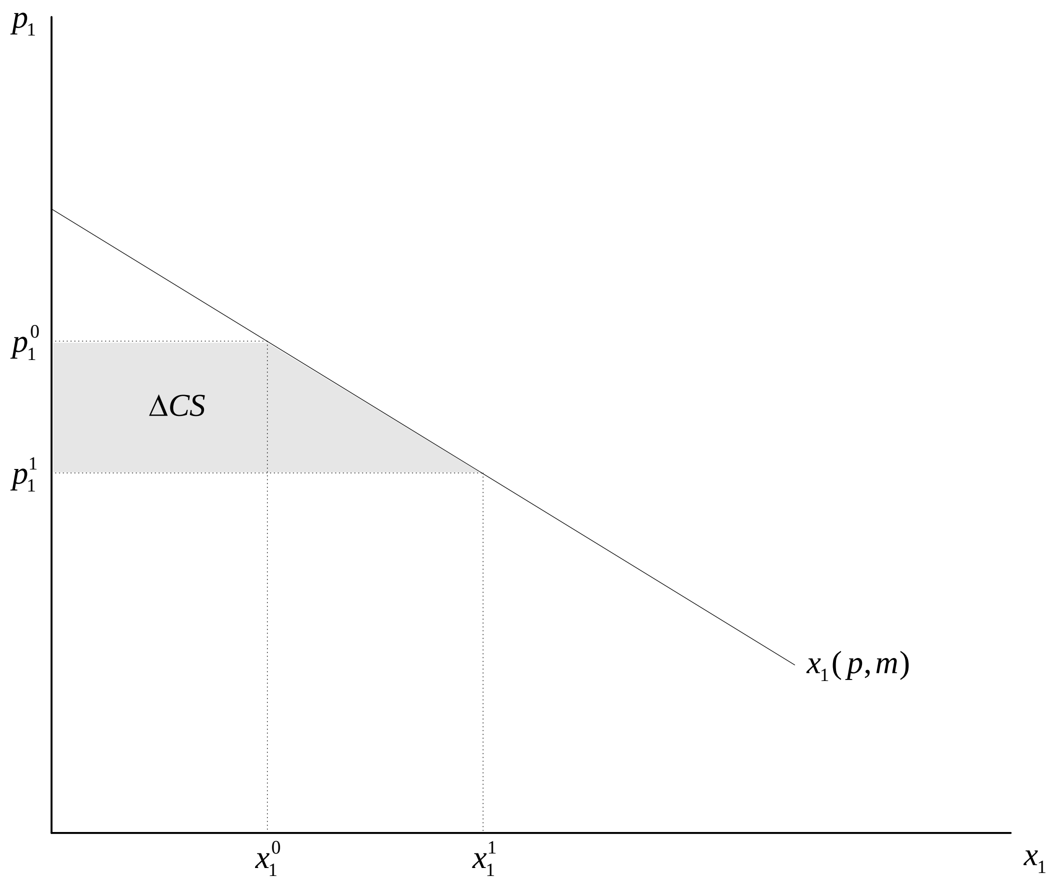


FIGURE 4.5

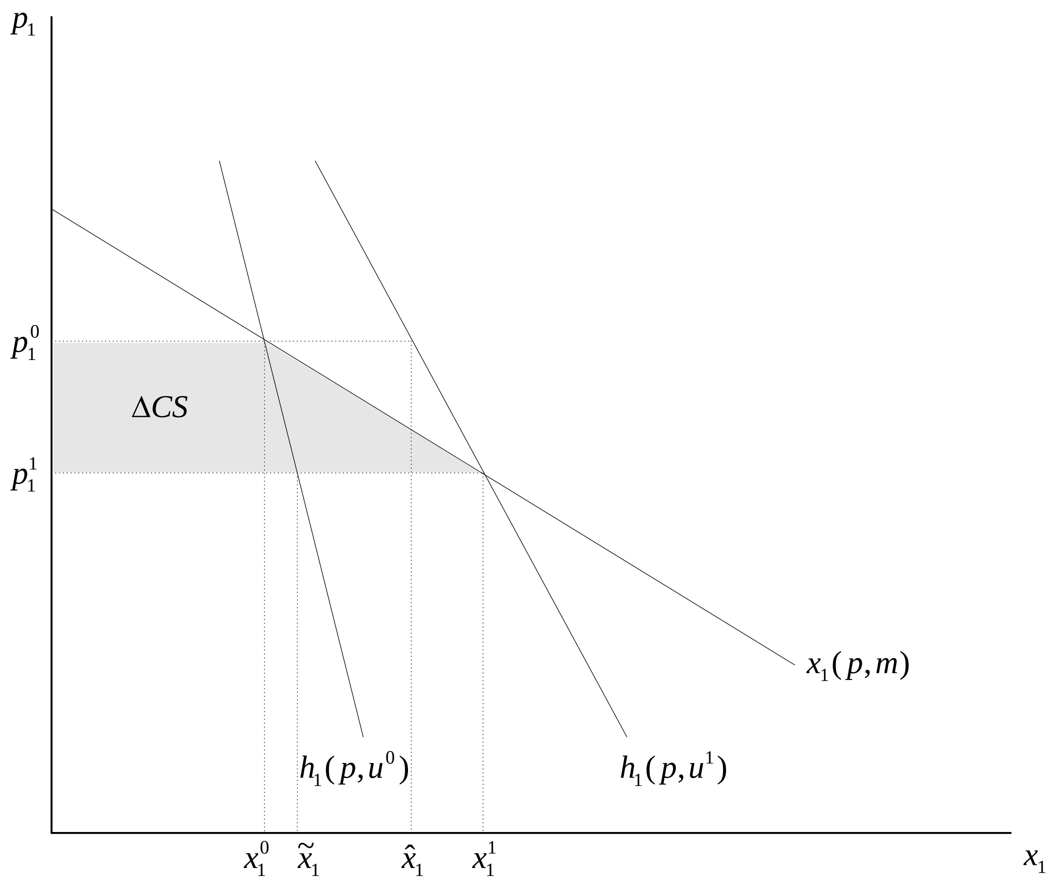


FIGURE 4.6

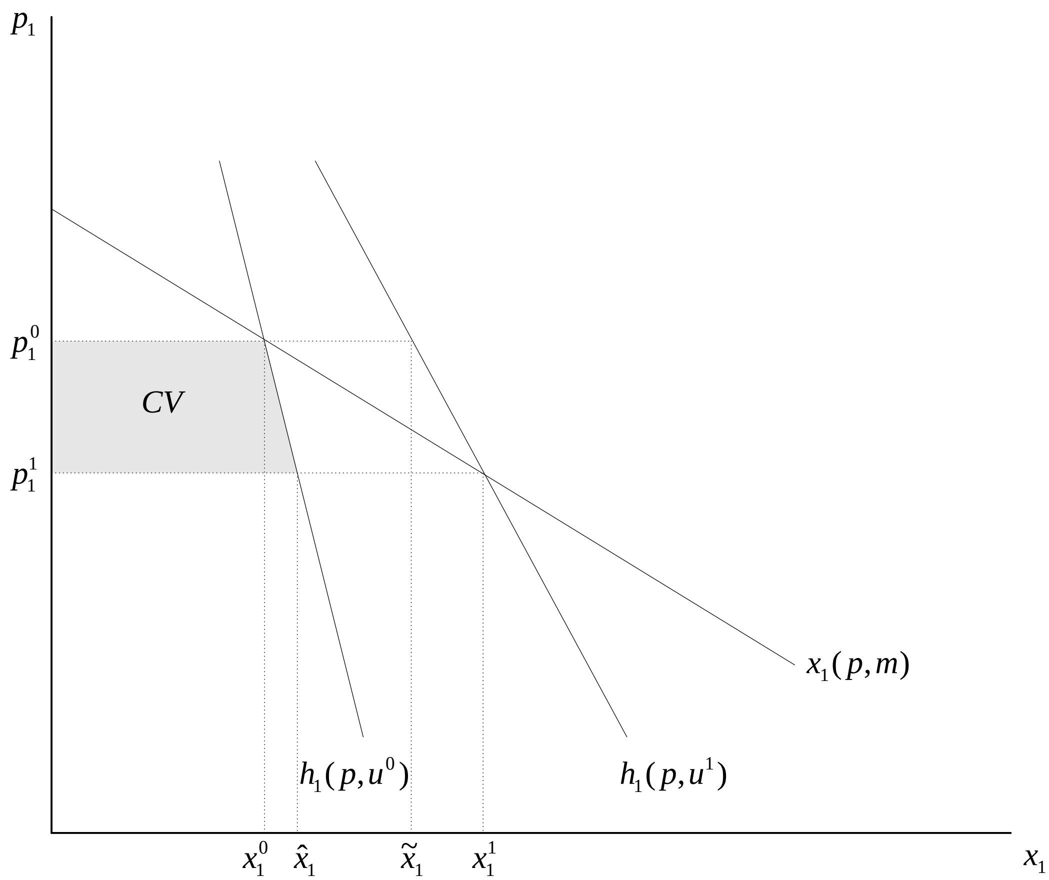


FIGURE 4.7

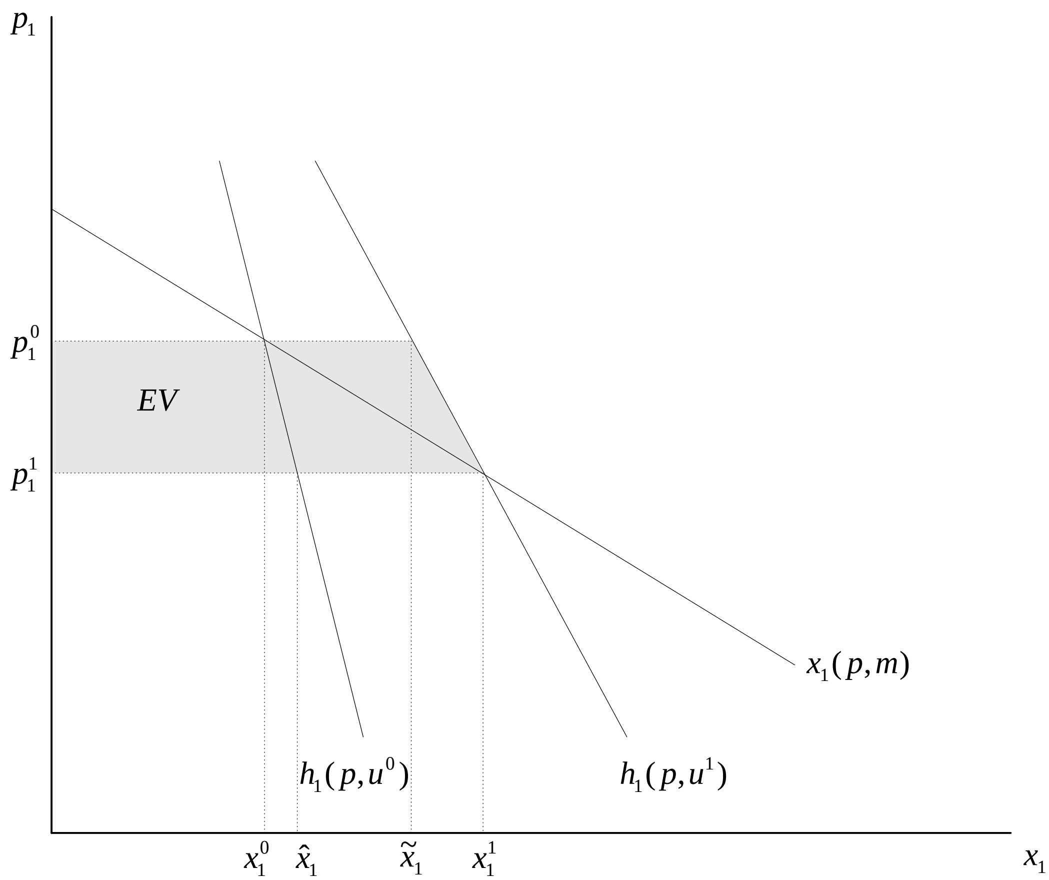


FIGURE 4.8

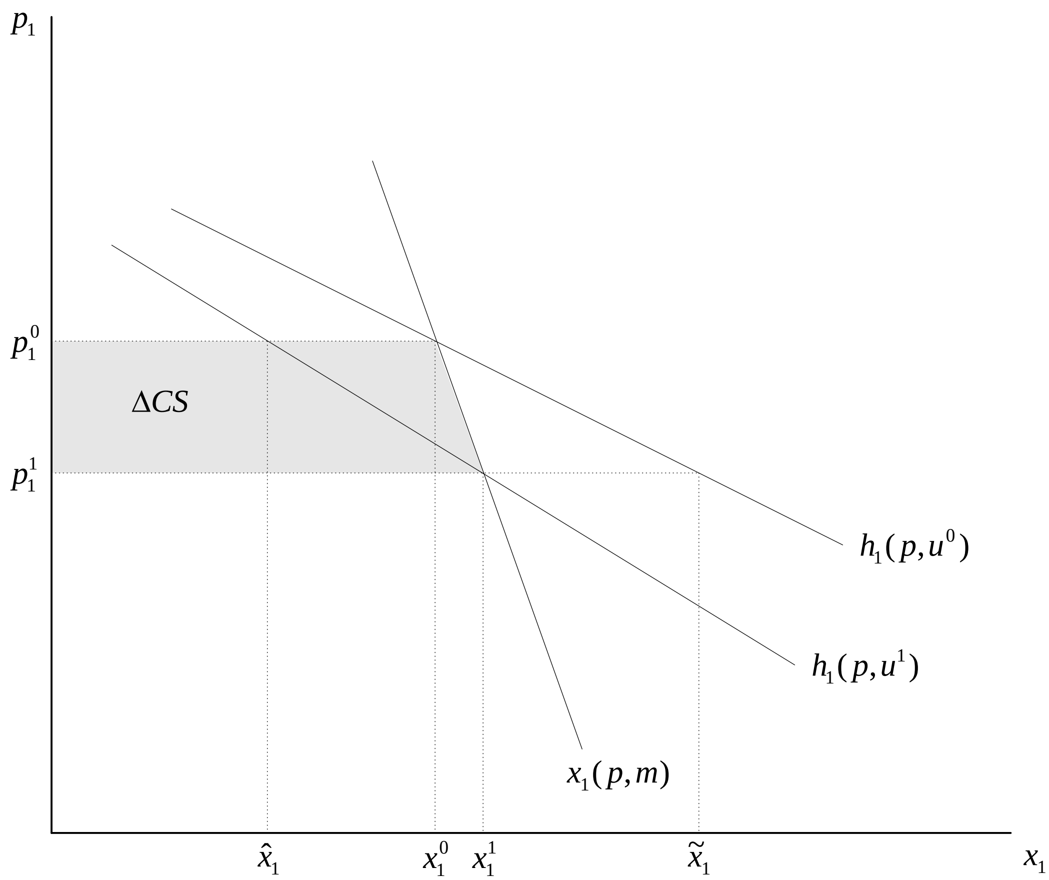


FIGURE 4.9

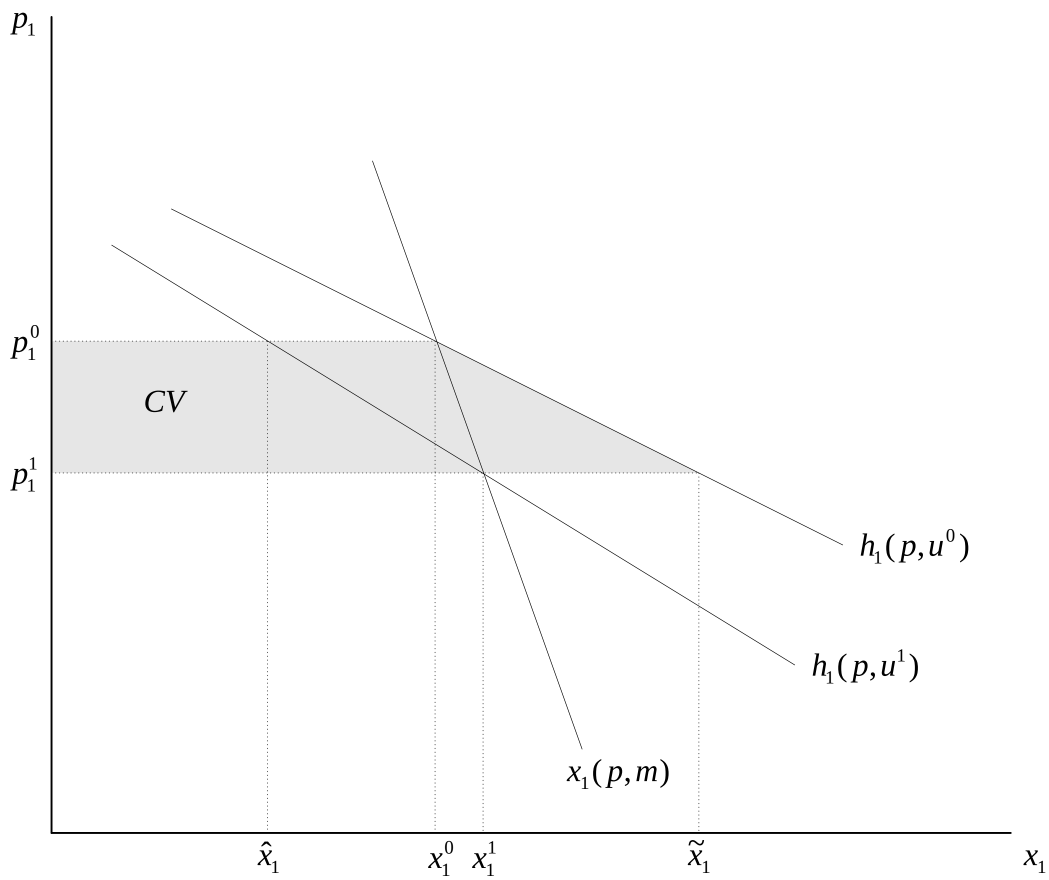


FIGURE 4.10

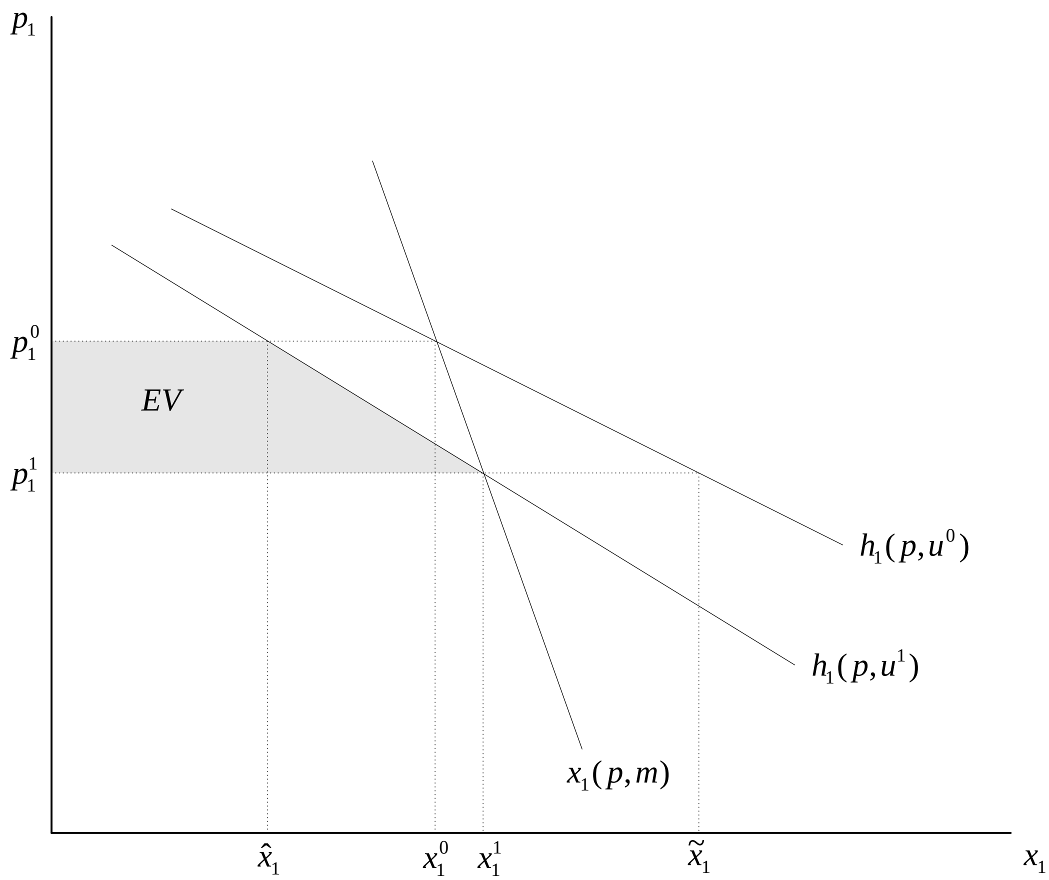


FIGURE 4.11

5. CHOICE UNDER UNCERTAINTY

5.1 Preliminaries: Definitions and Assumptions

A prospect (or lottery) Y is a set of state-contingent payoffs m_i (which we will interpret to be income or wealth) and associated probabilities π_i:

$$Y = \{m_1, m_2, ..., m_n; \pi_1, \pi_2, ..., \pi_n\} \text{ such that } \sum_{i=1}^{n} \pi_i = 1$$

A *certain prospect* is one for which $m_i = m \ \forall i$.

A prospect Y^j is the consequence of taking some action a^j. Thus, decision-making under uncertainty can be thought of as choosing between prospects.

The *expected value* of a prospect is

$$\mathrm{E}[Y] = \sum_{i=1}^{n} \pi_i m_i$$

5.2 Expected Utility

If preferences satisfy reasonable assumptions on preferences and perceptions over prospects (see the Appendix to this Chapter for more detail) then there exists a utility function $u(Y)$ – called a *Von Neumann-Morgenstern* utility function – defined over prospects such that $Y^1 \succ Y^2$ if and only if $u(Y^1) > u(Y^2)$, where $u(Y) = \sum_{i=1}^{n} \pi_i v(m_i)$, and $v(m)$ is the standard indirect utility function. This Von Neumann-Morgenstern utility function is unique up to an affine transformation.

This expected utility representation of preferences over prospects extends to continuous probability distributions $\pi(m)$ defined over outcomes $m \in M$, where

$$u(Y) = \int_{m \in M} v(m)\pi(m)dm$$

Note that we can embed the consumer demand theory from Topics 2 – 4 into this expected utility framework by interpreting $v(m_i)$ as the indirect utility function corresponding to income level m_i, where the dependence of $v(m_i)$ on prices is suppressed for notational simplicity.

In general, we will write $\mathrm{E}[v(m)]$ to denote the expected utility of a prospect, where *m* is a random variable corresponding to the payoffs (or outcomes) of the prospect.

<u>Example</u>

Suppose $v(m) = m^{\frac{1}{2}}$ and $Y = \{100, 64; \frac{1}{2}, \frac{1}{2}\}$. Then

$$u(Y) = \mathrm{E}[v(m)] = \frac{1}{2}v(100) + \frac{1}{2}v(64) = 9$$

5.3 Attitudes Towards Risk

Consider the following prospects:

$$Y^0 = \{100, 64; \tfrac{1}{2}, \tfrac{1}{2}\} \text{ with } \mathrm{E}[Y^0] = 82$$

$$\overline{Y} = \{82, 82; \pi, 1-\pi\} \text{ with } \mathrm{E}[\overline{Y}] = 82$$

These have identical expected values, but different degrees of *risk*; prospect Y^0 is riskier.

A *risk averse* agent dislikes risk. That is, $\mathrm{E}[v(m)] < v(\mathrm{E}[m])$.

A *risk neutral* agent is indifferent between prospects with the same expected values, regardless of risk. That is, $\mathrm{E}[v(m)] = v(\mathrm{E}[m])$.

A *risk loving* agent prefers risky prospects. That is, $\mathrm{E}[v(m)] > v(\mathrm{E}[m])$.

<u>Relation to the Curvature of $v(m)$</u>

1. Risk aversion $\Leftrightarrow$ $v(m)$ is strictly concave. That is,

$$v(\pi m_1 + (1-\pi)m_2) > \pi v(m_1) + (1-\pi)v(m_2)\ \forall \pi \in (0,1)$$

Example (see Figure 5.1):

$$Y^0 = \{100, 64; \tfrac{1}{2}, \tfrac{1}{2}\} \text{ and } v = m^{\frac{1}{2}}$$

$$u(Y^0) = \mathrm{E}[v(m)] = \frac{1}{2}(100)^{\frac{1}{2}} + \frac{1}{2}(64)^{\frac{1}{2}} = 9$$

$$v(\mathrm{E}[m]) = (\frac{1}{2}100 + \frac{1}{2}64)^{\frac{1}{2}} = (82)^{\frac{1}{2}} > 9$$

2. Risk neutral $\Leftrightarrow$ $v(m)$ is linear

3. Risk loving $\Leftrightarrow$ $v(m)$ is strictly convex

<u>Measures of Risk Aversion</u>

Using $v''(m)$ as a measure of risk aversion is not useful because it is not invariant to affine transformations. Instead we use either:

Arrow-Pratt measure of *absolute risk aversion*:

$$\alpha(m) = -\frac{v''(m)}{v'(m)}$$

Arrow-Pratt measure of *relative risk aversion*:

$$\rho(m) = -\frac{v''(m)m}{v'(m)}$$

In macroeconomic modeling, it is common to assume a utility function that exhibits constant relative risk aversion (CRRA):

$$v(m) = \frac{m^{1-\sigma}}{1-\sigma}$$

For this utility function, $\rho(m) = \sigma$.

5.4 Certainty-Equivalent Wealth and the Risk Premium

Consider a prospect

$$Y = \{m_1, m_2; \pi, 1-\pi\}$$

with expected utility

$$u(Y) = \mathrm{E}[v(m)] = \pi v(m_1) + (1-\pi)v(m_2)$$

Define the *certainty equivalent wealth* as $\hat{m}$ such that

$$v(\hat{m}) = \mathrm{E}[v(m)]$$

That is, $\hat{m}$ is the level of wealth, which if received with certainty, would yield the same utility as the uncertain prospect. See Figure 5.2.

<u>Example</u>

$$Y = \{100, 36; \tfrac{1}{2}, \tfrac{1}{2}\} \text{ and } v = m^{\frac{1}{2}}$$

$$\mathrm{E}[v(m)] = \frac{1}{2}(100)^{\frac{1}{2}} + \frac{1}{2}(36)^{\frac{1}{2}} = 8$$

Solve for $\hat{m}$ such that $[\hat{m}]^{\frac{1}{2}} = 8$. Thus, $\hat{m} = 64$.

The *risk premium* for a prospect is defined as

$$R \equiv \mathrm{E}[m] - \hat{m}$$

See Figure 5.2.

<u>Interpretation</u>

Suppose an agent must choose between a certain prospect that pays $\hat{m}$, and an uncertain prospect with expected value $\mathrm{E}[m]$ and certainty-equivalent payoff $\hat{m}$. For the agent to be indifferent between the two prospects, the uncertain prospect must have an associated expected value of $\hat{m} + R$, where *R* is the premium required to compensate for the risk. Note that $R = 0$ for a risk neural agent, and $R < 0$ for a risk lover.

5.5 The Demand for Insurance

Suppose an agent has current wealth *m* but with probability π she will suffer a loss *L*. She therefore faces an uncertain prospect:

$$Y = \{m, m - L; 1 - \pi, \pi\}$$

Suppose further that she can buy insurance against loss at price *r* per dollar of coverage *q*. How much insurance will she buy?

Her choice problem is to choose one prospect from a schedule of prospects:

$$Y(q) = \{m - rq, m - rq - L + q; 1 - \pi, \pi\}$$

and she makes that choice to maximize her expected utility:

$$\max_{q} \ \pi v(m - rq - L + q) + (1 - \pi) v(m - rq)$$

The associated first order condition is

$$\pi v'(m - rq^* - L + q^*)(1 - r) - (1 - \pi) v'(m - rq^*) r = 0$$

which can be written as a standard tangency condition:

$$\frac{v'(m - rq^* - L + q^*)}{v'(m - rq^*)} = \frac{r(1 - \pi)}{\pi(1 - r)}$$

<u>Profit for the Insurer</u>

In the event of a loss, the insurer makes

$$\Phi(\text{loss}) = rq - q$$

and in the event of no loss, the insurer makes

$$\Phi(\text{no loss}) = rq$$

Thus, expected profit for the insurer is

$$\mathrm{E}[\Phi] = \pi(rq - q) + (1-\pi)rq = q(r - \pi)$$

Under perfect competition, expected profit is zero. In that case, $r = \pi$. That is, the price of insurance is *actuarially fair*: price is just equal to expected cost.

Setting $r = \pi$ in the buyer's first-order condition yields

$$v'(m - \pi q^* - L + q^*) = v'(m - \pi q^*)$$

Suppose the agent is risk-averse. Then $v'' < 0$, and so $v'(m_1) = v'(m_2)$ iff $m_1 = m_2$. Thus, the first-order condition implies

$$m - \pi q^* - L + q^* = m - \pi q^*$$

which means $q^* = L$. That is, the agent buys full insurance.

Conversely, if $r > \pi$ then agent will buy less than full insurance ($q < L$), and face some residual risk

Note that we have assumed here that the agent cannot influence π (no moral hazard) and that the insurer can observe π (no adverse selection); we will later return to these issues and consider their implications for the market for insurance.

5.6 The Value of Information

Consider a setting where a risk-neutral agent faces an investment decision from which the returns are uncertain. In particular, the payoff from investing effort level e is θe, where θ is unknown. The cost of this investment is quadratic (and hence, strictly convex) in effort: $C(e) = ce^2$, where $c > 0$. Thus, net payoff to the agent is

$$P(e) = \theta e - ce^2$$

Suppose she believes that θ has two possible values: θ_L with probability π; and $\theta_H > \theta_L$ with probability $1-\pi$. Then the expected value of θ is

$$\pi\theta_L + (1-\pi)\theta_H \equiv \mu$$

No Information Acquisition: Choice Under Uncertainty

Consider her investment decision when facing this uncertainty. Since she is risk-neutral (by assumption), she will choose e to maximize her expected payoff

$$E[P(e)] = \pi(\theta_L e - ce^2) + (1-\pi)(\theta_H e - ce^2) = \mu e - ce^2$$

Her optimal investment is

$$e_0^* = \frac{\mu}{2c}$$

where the "0"subscript indicates a choice under uncertainty. The associated expected net payoff is

$$E[P(e_0^*)] = \frac{\mu^2}{4c}$$

Note that she will not actually receive this expected net payoff under any realization of θ. She will actually receive

$$P(e_0^*)\Big|_\theta = \frac{\theta\mu}{2c} - \frac{\mu^2}{4c} = \frac{\mu(2\theta - \mu)}{4c}$$

This is less than $E[P(e_0^*)]$ if the true value of θ is θ_L, and greater than $E[P(e_0^*)]$ if the true value of θ is θ_H.

Moreover, under either realization of θ, her investment choice is wrong *ex post* in the sense that her choice is not optimal given the realized value of θ. Her choice is nonetheless optimal *ex ante* (before the uncertainty is resolved) based on her beliefs about θ.

Choice Under Full Information

Now suppose the agent can purchase full information at cost k prior to making her investment choice. Her *ex post* investment-choice problem (that is, after she becomes informed), is to maximize

$$P(e) = \theta e - k - ce^2$$

and her optimal investment will be

$$e_1^* = \frac{\theta}{2c}$$

where the “1”subscript indicates a choice under full information. The associated net payoff is

$$P(e_1^*) = \frac{\theta^2}{4c} - k$$

Note that if $k = 0$ then this net payoff is strictly greater than the realized net payoff under any outcome when her choice is made under uncertainty. In particular, the difference in net payoff is

$$P(e_1^*) - P(e_0^*)\Big|_{\theta} = \frac{(\theta - \mu)^2}{4c} > 0 \quad \text{for} \quad \theta = \theta_L \quad \text{and for} \quad \theta = \theta_H$$

It is this difference in net payoff that makes information about θ valuable. The information allows the agent to make a better decision.

The Value of Information

To determine how much she is willing to pay to obtain the information before making her investment choice, we must compare the expected payoff under uncertainty with the *ex ante* expectation of her payoff under full information. The latter is calculated as follows.

If she obtains the information, then with probability π she will learn that $\theta = \theta_L$, and she will then go on to choose

$$e_1^*\Big|_{\theta = \theta_L} = \frac{\theta_L}{2c}$$

and receive net payoff

$$P(e_1^*)\Big|_{\theta = \theta_L} = \frac{\theta_L^2}{4c} - k$$

Conversely, with probability $1 - \pi$ she will learn that $\theta = \theta_H$, and she will then go on to choose

$$e_1^*\Big|_{\theta=\theta_H} = \frac{\theta_H}{2c}$$

and receive net payoff

$$P(e_1^*)\Big|_{\theta=\theta_H} = \frac{\theta_H^2}{4c} - k$$

Her *ex ante* expected net payoff from making an informed decision is

$$E[P(e_1^*)] = \pi P(e_1^*)\Big|_{\theta=\theta_L} + (1-\pi)P(e_1^*)\Big|_{\theta=\theta_H} = \frac{\pi\theta_L^2 + (1-\pi)\theta_H^2}{4c} - k$$

We can now calculate her WTP for the information as the maximum value of k that would make her just indifferent between being informed and remaining uniformed. We can find this value by setting $E[P(e_1^*)] = E[P(e_0^*)]$ and solving k. This yields

$$\bar{k} = \frac{\pi\theta_L^2 + (1-\pi)\theta_H^2 - \mu^2}{4c}$$

This expression can be written in terms of the variance of the Bernoulli distribution that describes her beliefs. That variance is

$$\sigma^2 = E[(\theta-\mu)^2] = \pi(\theta_L - \mu)^2 + (1-\pi)(\theta_H - \mu)^2$$

It is then straightforward to show that

$$\bar{k} = \frac{\sigma^2}{4c}$$

Note that WTP for the information is increasing in the degree of prior uncertainty, as measured by σ^2, and decreasing in the investment-cost parameter. The latter property reflects the fact that a high cost of investment reduces the level of investment undertaken under any beliefs, and that in turn reduces the difference that information makes to the investment choice. In particular, note that e_0^* and e_1^* both approach zero as $c \to \infty$, so in that limiting case, becoming informed can have no effect on the decision made. Hence, $\bar{k} \to 0$ as $c \to \infty$.

APPENDIX A5: PREFERENCES OVER PROSPECTS

Assumptions on Perceptions of Prospects

1. $\{m_1, m_2; 1, 0\} \sim \{m_1, m_1; \pi, 1-\pi\}$

   That is, the agent perceives as equivalent a prospect in which a payoff is received with probability one and a certain prospect with that same payoff.

2. $\{m_2, \{m_1, m_2; \pi, 1-\pi\}; 1-\omega, \omega\} \sim \{m_1, m_2; \pi\omega, 1-\pi\omega\}$

   That is, the agent's perception of a prospect depends only on *compound probabilities*.

Assumptions on Preferences over Prospects

1. Complete, reflexive and transitive.
2. Continuous: given three outcomes $m_1 \succ m_2 \succ m_3$, there exist prospects $Y^1 \sim Y^2$ such that $Y^1 = \{m_1, m_3; \pi, 1-\pi\}$ and $Y^2 = \{m_2; 1\}$.
3. Independence (or "sure thing principle"): for two prospects $Y^1 = \{m_1, m_2; \pi, 1-\pi\}$ and $Y^2 = \{m_3, m_2; \pi, 1-\pi\}$, $Y^1 \succ Y^2$ iff $m_1 \succ m_3$, regardless of m_2.
4. Positive Responsiveness: for two prospects $Y^1 = \{m_1, m_2; \pi^1, 1-\pi^1\}$ and $Y^2 = \{m_1, m_2; \pi^2, 1-\pi^2\}$, if $m_1 \succ m_2$ then $Y^1 \succ Y^2$ iff $\pi^1 > \pi^2$. That is, the prospect with the highest probability weight on the preferred outcome is the preferred prospect.

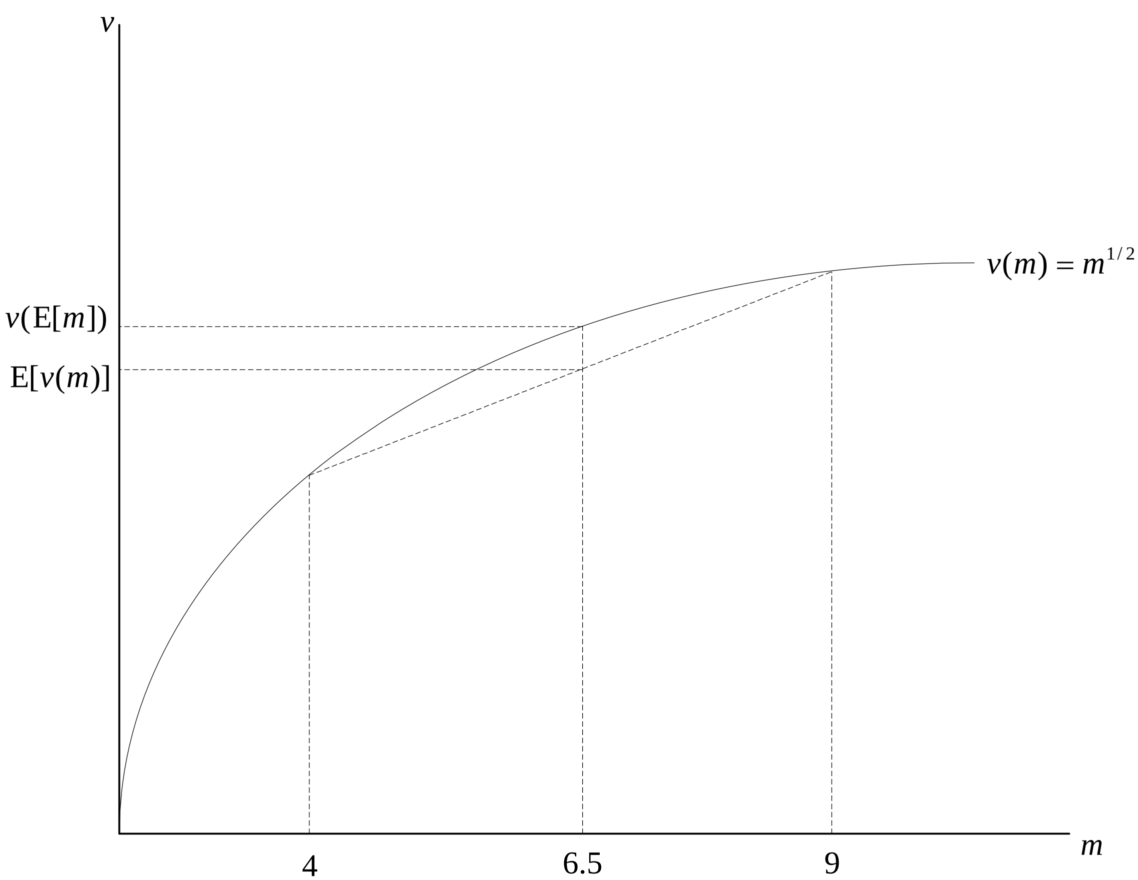


FIGURE 5.1

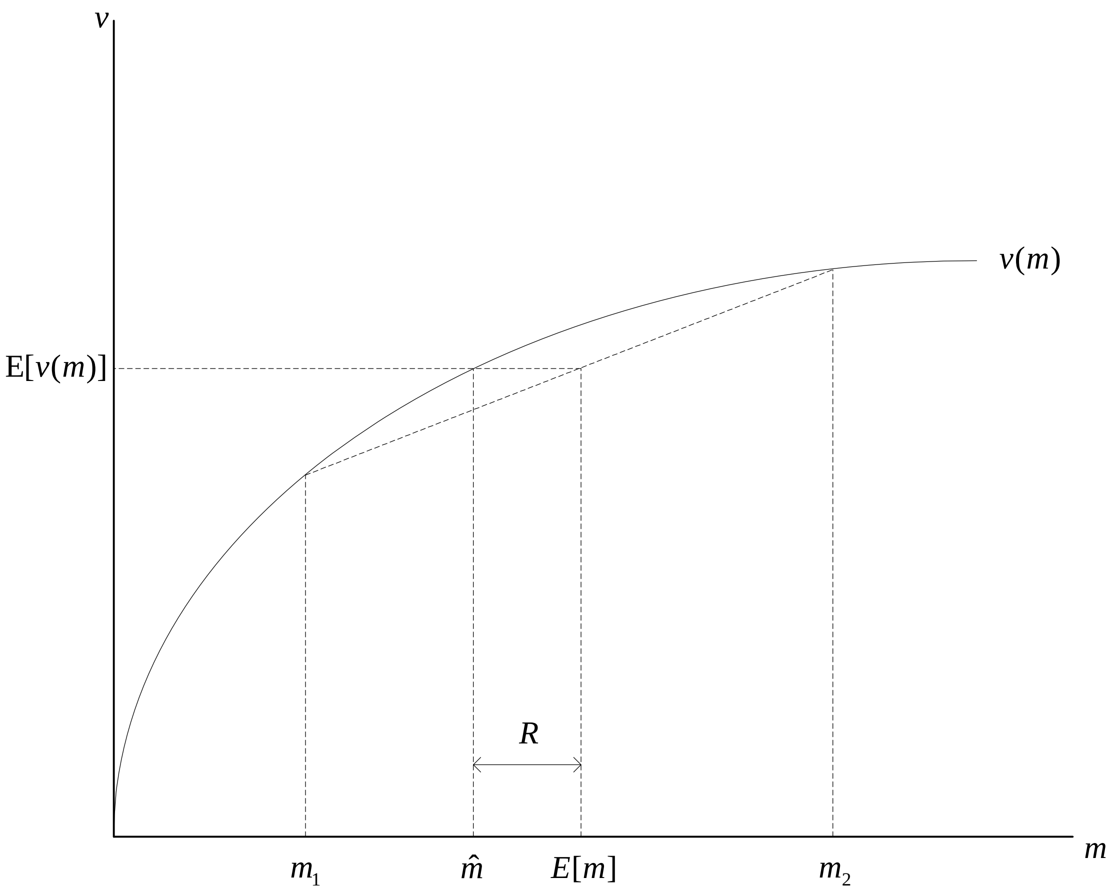


FIGURE 5.2

6. INTERTEMPORAL CHOICE

6.1 Intertemporal Preferences

Consider a consumer with preferences over a time path of composite consumption bundles $c_1, c_2, \ldots, c_T$. Under reasonable assumptions, these preferences can be represented by an intertemporal utility function $u(c_1, c_2, \ldots, c_T)$.

It is common to assume a time-separable form:

$$u(c_1, c_2, ..., c_T) = \sum_{t=1}^{T} \beta^{t-1} u(c_t)$$

where β is the *discount factor* and $u(c_t)$ is a time-invariant utility function (or felicity function). Note that we can express the discount factor as

$$\beta = \frac{1}{1+\rho}$$

where ρ is the *rate of time preference* or *discount rate*.

Note that this time-separable form imposes a common discount factor for all goods, the composite consumption of which in any period is c_t. It is therefore a restrictive representation of intertemporal preferences.

We can also express intertemporal utility in continuous time:

$$u(c) = \int_{t=0}^{T} u(c_t) e^{-\rho t} dt$$

where ρ is the discount rate.

6.2 A Simple Two-Period Model

A two period-lived agent has income m_1 when young and income m_2 when old. She has intertemporal preferences represented by

$$u = u(c_1) + \beta u(c_2)$$

Assume that income is storable; that is, income not consumed in period 1 can be saved for consumption in period 2.

Case 1: No Capital Market

Suppose there is no market for borrowing or lending. In this case the consumer's problem is

$$\max_{c_1 c_2} \; u(c_1) + \beta u(c_2)$$

$$\text{subject to} \quad \begin{array}{l} c_1 \le m_1 \\ c_2 \le m_2 + (m_1 - c_1) \end{array}$$

See Figure 6.1 for an agent who saves, and Figure 6.2 for an agent who does not save.

Case 2: Perfect Capital Market

Suppose the agent can borrow and lend on the market at interest rate r. In this case her choice problem is

$$\max_{c_1 c_2} \; u(c_1) + \beta u(c_2)$$

$$\text{subject to} \quad c_1 + \frac{c_2}{1+r} = m_1 + \frac{m_2}{1+r} \equiv w$$

Interpretation of the budget constraint: PV(consumption) = PV(lifetime income). Note that the maximum she can borrow in period 1 against future income is $\frac{m_2}{1+r}$, since she must pay that amount back with interest in period 2:

$$\left(\frac{m_2}{1+r}\right)(1+r) = m_2$$

See Figure 6.3. This reproduces the case from Figure 6.2 in which the agent did not save. Note that she now chooses to save due to the availability of interest on her savings, and she is better off because of it.

The Lagrangean for her choice problem is

$$L = u(c_1) + \beta u(c_2) + \lambda(w - c_1 - \frac{c_2}{1+r})$$

with associated FOCs:

$$u'(c_1) = \lambda$$

$$\beta u'(c_2) = \frac{\lambda}{1+r}$$

which can be written in ratio form as an *Euler equation*:

$$\frac{\beta u'(c_2)}{u'(c_1)} = \frac{1}{1+r}$$

This is simply a tangency condition: intertemporal $MRS =$ relative price of future and current consumption.

Writing $\beta = \frac{1}{1+\rho}$, we have

$$\frac{u'(c_2)}{u'(c_1)} = \frac{1+\rho}{1+r}$$

Thus, if $u'' < 0$, then

$$\rho > r \Rightarrow c_1 > c_2$$
$$\rho < r \Rightarrow c_1 < c_2$$

Note that in both cases, the agent could be a borrower or a lender when young, depending on her preferences and the relative value of m_1 and m_2. See Figure 6.3 for the case of a young lender, and Figure 6.4 for the case of a young borrower.

Case 3: Imperfect Capital Market

Suppose the borrowing rate is greater than the lending (or saving) rate: $r_B > r_S$. See Figure 6.5. This distortion could be caused by a variety of factors: for example, interest income is taxable, but interest payments are not tax deductible; imperfect competition; asymmetric information with respect to risk of default; transaction costs. If sufficiently large, such distortions can cause agents not to borrow or lend, when in the absence of the distortion they would (as illustrated in Figure 6.5).

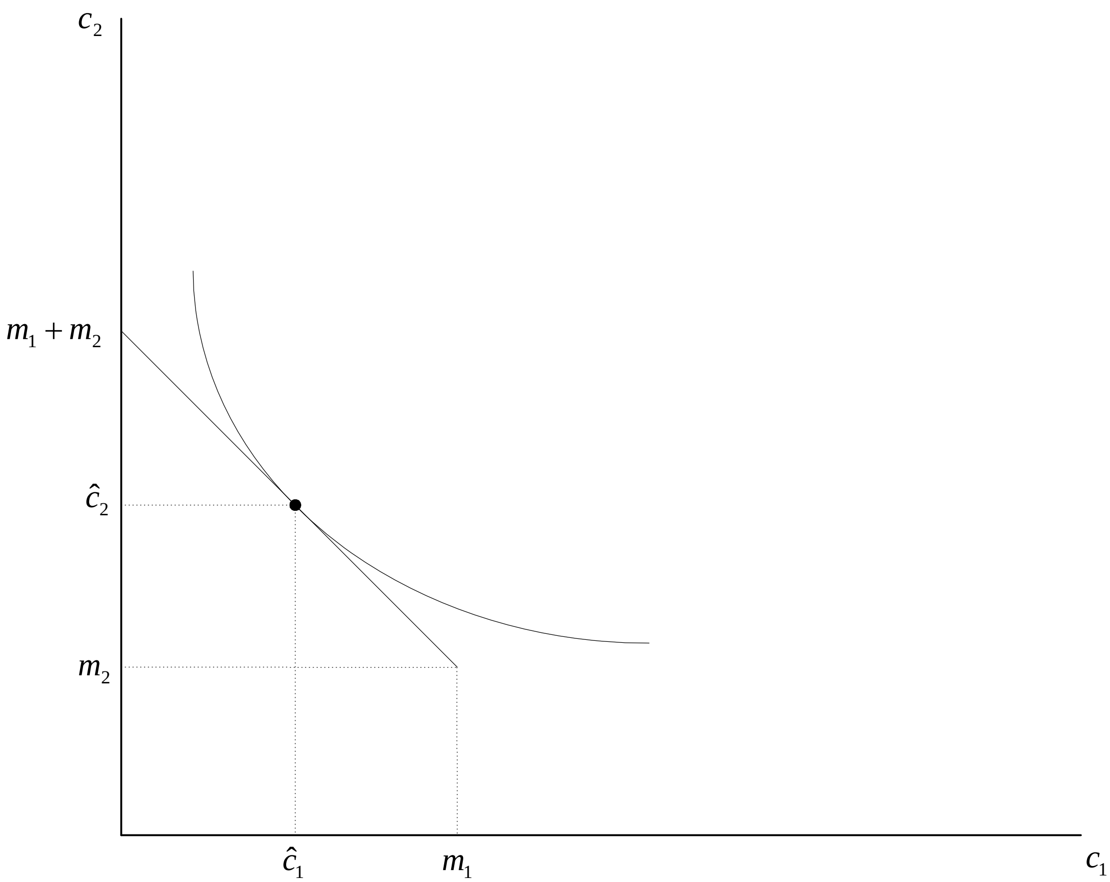


FIGURE 6.1

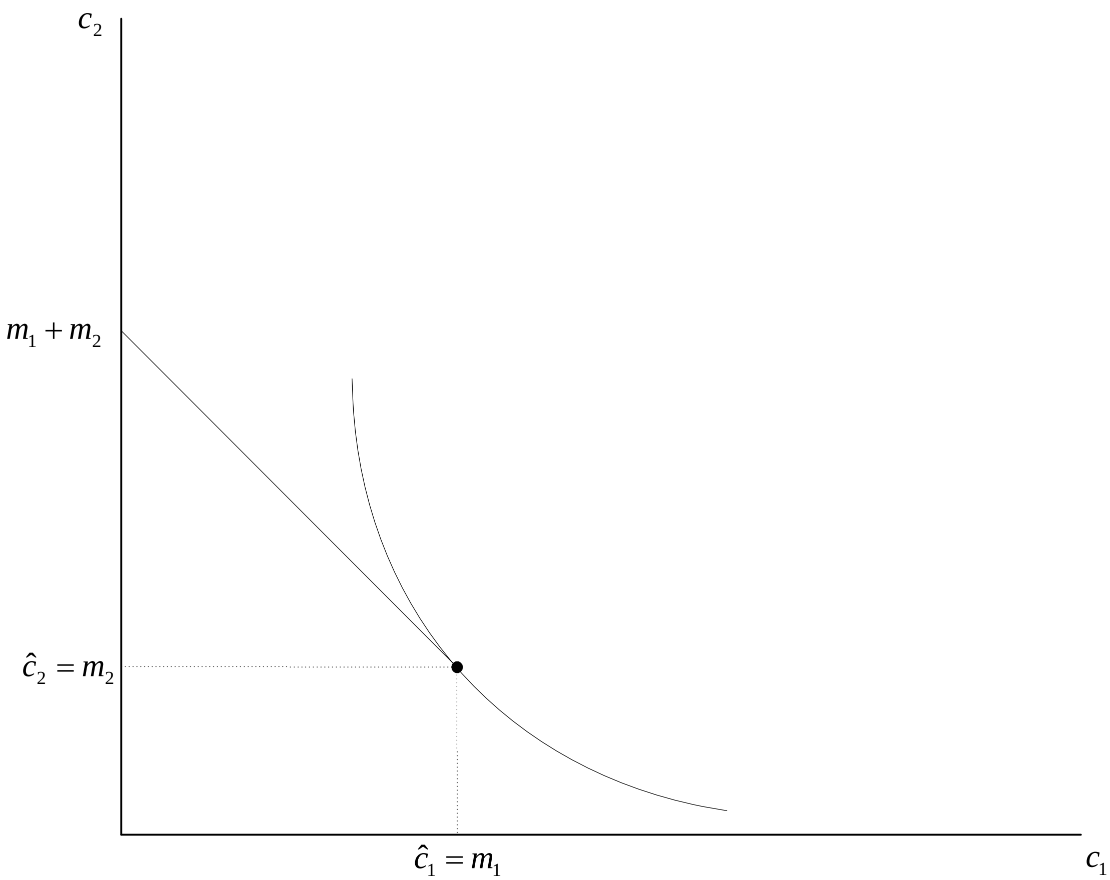


FIGURE 6.2

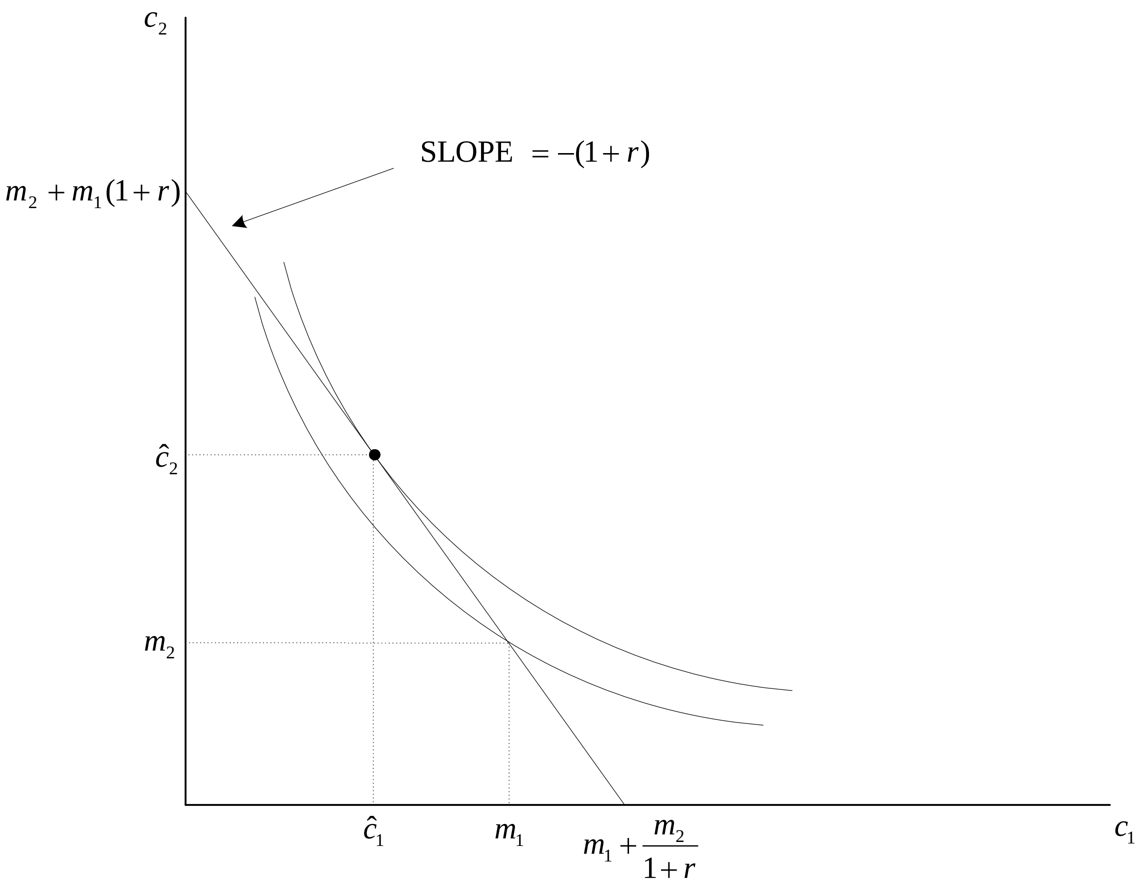


FIGURE 6.3

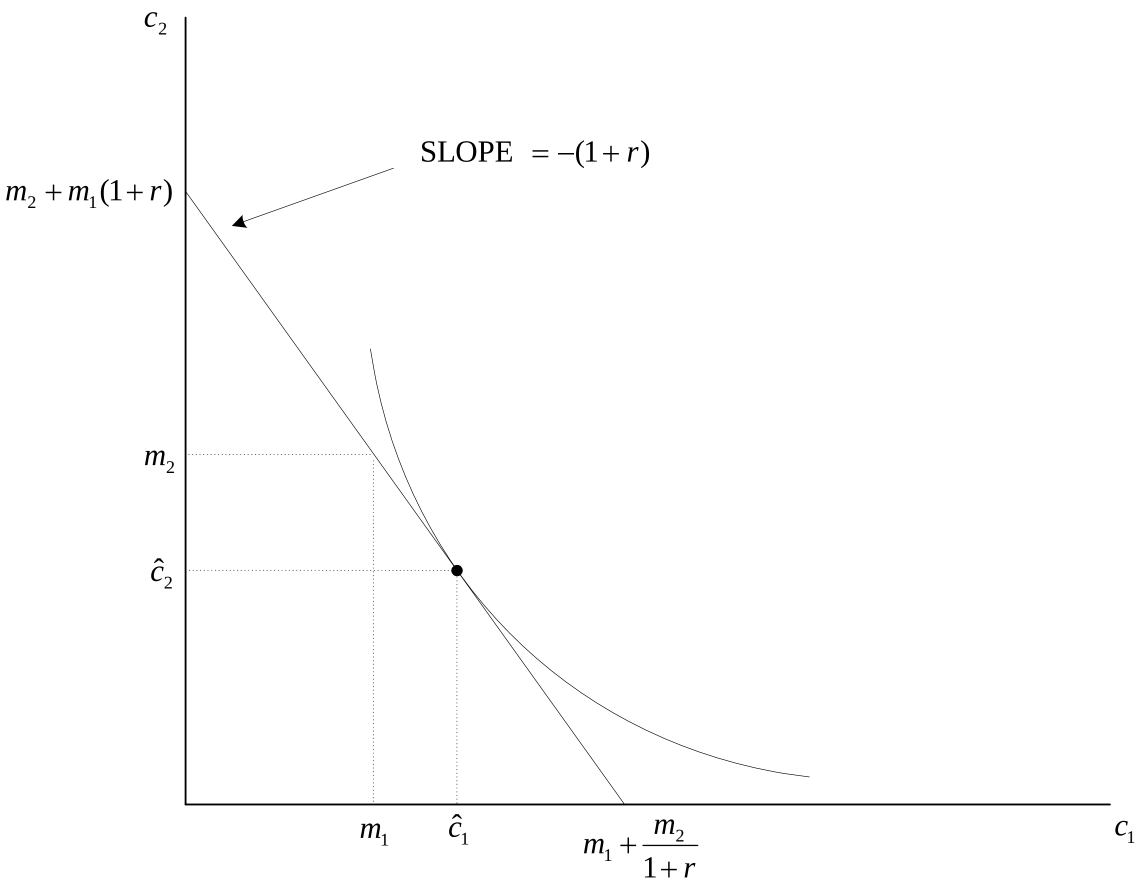
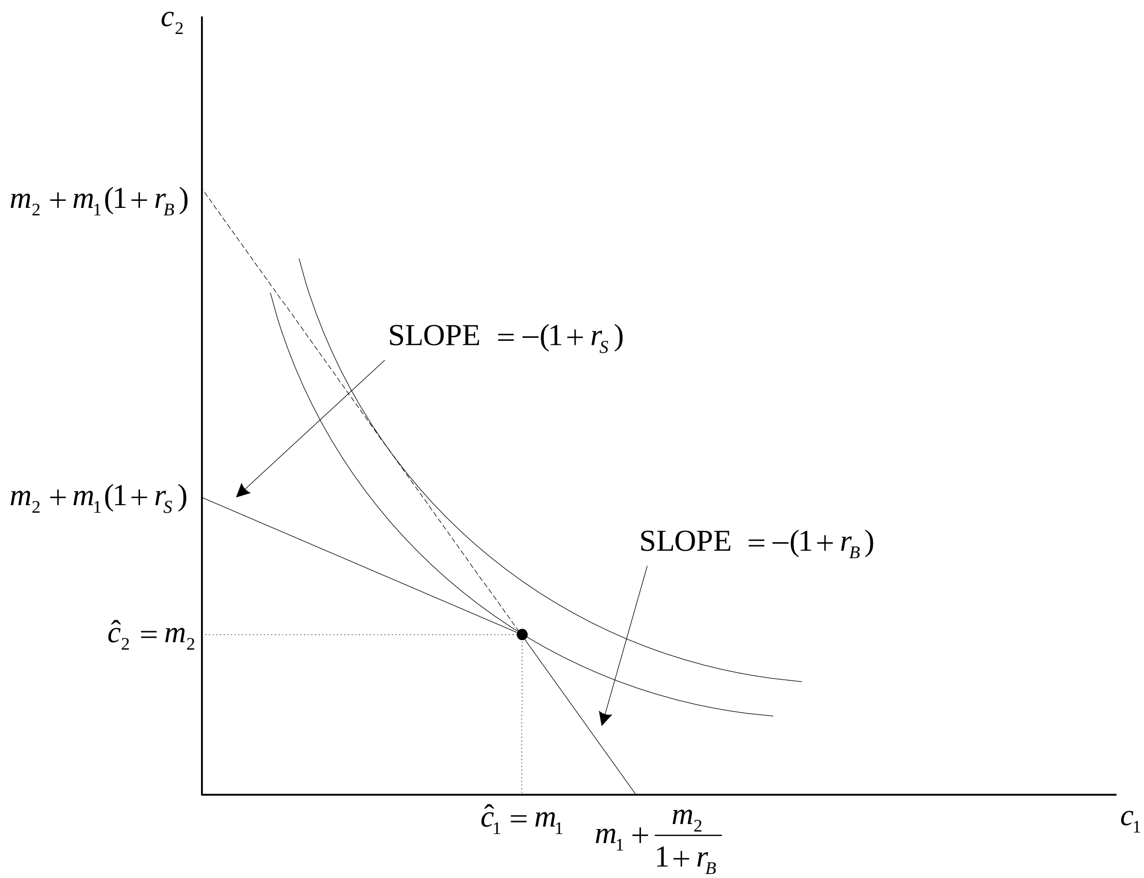


FIGURE 6.4



c_2
$m_2 + m_1(1+r_B)$
SLOPE $= -(1+r_S)$
$m_2 + m_1(1+r_S)$
SLOPE $= -(1+r_B)$
$\hat{c}_2 = m_2$
$\hat{c}_1 = m_1$
$m_1 + \dfrac{m_2}{1+r_B}$
c_1

FIGURE 6.5

7. PRODUCTION TECHNOLOGY

7.1 Technical Rate of Substitution

Consider a production function in two inputs x_1 and x_2:

$$y = f(x_1, x_2)$$

Total differentiation yields

$$dy = \frac{\partial f}{\partial x_1} dx_1 + \frac{\partial f}{\partial x_2} dx_2 = 0$$

Rearranging yields the *technical rate of substitution* (TRS):

$$\text{TRS} \equiv \frac{dx_2}{dx_1} = \frac{-\partial f}{\partial x_1} \bigg/ \frac{\partial f}{\partial x_2} \equiv \text{slope of an isoquant}$$

See Figure 7.1.

<u>Examples</u>

1. Cobb-Douglas

$$y = x_1^{\alpha} x_2^{\beta}$$

$$\frac{\partial f}{\partial x_1} = \alpha x_1^{\alpha-1} x_2^{\beta}$$

$$\frac{\partial f}{\partial x_2} = \beta x_1^{\alpha} x_2^{\beta-1}$$

$$TRS = -\frac{\alpha x_2}{\beta x_1}$$

2. Leontief (fixed proportions)

$$y = \min\{ax_1, bx_2\}$$

$$TRS = 0 \qquad \text{for } x_1 > \frac{b}{a} x_2$$

$$TRS = -\infty \qquad \text{for } x_1 < \frac{b}{a} x_2$$

See Figure 7.2.

7.2 Elasticity of Substitution

TRS measures the slope of an isoquant; the *elasticity of substitution* measures its curvature:

$$\sigma = \frac{\%\Delta(x_2/x_1)}{\%\Delta(TRS)} = \frac{\Delta(x_2/x_1)}{(x_2/x_1)} \Bigg/ \frac{\Delta TRS}{|TRS|}$$

This measures the rate at which the slope of an isoquant changes as we rotate a ray from the origin. See Figure 7.3.

For small changes we have

$$\sigma = \frac{d(x_2/x_1)}{d|TRS|} \cdot \frac{|TRS|}{(x_2/x_1)}$$

Moreover, any elasticity

$$\varepsilon = \frac{dy}{y} \Bigg/ \frac{dx}{x}$$

can be written as

$$\varepsilon = d[\log y]/d[\log x]$$

Thus, we can write

$$\sigma = \frac{d[\log(x_2/x_1)]}{d[\log|TRS|]}$$

<u>Example:</u> Cobb-Douglas

$$y = x_1^{\alpha} x_2^{\beta}$$

$$|TRS| = \frac{\alpha x_2}{\beta x_1}$$

Thus,

$$\frac{x_2}{x_1} = \frac{\beta}{\alpha}|TRS|$$

Taking the logarithm of both sides yields

$$\log(\frac{x_2}{x_1}) = \log(\frac{\beta}{\alpha}) + \log|TRS|$$

and so we have

$$\sigma = \frac{d[\log(x_2/x_1)]}{d[\log|TRS|]} = 1$$

7.3 Returns to Scale

What is the effect on output if all inputs are changed in the same proportion; that is, if the *scale* of production changes?

A technology exhibits *constant returns to scale* (CRS) iff

$$f(tx) = tf(x) \quad \forall t > 1$$

That is, the production function is homogeneous of degree 1.

Three reasons why a technology may not exhibit CRS:

(a) subdivision not possible (there is some minimum feasible scale)

(b) non-integer replication may not be possible

(c) an increase in scale may allow use of a more efficient technique (using the same inputs).

A technology exhibits *increasing returns to scale* (IRS) iff

$$f(tx) > tf(x) \quad \forall t > 1$$

and *decreasing returns to scale* (DRS) iff

$$f(tx) < tf(x) \quad \forall t > 1$$

In reality we offer observe *apparent* DRS because not all inputs have truly been scaled up (eg. managerial attention in a firm); the fixed factors become "congested".

Examples

1. Cobb-Douglas

$$y = x_1^{\alpha} x_2^{\beta}$$

$$f(tx) = (tx_1)^{\alpha}(tx_2)^{\beta} = t^{\alpha+\beta} x_1^{\alpha} x_2^{\beta}$$

Thus,

CRS iff $\alpha + \beta = 1$

DRS iff $\alpha + \beta < 1$

IRS iff $\alpha + \beta > 1$

2. Leontief

$$y = \min\{ax_1, bx_2\}$$

$$f(tx) = \min\{atx_1, btx_2\} = t\min\{ax_1, bx_2\}$$

Thus, CRS $\forall a, b$.

7.4 Homogeneous and Homothetic Technologies

Theorem

The TRS for a homothetic production function is independent of scale.

Proof. If $f(x)$ is homothetic then

$$f(x) = g[h(x)]$$

where $g' > 0$ and $h(x)$ is homogeneous. Thus,

$$\frac{\partial f}{\partial x_1} = g' \cdot \frac{\partial h}{\partial x_1}$$

$$\frac{\partial f}{\partial x_2} = g' \cdot \frac{\partial h}{\partial x_2}$$

and so

$$|TRS| = \frac{\partial f}{\partial x_1} \Big/ \frac{\partial f}{\partial x_2} = \frac{\partial h}{\partial x_1} \Big/ \frac{\partial h}{\partial x_2}$$

If $h(x)$ is homogeneous of degree *k*, then $\frac{\partial h}{\partial x_i}$ is homogeneous of degree *k*-1 (by Euler's theorem), so

$$\frac{\partial h(tx)}{\partial x_i} = t^{k-1}\frac{\partial h(x)}{\partial x_i}$$

Thus,

$$\frac{\partial h(tx)}{\partial x_1}\Big/\frac{\partial h(tx)}{\partial x_2} = t^{k-1}\frac{\partial h(x)}{\partial x_1}\Big/t^{k-1}\frac{\partial h(x)}{\partial x_2} = \frac{\partial h(x)}{\partial x_1}\Big/\frac{\partial h(x)}{\partial x_2}$$

which is independent of *t*.♣

<u>Geometric Interpretation</u>

Isoquants of homothetic functions have equal slope along a ray. See Figure 7.4.

7.5 A Production Function with Constant Elasticity of Substitution (CES)

$$y = (a_1 x_1^{\rho} + a_2 x_2^{\rho})^{\frac{1}{\rho}}$$

Fort this CES production function,

$$\sigma = \frac{1}{1-\rho}$$

To see this, note that

$$TRS = \frac{a_1}{a_2}\left(\frac{x_2}{x_1}\right)^{1-\rho}$$

So we can write

$$\frac{x_2}{x_1} = \left(\frac{a_2}{a_1}|TRS|\right)^{\frac{1}{1-\rho}}$$

Take logs to yield

$$\log\left(\frac{x_2}{x_1}\right) = \frac{1}{1-\rho}\log\left(\frac{a_2}{a_1}\right) + \frac{1}{1-\rho}\log(|TRS|)$$

Then

$$\sigma = \frac{d[\log(x_2/x_1)]}{d[\log|TRS|]} = \frac{1}{1-\rho}$$

<u>Special Cases</u>

1. Linear: $\rho = 1$

$$y = a_1 x_1 + a_2 x_2$$

In this case, $\sigma = \infty$. The isoquants are linear; see Figure 7.5.

2. Cobb-Douglas: $\rho \to 0$

At $\rho = 0$, the CES function is not defined, but note that

$$TRS_{CES} = -(\frac{x_1}{x_2})^{\rho-1} \text{ for } a_1 = a_2$$

Taking the limit we have

$$\lim_{\rho \to 0} TRS_{CES} = -(\frac{x_2}{x_1})$$

which is the TRS for a Cobb-Douglas production function when $a_1 = a_2$.

3. Leontief: $\rho \to -\infty$

$$TRS_{CES} = -(\frac{x_1}{x_2})^{-\infty} \text{ for } a_1 = a_2$$

$$= -(\frac{x_2}{x_1})^{\infty}$$

$$= -\infty \text{ if } x_2 > x_1$$

$$= 0 \text{ if } x_2 < x_1$$

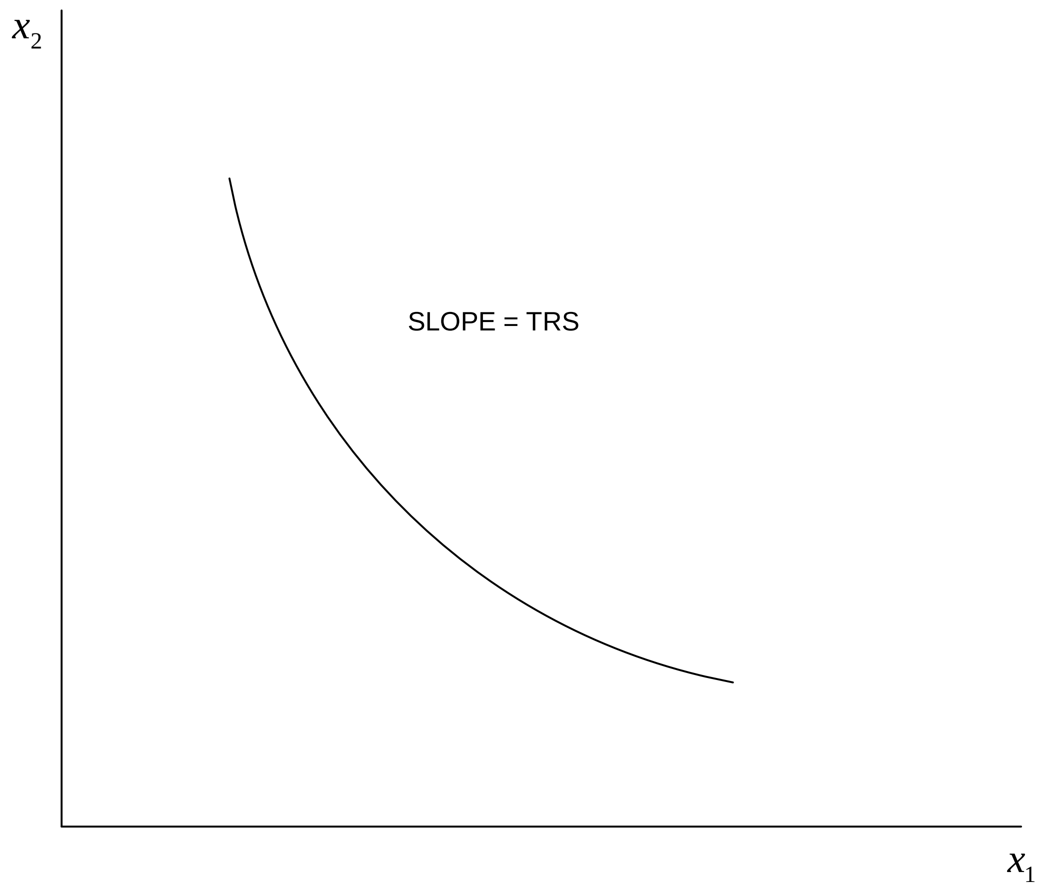


FIGURE 7.1

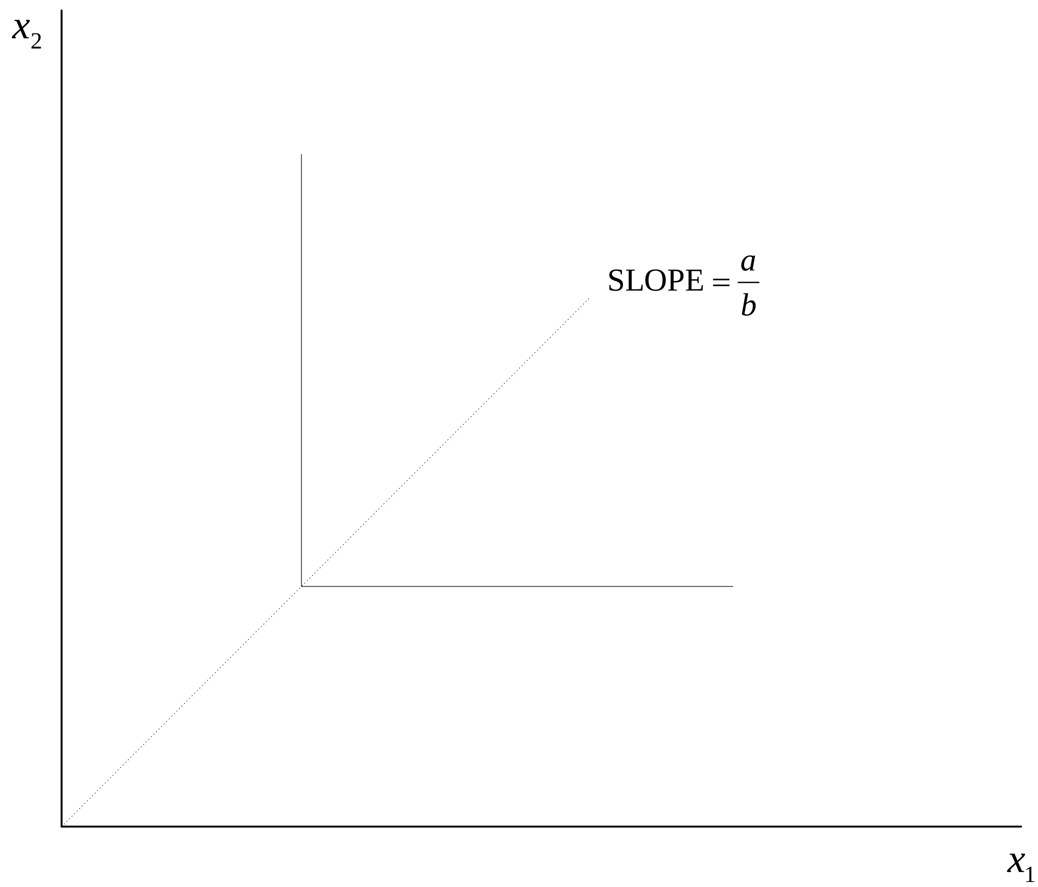


FIGURE 7.2

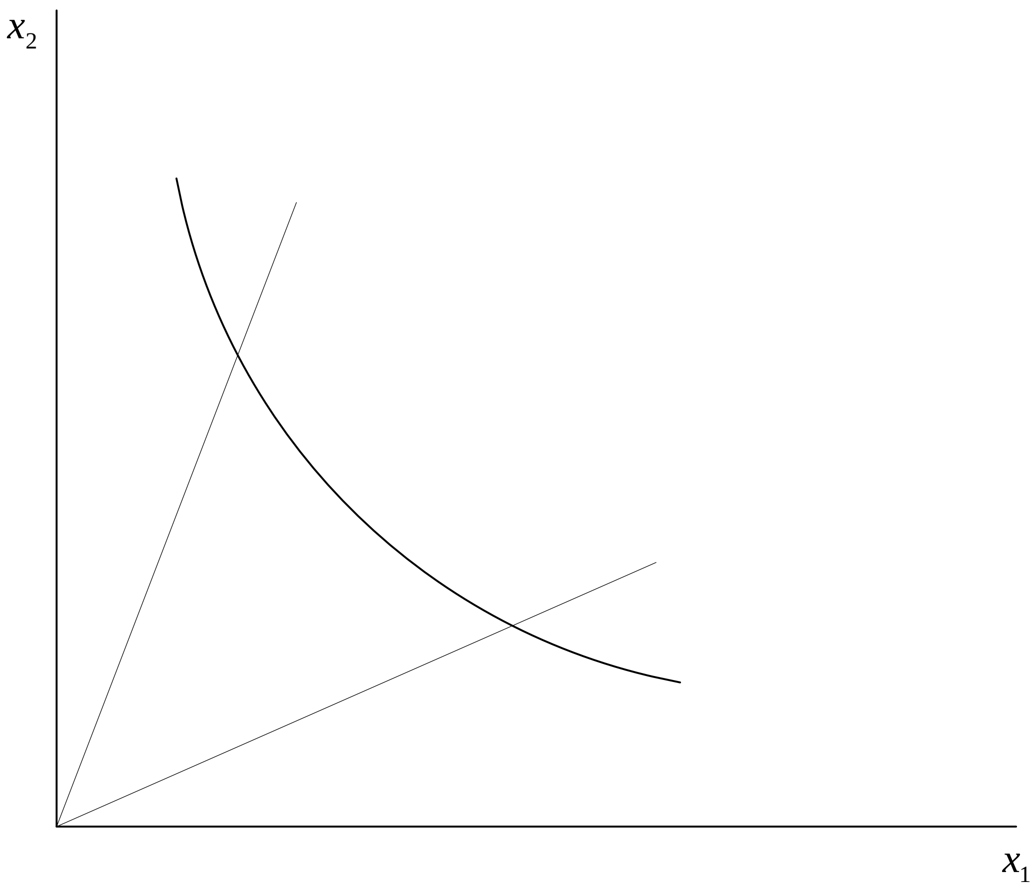


FIGURE 7.3

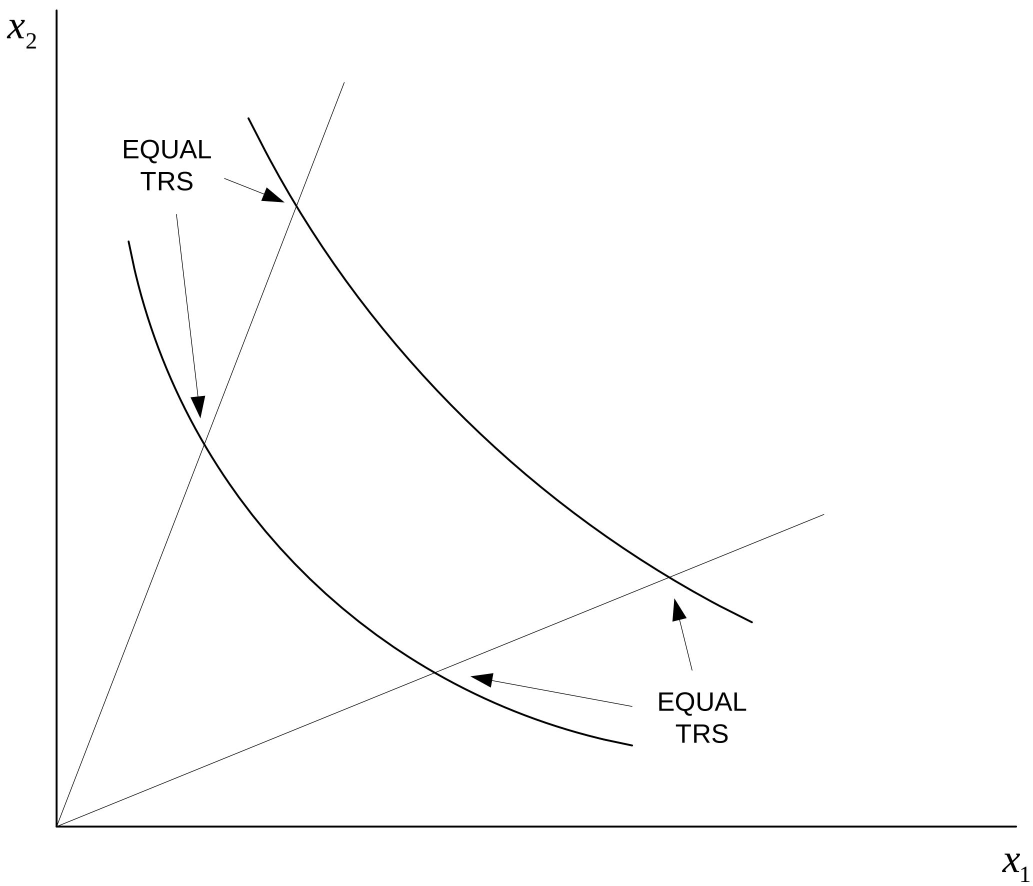


FIGURE 7.4

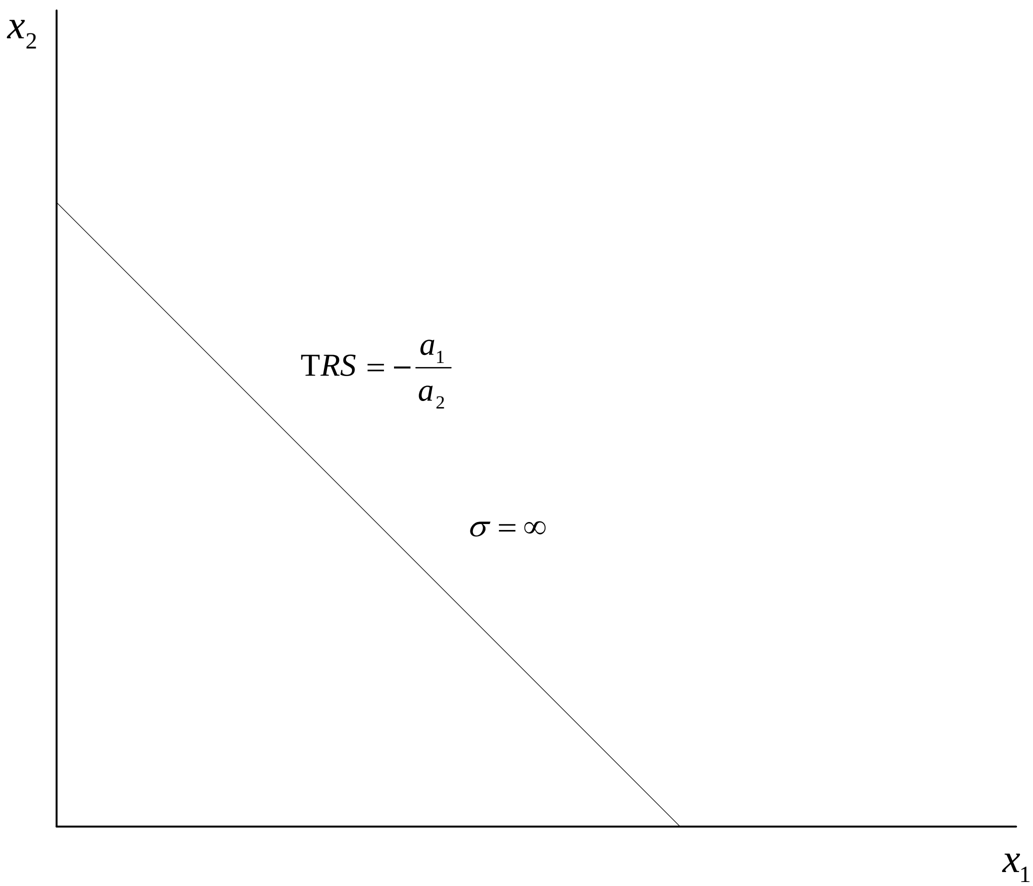


FIGURE 7.5

8. COST MINIMIZATION AND THE COST FUNCTION

8.1 The Cost Minimization Problem

Behavioral assumption: firms act to minimize cost. That is,

$$\min_{x} \; w(x)x \quad \text{subject to} \quad f(x) = y$$

Restrict attention to the case where the firm takes input prices as given: $w'(x) = 0$. That is, the firm is a price-taker on the input market. Thus, the first-order conditions for a minimum are

$$w_i - \lambda f_i = 0 \qquad \forall i$$

Taking the ratio of any pair *i* and *j* yields

$$\frac{f_i}{f_j} = \frac{w_i}{w_j}$$

or equivalently,

$$\frac{f_i}{w_i} = \frac{f_j}{w_j}$$

That is, the input mix is chosen to ensure that the marginal product per dollar of cost is equated across inputs.

Geometric interpretation: the slope of the isoquant is equal to the slope of the isocost line at the optimum. See Figure 8.1.

8.2 Conditional Input Demands

The solution to the cost minimization problem is

$$x_i(w, y) \text{ for } i = 1, \ldots, n$$

These are *conditional input demands* (or conditional factor demands). They are conditional on a particular level of output.

Note that the cost minimization problem is analogous to the expenditure minimization problem for the consumer, and the conditional input demands are analogous to the Hicksian demand curves.

8.3 The Cost Function

The cost function specifies the minimum cost of producing *y*:

$$c(w,y) = w \cdot x(w,y) = \sum_{i=1}^{n} w_i x_i(w,y)$$

Properties of the Cost Function

1. $c(w,y)$ is non-decreasing in *w* and increasing in *y*.
2. $c(w,y)$ is homogeneous of degree 1 in *w*.
3. $c(w,y)$ is concave in *w*.

These properties are analogous to the properties of the expenditure function; see section 2.5 in Chapter 2. Recall in particular the intuition for property 3: a passive reaction to an increase in *w* would leave input choices unchanged, and cost would rise linearly. The firm can generally do better than a passive reaction by *substituting* out of the input whose price has risen (except in the case of a fixed proportions technology).

Properties of the Conditional Input Demands

The following three properties follow from the definition of $c(w,y)$ as a minimum value function.

1. Negativity

$$\frac{\partial x_i}{\partial w_i} \leq 0$$

That is, the conditional own-price effect – the *substitution effect* – is non-positive.

Proof. By Shephard's lemma and concavity of $c(w,y)$.♣

2. Symmetry

$$\frac{\partial x_i}{\partial w_j} = \frac{\partial x_j}{\partial w_i}$$

That is, conditional cross-price effects are *symmetric*.

Proof. By Shephard's lemma and Young's theorem.♣

3. Homogeneity

$x(w, y)$ is homogeneous of degree 0 in w.

Proof. By Shephard's lemma and Euler's theorem.♣

8.4 Marginal Cost and Average Cost

$$\text{Marginal cost} \equiv MC = \frac{\partial c(w, y)}{\partial y}$$

Note that MC is equal to the Lagrange multiplier from the cost-minimization problem.

$$\text{Average cost} \equiv AC = \frac{c(w, y)}{y}$$

Theorem 1

If MC is rising then MC and AC are equal at minimum AC.

Proof. Consider the problem

$$\min_y \quad AC$$

the first-order condition for which is

$$\frac{\partial \left(\dfrac{c(w, y)}{y} \right)}{\partial y} = 0$$

That is,

$$\frac{y\dfrac{\partial c(w,y)}{\partial y}-c(w,y)}{y^2}=0$$

This yields a minimum iff $\partial^2 c(w,y)/\partial y^2 > 0$ (that is, iff MC is rising). Rearranging the first-order condition yields

$$\frac{\partial c(w,y)}{\partial y}=\frac{c(w,y)}{y}$$

That is, MC =AC. If AC has no interior minimum then it is minimized at $y=0$, where MC =AC.♣

<u>Short-Run vs. Long-Run Cost</u>

In the short-run (SR) at least one input is fixed; that is, its level cannot be chosen freely. In the long-run (LR) all inputs can be varied.

Suppose input x_n is fixed in the SR. Then the SR cost-minimization problem is

$$\min_{x_{i\neq n}} \sum_{i\neq n} w_i x_i + w_n x_n \quad \text{subject to} \quad f(x)=y$$

and the associated cost function is

$$c(w,y,x_n)=\sum_{i\neq n} w_i x_i(w,y,x_n)+w_n x_n$$

This can be interpreted as

Total Cost (TC) = Total Variable Cost (TVC) + Total Fixed Cost (TFC)

Note that in the LR there are no fixed costs.

We will use the notations SRMC and SRAC to denote SR costs, and MC and AC to denote LR costs. Thus, we have SRAC = AVC + AFC.

8.5 Cost and Returns to Scale

Suppose the production function is homogeneous; thus, it exhibits either DRS, CRS or IRS. What can we say about the associated long run cost function? (Note that returns to scale is by definition a LR notion).

Theorem 2

If $f(x)$ is homogeneous of degree *k* then $c(w, y)$ is homogenous of degree $(1/k)$ in *y*.

Proof. Let $y = f(x)$. If $f(x)$ is homogeneous of degree *k* then

(8.1) $$f(tx) = t^k f(x) = t^k y$$

If $c(w, y)$ is the cost of *y*, then the cost of $f(tx)$ must be $tc(w, y)$. Thus, by (8.1)

(8.2) $$c(w, t^k y) = tc(w, y)$$

Define $\phi = t^k$. Then $t = \phi^{1/k}$. Making this substitution in (8.2) and reversing the equation yields

$$c(w, \phi y) = \phi^{1/k} c(w, y)$$

That is, $c(w, y)$ is homogeneous of degree $(1/k)$ in *y*. ♣

Example

Consider the Cobb-Douglas production function:

$$f(x) = x_1^{\alpha} x_2^{\beta}$$

Note that this is homogeneous of degree $(\alpha + \beta)$. The cost-minimization problem is

$$\min_x \quad w_1 x_1 + w_2 x_2 \quad \text{subject to} \quad y = x_1^{\alpha} x_2^{\beta}$$

The first-order conditions for a minimum are

$$w_1 = \lambda \alpha x_1^{\alpha-1} x_2^{\beta} \quad \text{and} \quad w_2 = \lambda \beta x_1^{\alpha} x_2^{\beta-1}$$

where λ is the Lagrange multiplier. Together with the production constraint, these conditions solve for the conditional input demands:

$$x_1(w, y) = y^{1/(\alpha+\beta)} \left(\frac{\alpha w_2}{\beta w_1} \right)^{\beta/(\alpha+\beta)}$$

$$x_2(w,y) = y^{1/(\alpha+\beta)}\left(\frac{\beta w_1}{\alpha w_2}\right)^{\alpha/(\alpha+\beta)}$$

Substitution back into the objective function then yields the cost function:

$$c(w,y) = y^{1/(\alpha+\beta)}\left(w_1\left(\frac{\alpha w_2}{\beta w_1}\right)^{\beta/(\alpha+\beta)} + w_2\left(\frac{\beta w_1}{\alpha w_2}\right)^{\alpha/(\alpha+\beta)}\right)$$

which we will write as

$$c(w,y) = y^{1/(\alpha+\beta)}\omega(w_1,w_2)$$

since our focus here is on the relationship between $c(w,y)$ and *y*. This cost function is homogeneous of degree $1/(\alpha+\beta)$ in *y*.♣

Theorem 3

If $f(x)$ is homogeneous of degree

(a) $k<1$ (that is, it exhibits DRS) then $c(w,y)$ is strictly convex in *y*;

(b) $k=1$ (that is, it exhibits CRS) then $c(w,y)$ is linear in *y*; and

(c) $k>1$ (that is, it exhibits IRS) then $c(w,y)$ is strictly concave in *y*.

Proof. If $f(x)$ is homogeneous of degree *k* then by Theorem 2 and Euler's theorem, $\partial c(w,y)/\partial y$ is homogenous of degree $(1/k)-1$ in *y*. That is,

$$\partial c(w,ty)/\partial y = t^{(1/k)-1}\partial c(w,y)/\partial y \quad \forall t>1$$

which we can write as

$$\frac{\partial c(w,ty)/\partial y}{\partial c(w,y)/\partial y} = t^{(1/k)-1} \quad \forall t>1$$

Thus, $\partial c(w,y)/\partial y$ is positively sloped if $k<1$ (in which case $t^{(1/k)-1}>1$ for $t>1$), zero-sloped if $k=1$ (in which case $t^{(1/k)-1}=1$ for $t>1$), and negatively sloped if $k>1$ (in which case $t^{(1/k)-1}<1$ for $t>1$). See Figure 8.2 for the case of $k<1$. ♣

Corollary

If $f(x)$ exhibits

(a) DRS then MC and AC are increasing in y, and AC < MC. See Figure 8.3.[1]

(b) CRS then MC and AC are constant, and MC = AC. See Figure 8.4.

(c) IRS then MC and AC are decreasing in y, and AC > MC. See Figure 8.5.

An alternative way to see these properties of AC is to note that if $c(w, y)$ is homogenous of degree $(1/k)$ in y, then

$$\frac{c(w,ty)}{ty} = \frac{t^{1/k}c(w,y)}{ty} = \frac{t^{(1/k)-1}c(w,y)}{y}$$

This it, AC is homogeneous of degree $(1/k)-1$ in y. Thus, if $k<1$ then $t^{(1/k)-1}>1$ for $t>1$, and so $AC(ty)>AC(y)$ for $t>1$. That is, AC is rising. Conversely, if $k>1$ then $t^{(1/k)-1}<1$ for $t>1$, and so $AC(ty)<AC(y)$ for $t>1$. That is, AC is falling.

Note that a production function need not be homogeneous; that is, it may not exhibit CRS, IRS or DRS. In practice, AC is often U-shaped, as illustrated in Figure 8.6. Why? There are no fixed costs in the LR so this cannot explain it. However, there may be quasi-fixed costs. A *quasi-fixed cost* is a non-variable cost that must be incurred if and only if there is a positive level of output; that is, it is independent of output for $y>0$ but is zero at $y=0$. An important example: a learning cost associated with the first unit of production.

[1] Figures 8.3 – 8.5 depict the AC and MC curves for the Cobb-Douglas production function.

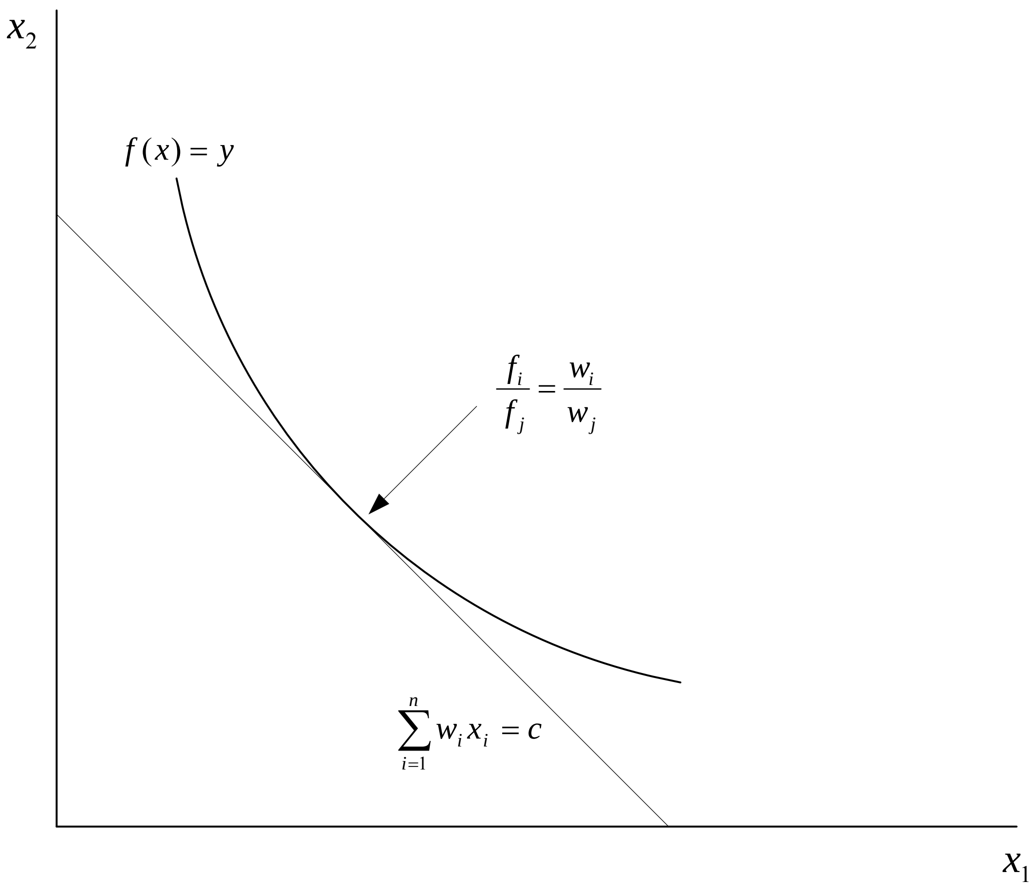


FIGURE 8.1

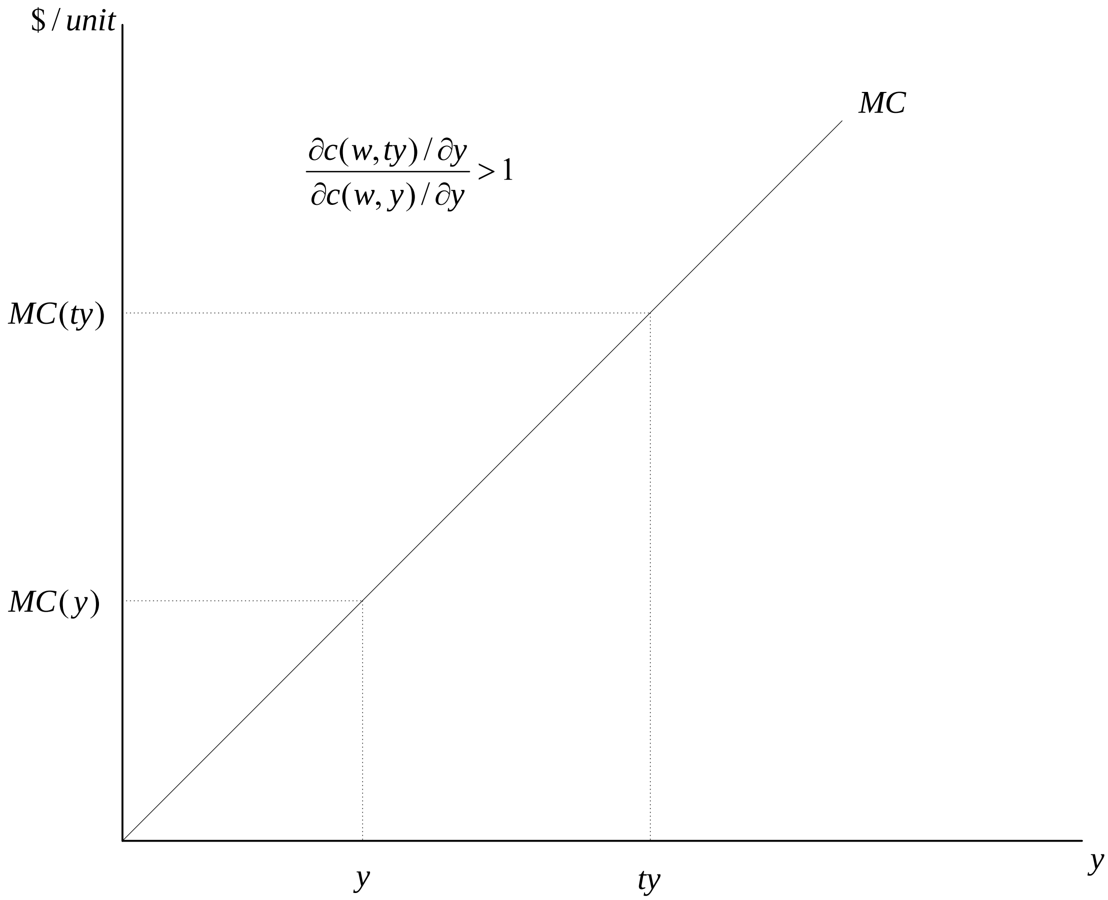

$\$/unit$
MC
$\frac{\partial c(w,ty)/\partial y}{\partial c(w,y)/\partial y} > 1$
$MC(ty)$
$MC(y)$
y
ty
y

FIGURE 8.2

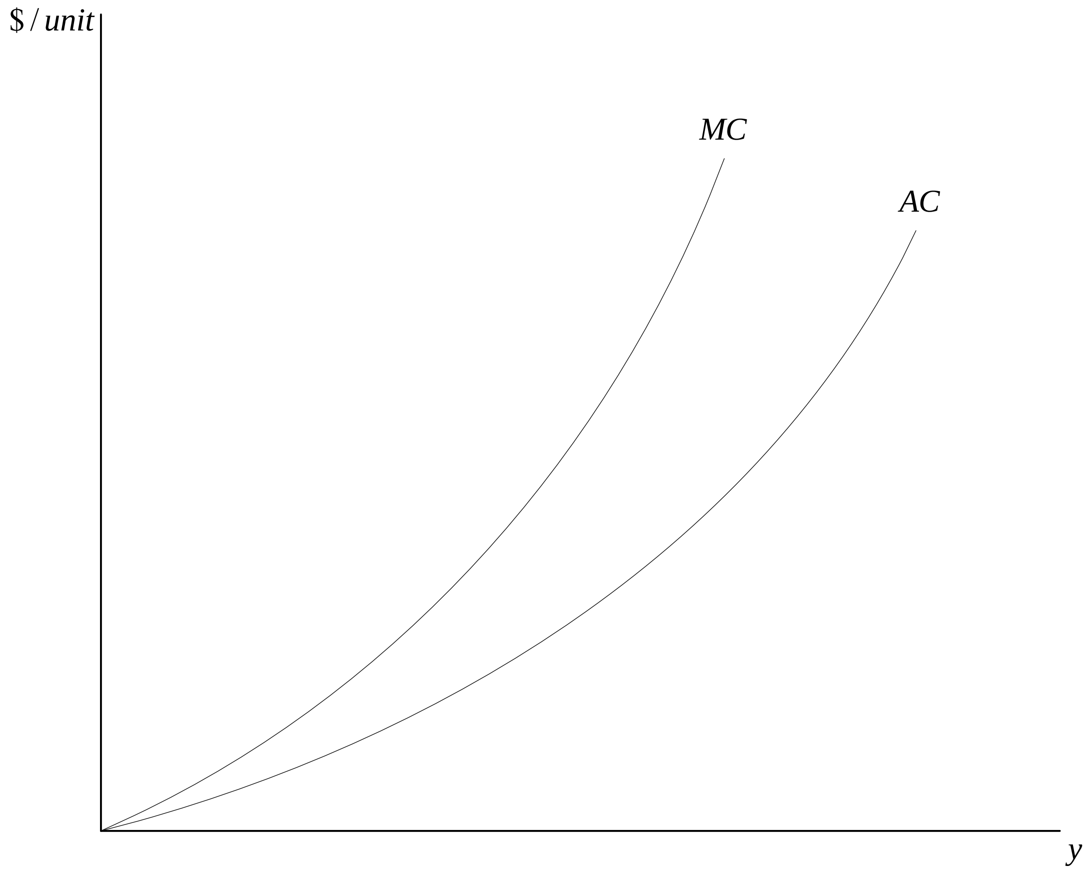


FIGURE 8.3

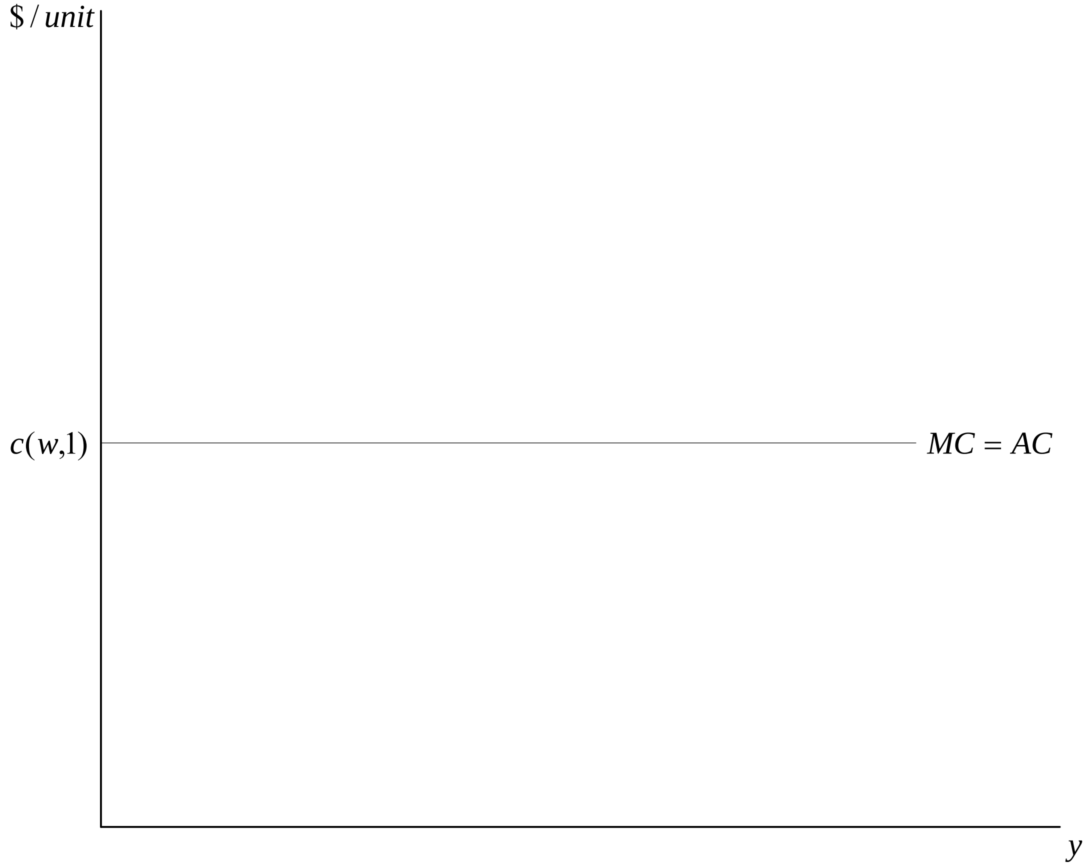


FIGURE 8.4

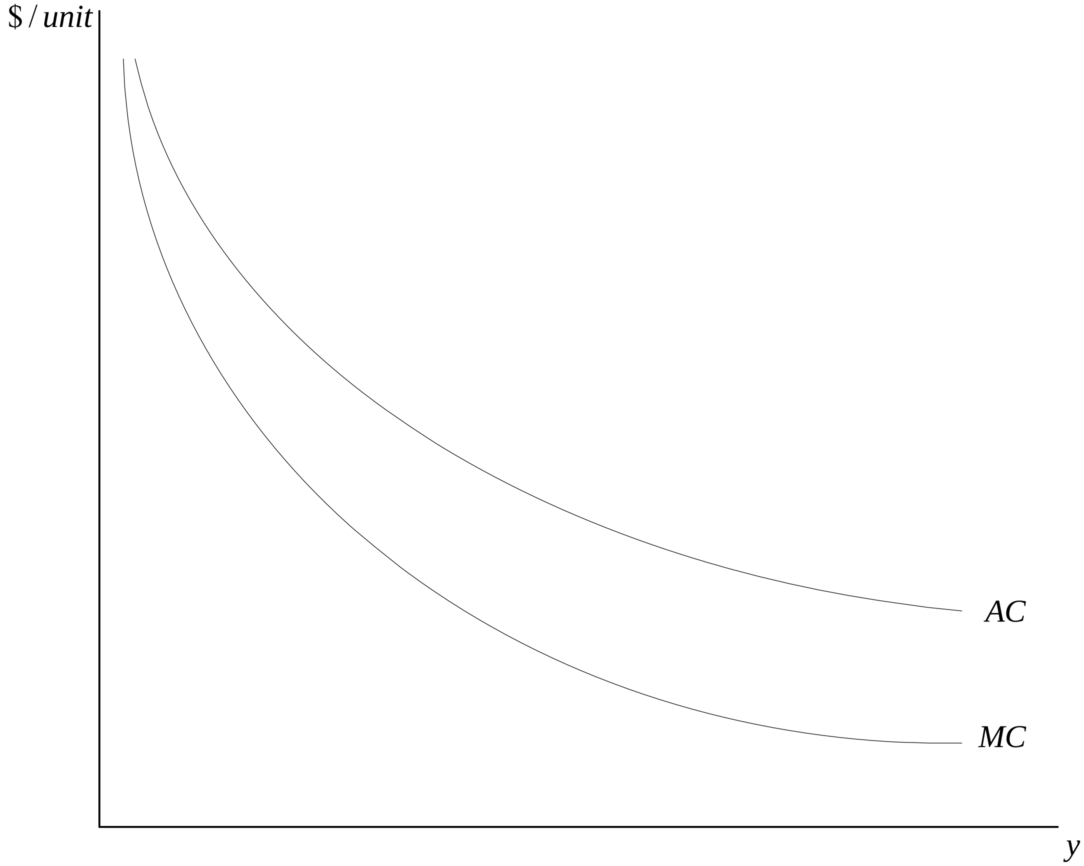


FIGURE 8.5

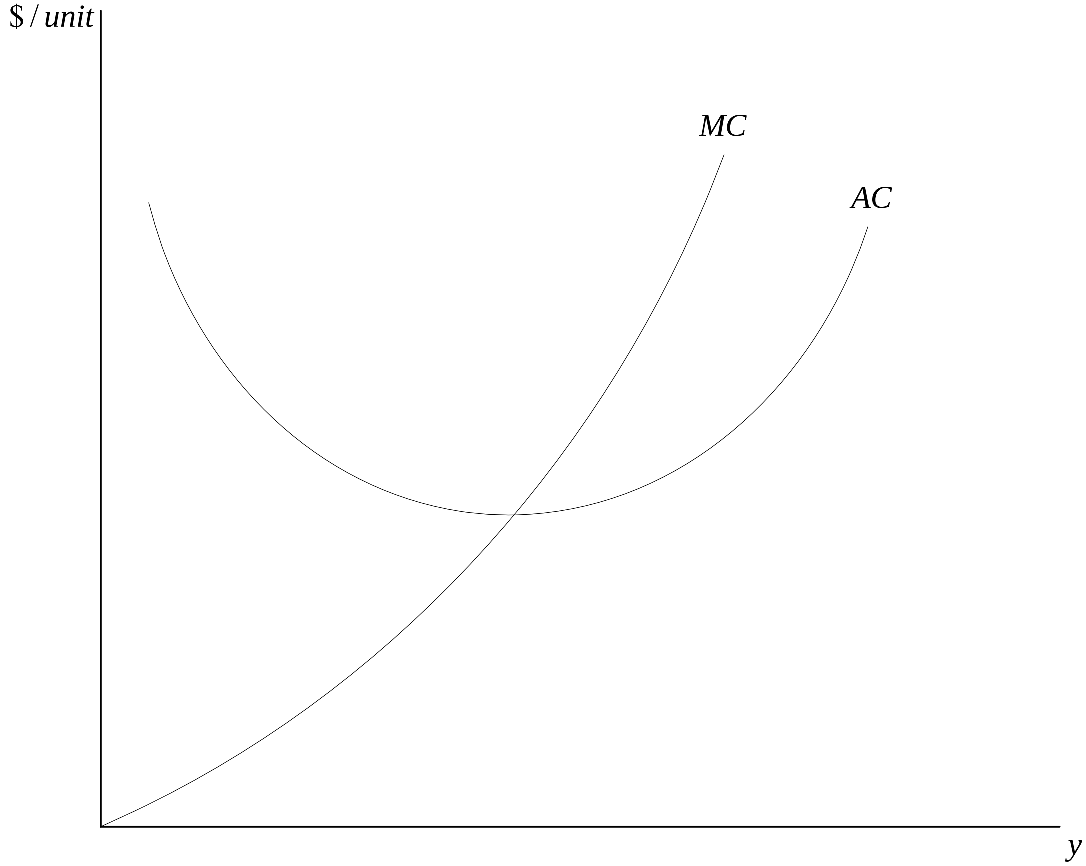


FIGURE 8.6

9. PROFIT MAXIMIZATION AND THE COMPETITIVE FIRM

9.1 The "Competitive" Firm

The "competitive" firm is a price-taker in its input markets and in its product market; that is, its own output choice has no impact on market prices because it is so small relative to the market.[1] Behavioral assumption: the firm acts to maximize profit.

There are two equivalent ways to derive the firm's input and output choices:

(1) direct profit maximization to obtain (unconditional) input demands and output;

(2) a two stage procedure in which we first derive the cost function (recall Topic 8) and then use the cost function in the profit maximization problem to derive supply.

The two approaches are equivalent. The second is somewhat more general because it allows us to examine profit maximization under a variety of product markets structures using the same cost function.

We begin with the first approach, and then follow the second approach in section 9.5.

9.2 Input Demands and the Supply Function

The direct profit maximization problem is

$$\max_x \quad pf(x) - wx$$

The first-order conditions are

$$pf_i = w_i \qquad \forall i$$

Interpretation: the value marginal product (VMP) of each input is equated to its price.

Solution of the first-order conditions yields the *input demands* (or factor demands):

$$x_i(p, w) \text{ for } i = 1, \ldots, n$$

Substitution of the input demands into the production function yields the *supply function*:

$$y(p,w) = f(x(p,w))$$

<u>Example</u>: Cobb-Douglas

$$f(x) = x_1^a x_2^b$$

The profit maximization problem is

$$\max_x \quad p x_1^a x_2^b - w_1 x_1 - w_2 x_2$$

with first-order conditions

(9.1) $$a p x_1^{a-1} x_2^b = w_1$$

(9.2) $$b p x_1^a x_2^{b-1} = w_2$$

Take the ratio of (9.1) and (9.2) to obtain

(9.3) $$\frac{a x_2}{b x_1} = \frac{w_1}{w_2}$$

Rearrange (9.3) to obtain:

(9.4) $$x_2 = \frac{b w_1 x_1}{a w_2}$$

Substitute (9.4) into (9.2) and rearrange to obtain the input demand for x_1:

(9.5) $$x_1(p,w) = \left(\frac{bp}{w_2}\right)^{\frac{1}{1-a-b}} \left(\frac{a w_2}{b w_1}\right)^{\frac{1-b}{1-a-b}}$$

Then substitute (9.5) into (9.4) to obtain $x_2(p,w)$. Then construct the supply function:

$$y(p,w) = x_1(p,w)^a \, x_2(p,w)^b$$

[1] We will see in Chapter 13 (Section 13.6) that price-taking behaviour can be viewed as a limiting case of an oligopoly where the number of firms approaches infinity.

Second-Order Conditions

The first-order conditions for profit maximization are necessary and sufficient for a global interior maximum if $pf(x) - wx$ is concave in *x*. This requires that $f(x)$ be concave. Uniqueness requires strict concavity.

Example: Cobb-Douglas

$$H = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

$$= \begin{bmatrix} a(a-1)x_1^{a-2}x_2^b & abx_1^{a-1}x_2^{b-1} \\ abx_1^{a-1}x_2^{b-1} & b(b-1)x_1^a x_2^{b-2} \end{bmatrix}$$

Strict concavity requires (i) $f_{11} < 0$, and (ii) $f_{11}f_{22} - f_{21}f_{12} > 0$. That is,

(i) $f_{11} < 0$: $a < 1$

(ii) $f_{11}f_{22} - f_{21}f_{12} = ab(a-1)(b-1)x_1^{2(a-1)}x_2^{2(b-1)} - a^2b^2x_1^{2(a-1)}x_2^{2(b-1)} > 0$: $a + b < 1$

Thus, we require diminishing marginal products for each factor, and DRS.

9.3 The Profit Function

The maximum value function for the profit-maximization problem is the *profit function*:

$$\pi(p,w) = py(p,w) - wx(p,w)$$

Properties of the Profit Function

1. non-decreasing in *p*; non-increasing in *w*
2. homogeneous of degree 1 in *p* and *w*
3. convex in *p* and *w*

These properties do not rely on concavity of $f(x)$; they follow from the definition of $\pi(p,w)$ as a maximum value function.

Proof of Property 3

Define

$$p'' = tp + (1-t)p'$$

$$w'' = tw + (1-t)w'$$

Then

$$\pi(p'',w'') = p''y(p'',w'') - w''x(p'',w'')$$
$$= [tp + (1-t)p']y(p'',w'') - [tw + (1-t)w']x(p'',w'')$$
$$= t[py(p'',w'') - wx(p'',w'')] + (1-t)[p'y(p'',w'') - w'x(p'',w'')]$$

By definition of $\pi(p'',w'')$ as a maximum value function.

$$py(p'',w'') - wx(p'',w'') \leq \pi(p,w)$$

$$p'y(p'',w'') - w'x(p'',w'') \leq \pi(p',w')$$

Thus,

$$\pi(p'',w'') \leq t\pi(p,w) + (1-t)\pi(p',w')$$

That is, $\pi(p,w)$ is convex in *p* and *w*.♣

Intuition: if the firm reacted to a change in *w* or a change in *p* by keeping inputs and output unchanged, then profit would be linear in *w* and *p*. However, the firm can generally do better than that by adjusting input and output optimally in response to price changes.

Hotelling's Lemma

(i) $$\frac{\partial \pi(p,w)}{\partial p} = y(p,w)$$

(ii) $$\frac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w)$$

if these derivatives exist.

Proof. These are just applications of the envelope theorem. Alternatively, they can be proved by brute force. Consider (ii):

$$\pi(p,w) = py(p,w) - w_i x_i(p,w) - \sum_{j \neq i} w_i x_j(p,w)$$

Therefore

$$\frac{\partial \pi(p,w)}{\partial w_i} = p\left(\sum_{j=1}^{n} \frac{\partial f}{\partial x_j} \cdot \frac{\partial x_j}{\partial w_i}\right) - x_i(p,w) - \sum_{j=1}^{n} w_j \frac{\partial x_j}{\partial w_i}$$

$$= \left(\sum_{j=1}^{n}\left(p\frac{\partial f}{\partial x_j} - w_j\right)\frac{\partial x_j}{\partial w_i}\right) - x_i(p,w)$$

But at the optimum:

$$p\frac{\partial f}{\partial x_j} - w_j = 0 \quad \forall j$$

Therefore

$$\frac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w) \clubsuit$$

9.4 Properties of the Supply Function and the Input Demands

1. Monotonicity

(i) $\frac{\partial y(p,w)}{\partial p} \geq 0$

(ii) $\frac{\partial x_i(p,w)}{\partial w_i} \leq 0$

Proof. By Hotelling's lemma:

$$\frac{\partial y(p,w)}{\partial p} = \frac{\partial^2 \pi(p,w)}{\partial p^2}$$

This is non-negative by convexity of $\pi(p,w)$. Similarly, by Hotelling's lemma:

$$\frac{\partial x_i(p,w)}{\partial w_i} = \frac{-\partial^2 \pi(p,w)}{\partial w_i^2}$$

This is non-positive, also by convexity of $\pi(p,w)$.♣

2. <u>Homogeneity</u>

$y(p,w)$ and $x(p,w)$ are homogeneous of degree zero in *p* and in *w*.

Proof. Recall that $\pi(p,w)$ is homogeneous of degree 1 in *p* and *w*. By Hotelling's lemma

$$y(p,w) = \frac{\partial \pi(p,w)}{\partial p}$$

Then by Euler's theorem, $y(p,w)$ is homogeneous of degree zero in *p* and *w*. Similarly for $x(p,w)$.♣

3. <u>Symmetry</u>

$$\frac{\partial x_i(p,w)}{\partial w_j} = \frac{\partial x_j(p,w)}{\partial w_i}$$

Proof. By Hotelling's lemma

$$\frac{\partial x_i(p,w)}{\partial w_j} = \frac{\partial^2 \pi(p,w)}{\partial w_i \partial w_j}$$

The result then follows from Young's theorem.♣

<u>Relationship to the Production Function</u>

A sufficient condition for existence of the derivatives in Hotelling's lemma is concavity of the production function. Strict concavity of $f(x)$ implies strict convexity of $\pi(p,w)$, which in turn implies:

$$\frac{\partial y(p,w)}{\partial p} > 0$$

$$\frac{\partial x_i(p,w)}{\partial w_i} < 0$$

9.5 Profit Maximization Using the Cost Function

The profit-maximization problem:

$$\max_{y} \quad py - c(w, y)$$

The first-order condition is

$$\frac{\partial c(w, y)}{\partial y} = p$$

That is, $MC = p$. This condition solves for the supply function, $y(p, w)$.

The second-order condition is

$$\frac{\partial^2 c(w, y)}{\partial y^2} > 0$$

That is, the cost function is strictly convex in *y*. This requires that $f(x)$ is strictly concave. If the production function is homogenous, then concavity requires DRS.

Under CRS there is a continuum of profit maximizing outputs – the argmax is not unique – and under IRS there is no finite argmax.

Important Identities

1. $x(w, y(p, w)) \equiv x(p, w)$

2. $pf\left(x(w, y(p, w))\right) - c(w, y(p, w)) \equiv \pi(p, w)$

Fixed Costs and Supply

Suppose the cost function is

$$c(w, y) = c_v(w, y) + F$$

where $c_v(w, y)$ is total variable cost, and *F* is fixed cost.[2] The firm will produce $y > 0$ if and only if

[2] Note that a fixed cost is incurred in the SR even if $y = 0$. In contrast, a *quasi-fixed cost* is independent of output for $y > 0$ but is zero at $y = 0$.

$$py(p) - c_v(w, y(p)) - F \geq -F$$

since profit is $-F$ at $y = 0$. Thus, production at $y > 0$ is worthwhile if and only if

$$p \geq \frac{c_v(w, y(p))}{y(p)}$$

That is, price must be no less than AVC. Thus, the SR supply curve for the competitive firm is the SRMC curve above AVC.

In the LR there are no fixed costs, and the firm will continue to supply $y > 0$ if $p \geq AC$. Thus, the LR supply curve is the MC above AC.

See Figure 9.1 for a summary of the key relationships in the theory of competitive firm.

See the Appendix to this Chapter for a summary of results on the relationship between the production function, the cost function and the supply curve for a competitive firm.

APPENDIX A9: A SUMMARY OF RESULTS ON PRODUCTION, COST, AND COMPETITIVE SUPPLY

A9.1 Cost and Supply for Homogeneous Production Functions

Long Run Cost

- A homogeneous production function exhibits either CRS, IRS or DRS.
- MC and AC are both either constant (CRS), decreasing (IRS) or increasing (DRS).
- Neither MC nor AC can be U-shaped in the LR if the production function is homogeneous.

Long Run Supply

- With IRS, there is no competitive supply curve; competitive profit-maximization at $MC = p$ yields negative profit, at any price.
- With CRS, the supply curve is not uniquely defined; $MC = p$ does not yield a unique solution for output.
- With DRS, the LR supply curve is the entire MC curve, since MC lies above AC for all $y > 0$.

Short Run Cost

Suppose at least one factor is fixed; then there exists a fixed cost (in the SR). In this case:

- SRAC can be U-shaped under CRS and IRS, and must be U-shaped under DRS.
- If the SRAC is U-shaped, it is due to the combination of an increasing AVC and decreasing AFC; AVC itself cannot be U-shaped.
- In all cases, SRMC and AVC are both monotonic

Short Run Supply

- With CRS or DRS, SRMC is positively sloped, and the SR supply curve is the entire SRMC curve, since SRMC lies above AVC for all $y > 0$.
- With IRS, SRMC is positively sloped if and only if the returns to scale are not too strong, and in that case the SR supply curve is the entire SRMC curve.

- With strong IRS, SRMC is constant or negatively sloped, and the SR supply curve is either non-unique or undefined, respectively.

<u>Example</u>

Consider the Cobb-Douglas production function:

$$f(x) = x_1^a x_2^b$$

where $a > 0$ and $b > 0$. This function is homogeneous of degree $a + b$; it exhibits DRS if $a + b < 1$, CRS if $a + b = 1$ and IRS if $a + b > 1$. Recall from Section 8.5 in Chapter 8 that the associated LR cost function is

$$c(w, y) = y^{1/(a+b)} \omega(w_1, w_2)$$

LR marginal cost (MC) is

$$\frac{\partial c(w, y)}{\partial y} = y^{(1-a-b)/(a+b)} \frac{\omega}{a+b}$$

The slope of this MC is

$$\frac{\partial^2 c(w, y)}{\partial y^2} = y^{(1-2a-2b)/(a+b)} \frac{\omega(1-a-b)}{(a+b)^2}$$

Thus, MC is positively sloped iff $a + b < 1$ (DRS), constant iff $a + b = 1$ (CRS), and negatively sloped iff $a + b > 1$ (IRS).

LR average cost (AC) is

$$\frac{c(w, y)}{y} = y^{(1-a-b)/(a+b)} \omega$$

The slope of this AC is

$$\frac{\partial \left(\dfrac{c(w, y)}{y} \right)}{\partial y} = y^{(1-2a-2b)/(a+b)} \frac{\omega(1-a-b)}{(a+b)}$$

Thus, AC is positively sloped iff $a + b < 1$ (DRS), constant iff $a + b = 1$ (CRS), and negatively sloped iff $a + b > 1$ (IRS).

Two additional points are noteworthy:

(1) $$\frac{AC}{MC} = a + b$$

Thus $AC < MC$ iff $a + b < 1$ (DRS); $AC = MC$ iff $a + b = 1$ (CRS); and $AC > MC$ iff $a + b > 1$ (IRS). Recall Figures 8.3 – 8.5 from Chapter 8.

(2) $$\frac{\text{slope of } AC}{\text{slope of } MC} = a + b$$

Thus, *AC* is flatter than *MC* iff $a + b < 1$ (DRS); *AC* and *MC* have the same slope iff $a + b = 1$ (CRS); and *AC* is steeper than *MC* iff $a + b > 1$ (IRS). Recall Figures 8.3 – 8.5 from Chapter 8.

Now consider the SR cost functions. Suppose x_2 is fixed at $\bar{x}_2$. Then the cost minimization problem is trivial; there is only one choice of x_1 that can yield output y when x_2 is fixed:

$$x_1(w, y, \bar{x}_2) = \left(\frac{y}{\bar{x}_2^b} \right)^{1/a}$$

Then the SR cost function is

$$c(w, y, \bar{x}_2) = \left(\frac{w_1}{\bar{x}_2^{b/a}} \right) y^{1/a} + w_2 \bar{x}_2$$

For notational convenience, write this as

$$c(w, y, \bar{x}_2) = \varphi y^{1/a} + F$$

where $\varphi y^{1/a}$ is total variable cost (TVC) and *F* is total fixed cost (TFC).

SR marginal cost (SRMC) is

$$\frac{\partial c(w, y, \bar{x}_2)}{\partial y} = \frac{\varphi}{a} y^{(1-a)/a}$$

The slope of SRMC is

$$\frac{\partial^2 c(w, y, \bar{x}_2)}{\partial y^2} = \frac{\varphi(1-a)}{a^2} y^{(1-2a)/a}$$

This slope is positive iff $a < 1$. This condition must hold if $f(x)$ exhibits DRS or CRS, since in that case $a + b \leq 1$ (and recall that $b > 0$). Note also that this slope *can* be negative under IRS since it is possible that $a < 1$ even when $a + b > 1$.

SR average cost (SRAC) is

$$\frac{c(w, y, \bar{x}_2)}{y} = \varphi y^{(1-a)/a} + \frac{F}{y}$$

where the first component is average variable cost (AVC) and the second component is average fixed cost (AFC).

The slope of SRAC is

$$\frac{\partial\left(\frac{c(w, y, \bar{x}_2)}{y}\right)}{\partial y} = \frac{\varphi(1-a)}{a} y^{(1-2a)/a} - \frac{F}{y^2}$$

Note that the first term is monotonic in *y*. Thus, AVC cannot be U-shaped; it is increasing if $a < 1$, constant if $a = 1$, and decreasing if $a > 1$. However, the presence of the fixed cost means that SRAC can be U-shaped. In particular, SRAC is U-shaped iff $a < 1$. This must be true under DRS and CRS, and can be true even under IRS.

Note also that

$$\frac{SRMC}{AVC} = \frac{(1-a)}{a^2 y}$$

This is greater than one for $a < 1$. Thus, $SRMC > AVC$ in that case. This means that the SR supply curve is the entire SRMC when $a < 1$, since it lies everywhere above AVC. That is, the firm will supply a positive amount (in the SR) at any positive price even it makes a loss, since revenue will exceed variable costs.

A9.2 Cost and Supply for Non-Homogeneous Production Functions: An Example

In this case it is possible to have U-shaped MC and AC, even in the LR. However, not all non-homogeneous production functions will yield U-shaped cost curves, either in the SR or the LR; depends on the specific nature of the function.

We will examine a modified form of the Cobb-Douglas production function:

$$f(x) = \delta x_1 x_2$$

where δ is a productivity factor.

If δ is a constant then this is a regular C-D production function, exhibiting IRS (since $a = 1$ and $b = 1$). If δ is not a constant then the production function is non-homogenous.

Suppose the productivity of all factors declines as the scale of production rises, reflecting **limited managerial attention** (LMA). In particular, suppose

$$\delta = \frac{1}{1 + x_1 x_2}$$

In this case, the production function becomes

$$f(x) = \frac{x_1 x_2}{1 + x_1 x_2}$$

This function is non-homogeneous.

<u>Long Run Cost in the LMA Example</u>

- Both MC and AC are U-shaped because productivity eventually must decline as managerial attention is spread too thin.

<u>Long Run Supply in the LMA Example</u>

- The LR competitive supply curve in the case of U-shaped AC, is the MC curve above the AC curve; the firm will not produce if $p < AC$.

Short Run Cost in the LMA Example

- In the SR there are fixed costs. Recall that fixed costs can cause SRAC to be U-shaped even if AC is not U-shaped in the LR. In the LMA example, SRAC must be U-shaped.
- Even though MC and AC are U-shaped, SRMC and AVC are not U-shaped.
- Simple textbook presentations often claim erroneously that if MC is U-shaped then SRMC must also be U-shaped. The LMA example provides a counter-example.

Short Run Supply

- The SR competitive supply curve is the SRMC curve above AVC.

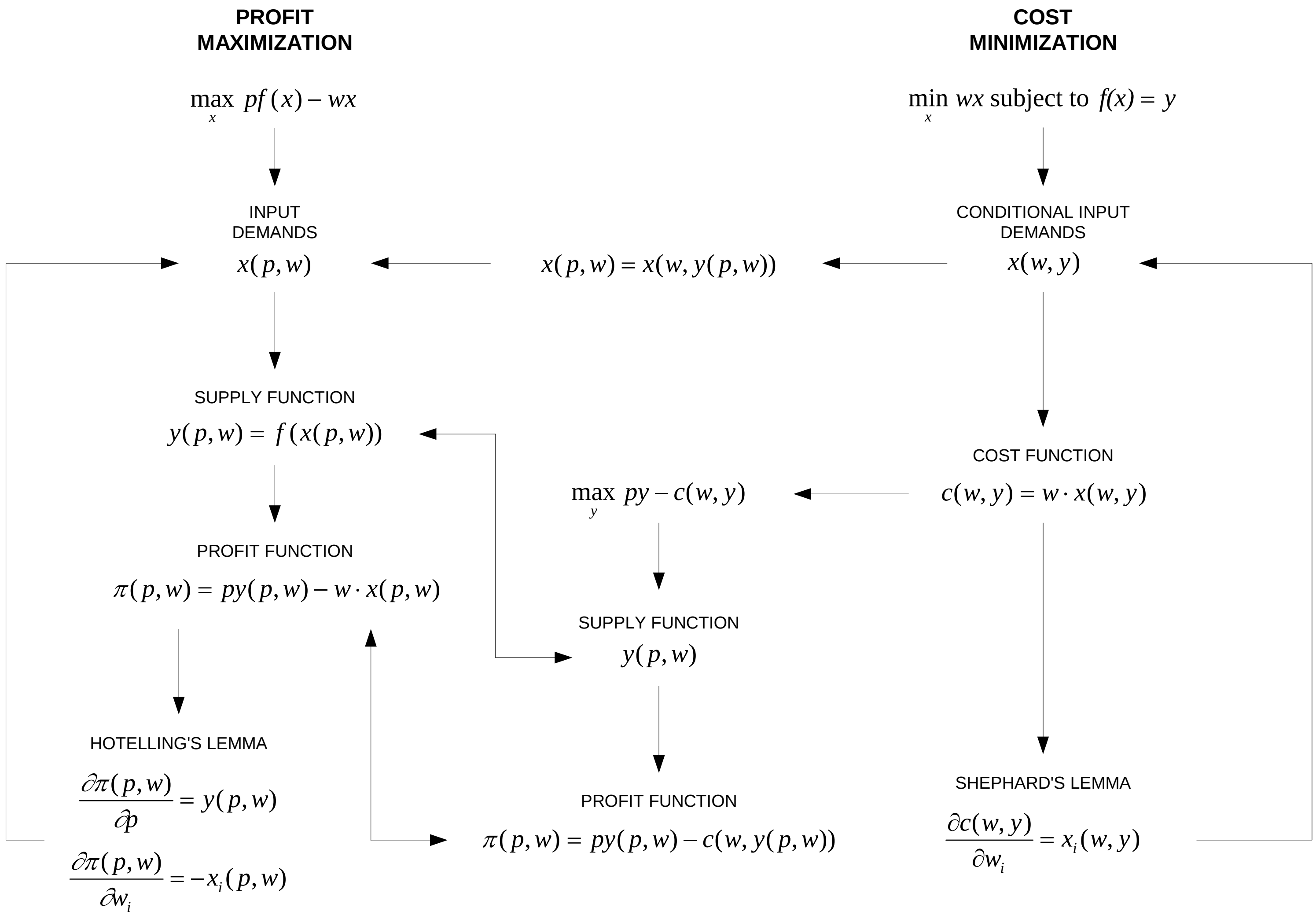

PROFIT MAXIMIZATION
COST MINIMIZATION
$\max_x \; pf(x) - wx$
$\min_x \; wx$ subject to $f(x) = y$
INPUT DEMANDS
$x(p,w)$
$x(p,w) = x(w, y(p,w))$
CONDITIONAL INPUT DEMANDS
$x(w,y)$
SUPPLY FUNCTION
$y(p,w) = f(x(p,w))$
COST FUNCTION
$c(w,y) = w \cdot x(w,y)$
$\max_y \; py - c(w,y)$
PROFIT FUNCTION
$\pi(p,w) = py(p,w) - w \cdot x(p,w)$
SUPPLY FUNCTION
$y(p,w)$
HOTELLING'S LEMMA
$\frac{\partial \pi(p,w)}{\partial p} = y(p,w)$
$\frac{\partial \pi(p,w)}{\partial w_i} = -x_i(p,w)$
PROFIT FUNCTION
$\pi(p,w) = py(p,w) - c(w, y(p,w))$
SHEPHARD'S LEMMA
$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y)$

FIGURE 9.1

10. COMPETITIVE MARKETS AND SOCIAL SURPLUS

10.1 Industry Supply

The *industry supply function* is the sum of the individual firm supply functions:

$$Y(p) \equiv \sum_{i=1}^{n} y_i(p)$$

where the dependency of y_i on *w* is suppressed here for notational simplicity.

<u>Examples</u>

1. Two firms with cost functions $c_1(y) = y^2$ and $c_2(y) = 2y^2$. For firm 1:

$$\frac{\partial c_1(y)}{\partial y} = p \quad \Rightarrow \quad y_1(p) = \frac{p}{2}$$

For firm 2:

$$\frac{\partial c_2(y)}{\partial y} = p \quad \Rightarrow \quad y_2(p) = \frac{p}{4}$$

Industry supply:

$$Y(p) = \frac{p}{2} + \frac{p}{4} = \frac{3p}{4}$$

2. *n* firms each with cost function $c(y) = y^2 + 1$ (so AC is U-shaped). For each firm,

$$\frac{\partial c(y)}{\partial y} = p \quad \Rightarrow \quad \begin{cases} y(p) = \dfrac{p}{2} & \text{if } p \geq 2 \\ y(p) = 0 & \text{if } p < 2 \end{cases}$$

Industry supply:

$$Y(p) = \frac{np}{2} \quad \text{if} \quad p \geq 2$$

$$Y(p) = 0 \quad \text{if} \quad p < 2$$

10.2 Market Equilibrium in the Short Run

Equilibrium price equates supply and demand:

$$Y(p) \equiv \sum_{j=1}^{n} y_j(p) = \sum_{i=1}^{I} x_i(p) \equiv X(p)$$

<u>Example</u>

Suppose there are *n* identical firms each with cost function $c(y) = y^2$. Industry supply is

$$Y(p) = \frac{np}{2}$$

Suppose aggregate demand is linear:

$$X(p) = a - bp$$

In equilibrium:

$$\frac{np}{2} = a - bp$$

Solving for *p*:

$$p^* = \frac{2a}{n + 2b}$$

<u>Equilibrium Price and the Number of Firms</u>

How does p^* vary with *n*? In general, equilibrium with identical firms requires

$$ny(p) = X(p)$$

Differentiate both sides with respect to *n*:

$$ny'(p)p'(n) + y(p) = X'(p)p'(n)$$

Collecting terms:

$$p'(n) = \frac{y(p)}{X'(p) - ny'(p)} < 0 \quad \text{for} \quad X'(p) < 0 \quad \text{and} \quad y'(p) > 0$$

10.3 Free Entry and Long Run Equilibrium

A competitive market is characterized by price-taking firms and *free-entry* in the LR. Firms enter to the point where profit is driven to zero for the marginal firm; that is, $AC = p$.

Note that if firms are heterogeneous in some respect then existing firms may possibly earn positive profit even though profit is zero for the marginal firm . These infra-marginal firms earn *economic rent*; this is either a rent to being first, or a rent to being a low cost producer.

<u>Example with U-Shaped Costs</u>

Suppose cost for any incumbent firm or potential entrant is:

$$c(y) = 2y^2 + 8$$

and demand is given by

$$X(p) = 403 - 10p$$

Profit maximization implies $MC = p$:

$$4y = p$$

and associated individual supply:

$$y(p) = \frac{p}{4}$$

Zero profit implies $AC = p$:

$$2y(p) + \frac{8}{y(p)} = p$$

Solving for p yields:

$$p^* = 8$$

and individual supply is therefore

$$y(p^*) = \frac{8}{4} = 2$$

Equilibrium requires:

$$ny(p^*) = X(p^*)$$

or

$$2n = 403 - 80$$

Solving for *n*:

$$n^* = 161.5$$

But *n* must be an integer. Thus,

$$\hat{n} = 161$$

Therefore

$$\hat{Y}(p) = 161 \cdot y(p)$$

Equating supply with demand, we have

$$161\frac{\hat{p}}{4} = 403 - 10\hat{p}$$

Solving for $\hat{p}$ yields

$$\hat{p} = 8.02$$

and all firms earn positive profit.

<u>Cost in Equilibrium</u>

Recall from Section 8.4 in Chapter 8 that when AC is U-shaped, minimum AC occurs where $MC = AC$. Thus, ignoring integer problems, AC is minimized in CE when AC is U-shaped.

<u>Returns to Scale and LR Equilibrium</u>

(a) CRS: $MC = AC$

Price is determined by

$$p^* = AC$$

Industry output is determined by aggregate demand:

$$Y = X(p^*)$$

but individual firm output and the number of firms is indeterminate, since the profit-maximization problem does not have a unique solution.

(b) DRS: $MC > AC \quad \forall y$

Any $p > 0$ yields positive profit at $MC = p$. (See Figure 10.1). Thus, equilibrium occurs at minimum feasible production for each firm.

(c) IRS: $MC < AC \quad \forall y$

In this case, profit maximization ($MC = p$) yields negative profit at any price. Thus, a competitive equilibrium does not exist under IRS. The unregulated market structure will be characterized by a small number of firms, and these firms will not act as a price-takers.

Under IRS, production at the lowest possible AC requires a monopoly, where one firm supplies the entire market. This is *natural monopoly*. See Figure 10.2. Such a firm will tend to exploit its monopoly power (see Chapter 12), and hence, natural monopoly is usually regulated.

10.4 Competitive Equilibrium and Social Surplus: Partial Equilibrium Analysis

Consider a representative consumer with quasi-linear preferences:

$$u(x) = h(x_1) + x_2$$

where $h(x_1)$ is some function with $h'(x_1) > 0$. Let $p_1 = p$ and $p_2 = 1$; that is, x_2 is the numeraire good. Utility maximization:

$$L = h(x_1) + x_2 + \lambda\left(m - px_1 - x_2\right)$$

The interior solution yields demand curves

$$x_1(p) \quad \text{such that} \quad h'(x_1) = p$$

$$x_2(p,m) = m - px_1(p)$$

Focus on equilibrium in the market for x_1. Consider a representative firm with cost function $c(y)$ in the production of x_1. Competitive supply is given by

$$y(p) \text{ such that } c'(y) = p$$

Thus, in competitive equilibrium (CE):

$$c'(x_1) = h'(x_1)$$

Now compare this CE with the solution to the following centralized planning problem:

$$\max_{x_1, x_2} \quad h(x_1) + x_2 \text{ subject to } c(x_1) + x_2 = e$$

where *e* is the economy's endowment of the numeraire good, which can be consumed directly as x_2 or transformed into x_1 according to the cost function $c(x_1)$. The objective is to maximize social surplus subject to the production technology in the economy.

Substituting directly for the constraint, we have

$$\max_{x_1} \quad h(x_1) + e - c(x_1)$$

with first order condition

$$h'(x_1) = c'(x_1)$$

This states that marginal social benefit is equal to marginal social cost. Thus, the CE and the planning solution coincide; the CE maximizes social surplus.

The same result can be obtained via a somewhat different approach. Consider a single market, for good *x*. Inverse demand is given by $p(x)$. Suppose the market quantity and price are $x = q$ and $p = p(q)$ respectively. Then *consumer surplus* is given by

$$CS = \int_0^q p(x)dx - p(q)q$$

See Figure 10.3.

The competitive supply curve is the marginal cost curve: $c'(x)$. *Producer surplus* is given by

$$PS = p(q)q - \int_0^q c'(x)dx$$

See Figure 10.4.

Thus, social surplus is given by

$$SS \equiv CS + PS = \int_0^q p(x)dx - \int_0^q c'(x)dx$$

Now consider a planning problem in which *q* is chosen to maximize SS:

$$\max_q \quad \int_0^q p(x)dx - \int_0^q c'(x)dx$$

The first order condition (using Leibnitz's rule for differentiating an integral; see technical note below):

$$p(q) = c'(q)$$

That is, $P = MC$; social surplus is maximized at the CE.

Technical Note: Leibnitz's Rule

For the definite integral

$$V(r) = \int_{a(r)}^{b(r)} f(x,r)dx$$

The derivative with respect to *r* is

$$V'(r) = f\big(b(r),r\big)b'(r) - f\big(a(r),r\big)a'(r) + \int_{a(r)}^{b(r)} \big(\partial f(x,r)/\partial r\big)dx$$

Proof. Evaluation of the definite integral yields

$$V(r) = F(b(r),r) - F(a(r),r)$$

where $F(x)$ is the indefinite integral of $f(x)$. Differentiation with respect to *r* then yields the result.♣

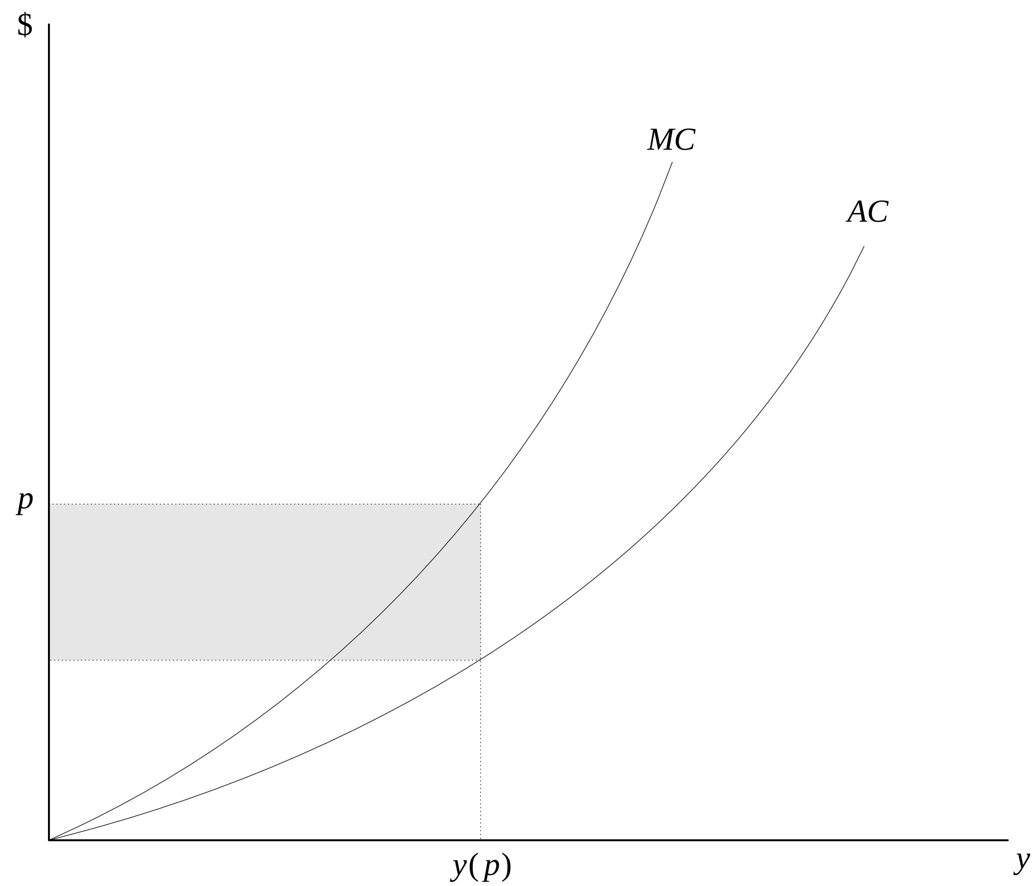


FIGURE 10.1

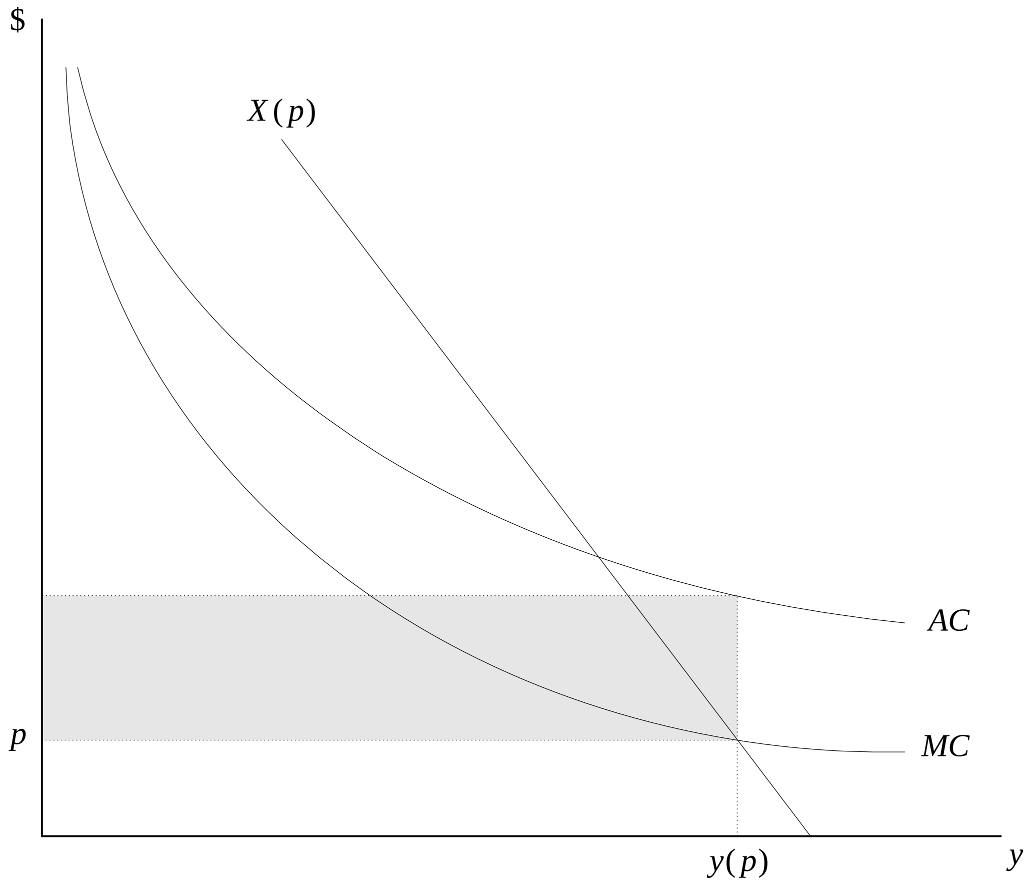


FIGURE 10.2

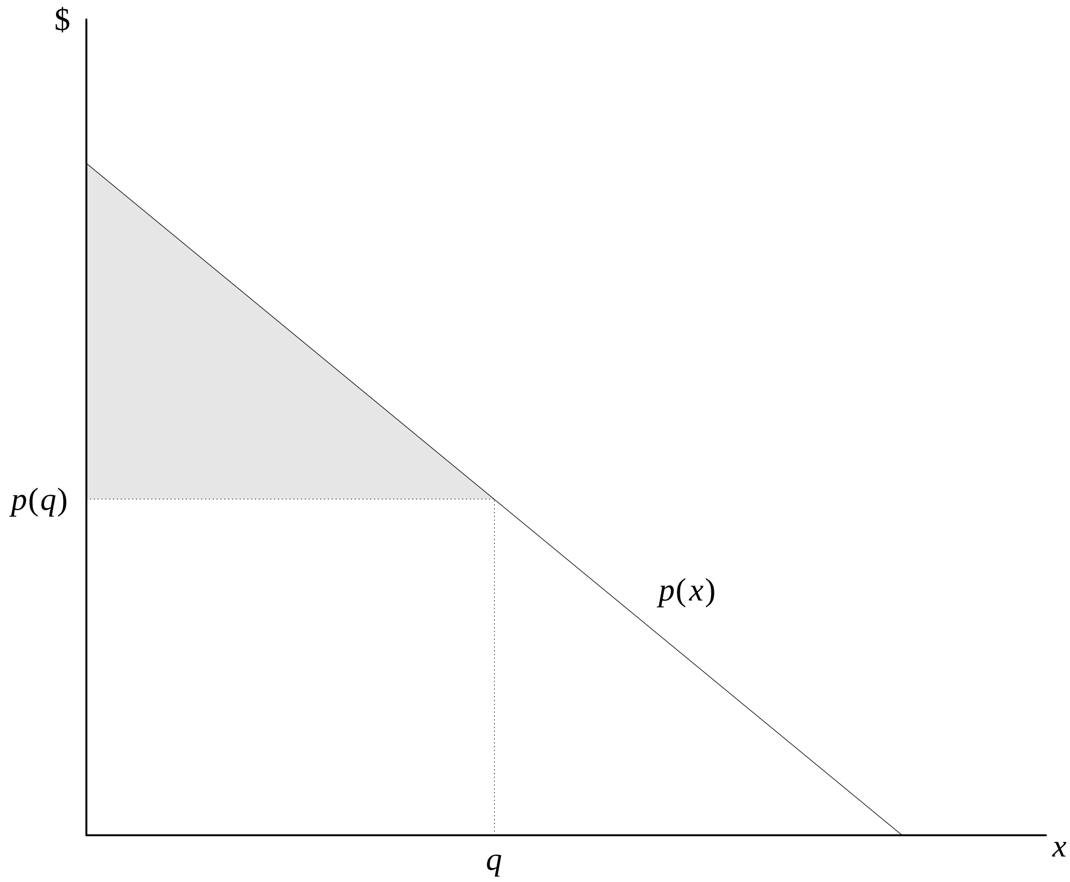


FIGURE 10.3

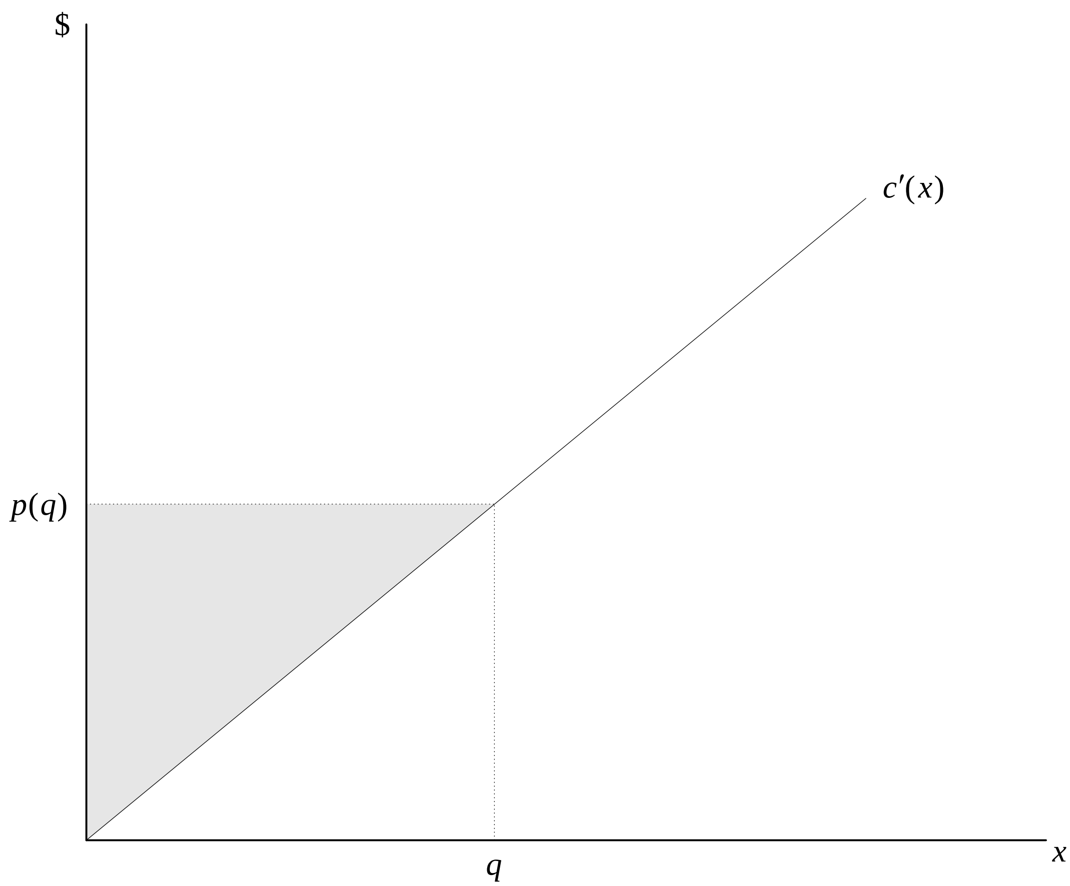


FIGURE 10.4

11. COMPETITIVE GENERAL EQUILIBRIUM IN A TWO-SECTOR ECONOMY

11.1 The Two-Sector Model

There are two factors of production (K and L), two consumption goods (X and Y), and h households (consumers of X and Y and suppliers of K and L). Resource constraints: $\bar{K}$ and $\bar{L}$ are fixed.

<u>Production</u>

$$X = F^x(K_x, L_x)$$

$$Y = F^y(K_y, L_y)$$

Assume CRS. This implies:

(i) zero profit in equilibrium (since $p = AC$ in equilibrium) and hence we do not need to be concerned with the ownership of firms among households and the distribution of dividends;

(ii) the number of firms is indeterminate.

Consider profit for a representative firm in sector i:

$$\pi_i = p_i F^i(K_i, L_i) - wL_i - rK_i$$

Profit maximization implies:

$$\frac{F_L^i}{F_K^i} = \frac{w}{r}$$

$$MC^i = p_i$$

from which we can derive factor demands in each sector,

$$K^i(p_i, w, r) \quad \text{and} \quad L^i(p_i, w, r)$$

and aggregate supply functions for each sector,

$$X^s(p_x, w, r) \quad \text{and} \quad Y^s(p_y, w, r)$$

Consumption

Household *j* is endowed with $\bar{L}^j$ and $\bar{K}^j$ such that

$$\sum_j \bar{L}^j = \bar{L} \quad \text{and} \quad \sum_j \bar{K}^j = \bar{K}$$

This means that wealth for household *j* is

$$M^j = w\bar{L}^j + r\bar{K}^j$$

Utility maximization for household *j*:

$$\max_{X^j, Y^j, l^j} \quad u^j(X^j, Y^j, l^j) \quad \text{subject to} \quad p_x X^j + p_y Y^j + wl^j = M^j$$

This yields the usual first-order conditions:

$$MRS^j_{x,y} = \frac{p_x}{p_y} \quad \text{and} \quad MRS^j_{l,y} = \frac{w}{p_y}$$

from which we can construct the Marshallian demand functions for *X* and *Y*,

$$X^j(p_x, p_y, w, r) \quad \text{and} \quad Y^j(p_x, p_y, w, r)$$

and labour supply,

$$L^j = \bar{L}^j - l^j(p_x, p_y, w, r)$$

From these we can construct aggregate demands for *X* and *Y* and aggregate labour supply:

$$X^D(p_x, p_y, w, r) = \sum_j X^j(p_x, p_y, w, r)$$

$$Y^D(p_x, p_y, w, r) = \sum_j Y^j(p_x, p_y, w, r)$$

$$L^S(p_x, p_y, w, r) = \sum_j L^j(p_x, p_y, w, r)$$

Market Equilibrium

In the goods market:

(11.1) $$X^S(p_x, w, r) = X^D(p_x, p_y, w, r)$$

(11.2) $$Y^S(p_y, w, r) = Y^D(p_x, p_y, w, r)$$

In the factor market:

(11.3) $$\sum_i K^i(p_i, w, r) = \overline{K}$$

(11.4) $$\sum_i L^i(p_i, w, r) = L^S(p_x, p_y w, r)$$

We have four equations and four unknowns (p_x, p_y, w and *r*). However, there are only three independent equations because the equations are linked by the household budget constraint. This linkage implies *Walras' law*:

Walras' Law

If there are *n* markets and (*n*-1) are in equilibrium then the n^{th} market must also be in equilibrium.

This means that our four equations can only be solved for three unknowns; one of the goods or factors must be specified as the numeraire. Convention: $p_y = 1$.

Economic interpretation: equilibrium identifies *relative prices* not absolute prices. More generally, all values in economics are relative; there are no absolute values.

11.2 Properties of Competitive Equilibrium

To simplify the presentation and to allow the use of some simple diagrams, we will henceforth abstract from the labour-leisure choice and set $l^j = 0 \ \ \forall j$. Thus, all available labour is used in production.

1. Efficiency in Production

A production allocation is efficient if it is not possible, by re-allocating available factors, to produce more of one good without producing less of another. This means the economy is on its *production possibility frontier* (PPF).

The PPF is the solution to the following planning problem:

$$\max_{K_x, L_x} \quad F^x(K_x, L_x)$$

$$\text{subject to} \begin{bmatrix} F^y(K_y, L_y) = Y \\ K_y + K_x = \overline{K} \\ L_x + L_y = \overline{L} \end{bmatrix}$$

Substitute the second and third constraints into the first, and construct the Lagrangean:

$$\Phi = F^x(K_x, L_x) + \lambda[Y - F^y(\overline{K} - K_x, \overline{L} - L_x)]$$

The first order conditions are

$$F_K^x + \lambda F_K^y = 0$$

$$F_L^x + \lambda F_L^y = 0$$

Taking the ratio yields the condition for efficiency in production:

$$\frac{F_K^x}{F_L^x} = \frac{F_K^y}{F_L^y}$$

Solution of the first order conditions, in combination with the constraint, yields the PPF. See Figure 11.1.

The PPF is often called a *transformation function* and is usually written in implicit form, $T(X, Y, \overline{K}, \overline{L}) = 0$. The slope of the PPF is called the *marginal rate of transformation* (MRT). The MRT can be thought of as measuring the marginal cost of producing *X* in terms of *Y*; that is, $MRT_{X,Y} = MC_X$.

Now consider the competitive equilibrium (CE). Recall that in CE,

$$\frac{F_L^i}{F_K^i} = \frac{w}{r} \qquad \forall i$$

Since *all firms face the same prices*, it follows that in CE

$$\frac{F_L^x}{F_K^x} = \frac{F_L^y}{F_K^y}$$

That is, the CE is production efficient.

2. <u>Efficiency in Consumption</u>

A consumption allocation is Pareto efficient if it is not possible, by re-allocating the available goods, to make one person better off without making someone else worse off. This means the economy is on its *utility possibility frontier* (UPF).

The UPF is the solution to the following planning problem (specified here for the case of two households, *A* and *B*):

$$\max_{X^A, Y^A} \quad u^A(X^A, Y^A)$$

$$\text{subject to} \begin{bmatrix} u^B(X^B, Y^B) = u^B \\ X^A + X^B = \overline{X} \\ Y^A + Y^B = \overline{Y} \end{bmatrix}$$

where $\{\overline{X}, \overline{Y}\}$ is some point on the PPF.

Substitute the second and third constraints into the first, and construct the Lagrangean:

$$\Phi = u^A(X^A, Y^A) + \lambda[u - u^B(\overline{X} - X^A, \overline{Y} - Y^A)]$$

The first order conditions are

$$u_x^A + \lambda u_x^B = 0$$

$$u_y^A + \lambda u_y^B = 0$$

Taking the ratio yields the condition for efficiency in consumption:

$$\boxed{MRS_{x,y}^A = MRS_{x,y}^B}$$

Solution of the first order conditions, in combination with the constraint, yields the UPF or *contract curve*, $u^A = U(\overline{X}, \overline{Y}, u^B)$. See Figure 11.2.

Now consider the competitive equilibrium (CE). Recall that in CE,

$$MRS_{x,y}^j = \frac{p_x}{p_y} \quad \forall j$$

Since *all consumers face the same prices*, it follows that in CE

$$MRS_{x,y}^A = MRS_{x,y}^B$$

That is, the CE is consumption efficient.

3. Overall Efficiency

Efficiency in production requires that the economy is on the PPF. Efficiency in consumption requires that the goods produced are allocated in a Pareto efficient manner. The final element of efficiency relates to *where* on the PPF the economy should be.

An allocation is Pareto efficient if it is not possible, by re-allocating the factors of production, to make one person better without making someone else worse off. This means the economy is on the *grand utility possibility frontier* (GUPF).

The GUPF is the solution to the following planning problem (specified here for the case of two households, *A* and *B*):

$$\max_{X,Y} \quad U(X,Y,u^B) \quad \text{subject to} \quad T(X,Y,\overline{K},\overline{L})=0$$

The first order conditions are

$$u_x^A + \lambda T_x = 0$$

$$u_y^A + \lambda T_y = 0$$

Taking the ratio yields the condition for overall allocative efficiency:

$$\boxed{\frac{u_x^A}{u_y^A} = \frac{T_x}{T_y}}$$

That is, $MRS_{x,y}^A = MRT_{x,y}$. Since we also have $MRS_{x,y}^A = MRS_{x,y}^B$ as an implicit property of the contract curve, overall we have

$$MRS_{x,y}^A = MRS_{X,Y}^B = MRT_{x,y}$$

See Figure 11.3.

Now consider the competitive equilibrium. Recall that in CE,

$$MC_x = p_x \quad \text{and} \quad MC_y = p_y$$

and

$$MRS^A_{x,y} = MRS^B_{X,Y} = \frac{p_x}{p_y}$$

Therefore, in CE

$$MRS^A_{x,y} = MRS^B_{X,Y} = \frac{MC_x}{MC_y}$$

Set *Y* as the numeraire good; that is, $MC_y = 1$. Thus, in CE we have

$$MRS^A_{x,y} = MRT_{x,y} = MRS^B_{x,y}$$

That is, the CE is Pareto efficient.

This result for our simple two sector economy generalizes to an economy with multiple factors, multiple goods, multiple households, uncertainty in outcomes, and intertemporal choices. It reflects a fundamental result in economics, the first welfare theorem (stated here without proof).

<u>The First Welfare Theorem</u>

In an economy where

(a) all agents are price-takers

(b) there are no IRS or indivisibilities

(c) there are no public goods

(d) there are no externalities

(e) information is symmetric between buyers and sellers,

every competitive equilibrium is Pareto efficient.

A closely related, and equally important result, is the second welfare theorem (also stated here without proof).

<u>The Second Welfare Theorem</u>

In an economy where (a) – (e) hold, any Pareto-efficient allocation can be supported as a competitive equilibrium with appropriate resource transfers.

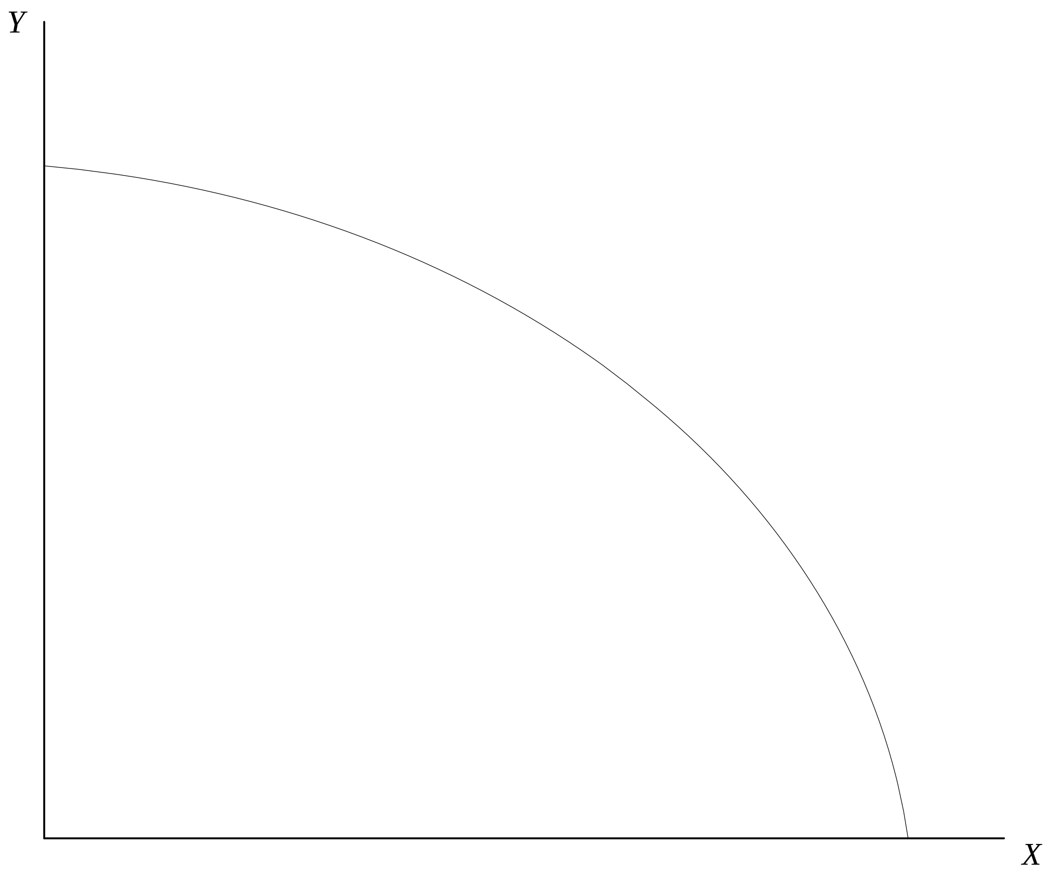


FIGURE 11.1

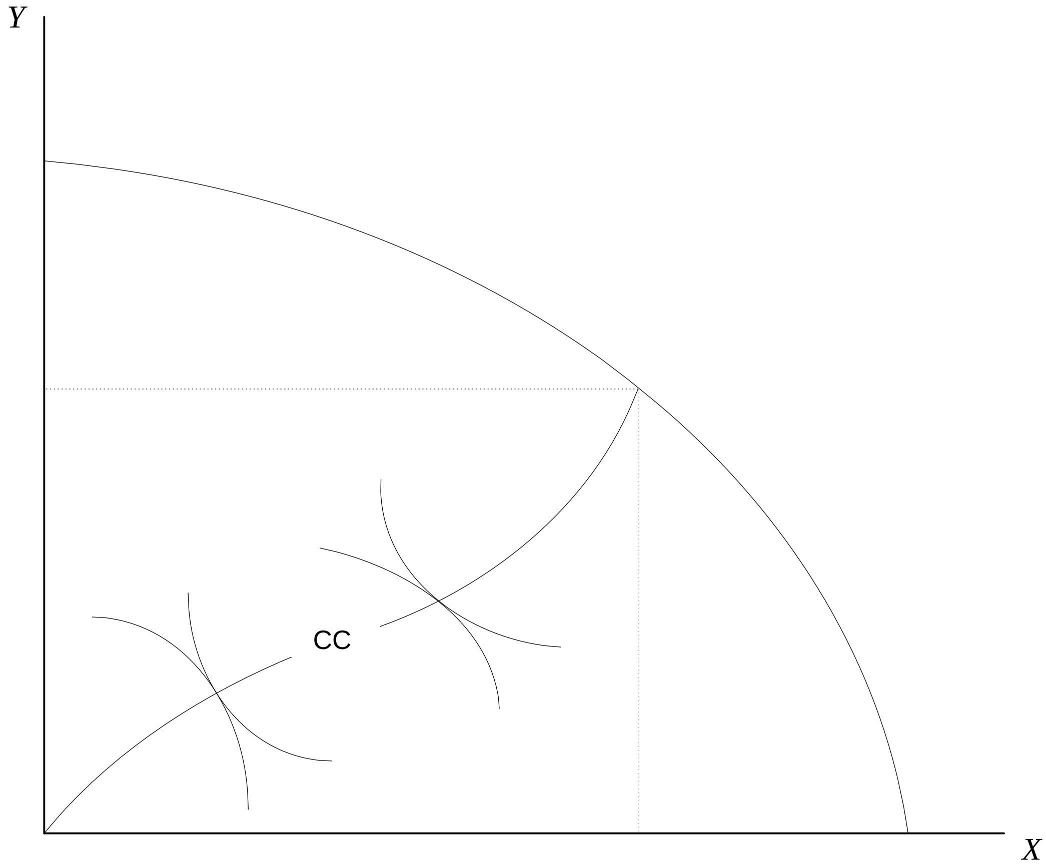


FIGURE 11.2

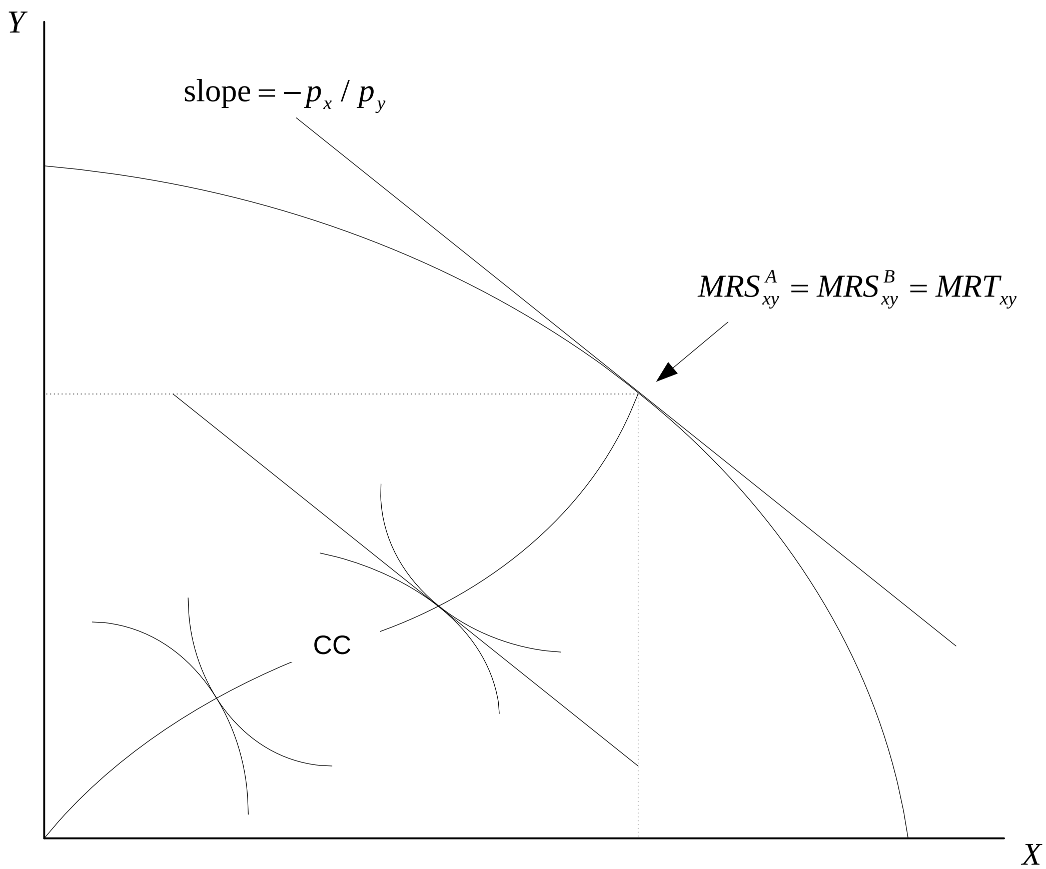


FIGURE 11.3

12. MONOPOLY

12.1 Introduction

Monopoly is where there is a single seller of the market good. Monopoly is due to *barriers to entry* (often due to economies of scale or scope, or network effects), or legislative protection (such as through patents).

We will deal first with the case of non-decreasing MC, and then consider the case of decreasing MC over the entire market range (natural monopoly).

<u>Assumptions</u>

1. The firm is a price-taker on factor markets. (In reality, a single seller on the product market often also has market power in the factor markets).
2. The firm is cost minimizing. (In reality, there may be "X-inefficiency": the absence of competitive pressure may mean that the firm does not minimize cost).

These assumptions mean that the cost function for the monopolist is independent of its position in the product market *per se;* the cost function simply reflects technological factors and factor prices, as it would for a "competitive" firm.

12.2 The Monopoly Problem

The firm chooses output to maximize profit:

$$\max_{y} \quad p(y)y - c(y)$$

where *p(y)* is market inverse demand and *c(y)* is the cost function.

The first-order condition is

(12.1) $$p(y) + yp'(y) = c'(y)$$

If $c''(y) \geq 0$ (that is, MC is non-decreasing) then the first-order condition defines a maximum.

Thus, if MC is non-decreasing then the monopolist produces where

marginal revenue (MR) = marginal cost (MC)

See Figure 12.1, where $\hat{y}$ is the monopoly output and $\hat{p}$ is the monopoly price.

Note that the monopolist does not have a supply function *per se*. It does not respond to a given price with an output choice as competitive firms do; it sets both price and output simultaneously. In particular, the monopolist chooses output $\hat{y}$ to maximize profit and then sets price $\hat{p} = p(\hat{y})$ to clear the market.

Note that equation (12.1) from above can be written as

$$p(y)\left(1+\frac{1}{\varepsilon}\right)=c'(y)$$

where $\varepsilon < 0$ is the elasticity of demand. A competitive firm *perceives* $\varepsilon = -\infty$; thus, it sets output such that $p = MC$. In contrast, the monopolist sets $p > MC$ for any $|\varepsilon| < \infty$.

12.3 Monopoly and Foregone Social Surplus

The monopoly outcome has an associated "deadweight loss" (DWL) since $p > MC$. See Figure 12.2. Each unit between $\hat{y}$ and y^* has a value to consumers greater than its production cost; this positive net *surplus* is foregone in the monopoly outcome. This foregone surplus is the DWL, the shaded area in Figure 12.2.

If the monopoly is due to patent protection then this welfare cost must be weighed against the *ex ante* incentives that monopoly rights over an invention create for research and development. Optimal patents are designed to just balance these two factors to maximize welfare in a dynamic context.

12.4 Price Discrimination

The monopolist may not set a single price. It may set a menu of prices that differ across units sold. There are three basic types of such price discrimination.

1. First Degree Price Discrimination

This is often called "perfect price discrimination": each unit is sold at a different price, set equal to the maximum WTP for that unit in the market. This yields an efficient outcome.

See Figure 12.3. The monopolist sells y^*; the last unit is sold at MC. The monopolist extracts the entire surplus from the market (the shaded area in the figure); there is no consumer surplus. Thus, while the outcome is efficient, it leads to a very skewed distribution of surplus.

2. Second Degree Price Discrimination

Less than perfect discrimination: different blocks of output are sold at different prices. Example: "buy one and get a second one at half price".

3. Third Degree Price Discrimination

The monopolist sets different prices for different groups of consumers or for different markets. Examples: a discount for students and seniors; generic versus brand-names; periodic sales. It is often motivated by different income levels among consumers and different associated WTPs. Note that price discrimination of this type requires identifiability of the different consumer groups and non-transferability of goods across those groups after sale.

Suppose there are two markets (or two types of consumers) with demands $p_1(y)$ and $p_2(y)$. The profit maximization problem is

$$\max_{y_1, y_2} \; p_1(y_1)y_1 + p_2(y_2)y_2 - c(y_1 + y_2)$$

with first order conditions

$$MR_1 = MC$$

$$MR_2 = MC$$

These imply

$$MR_1 = MR_2$$

Intuition: if revenue is higher on the marginal unit in one market, shift sales into that market until marginal revenue is equated across the two markets.

In which market is price higher? Since $MR_1 = MR_2$, we have

$$p_1(1+\frac{1}{\varepsilon_1}) = p_2(1+\frac{1}{\varepsilon_2})$$

Thus $p_1 > p_2$ if and only if $\varepsilon_1 > \varepsilon_2$. That is, price is higher in the less elastic the market. (Note that $\varepsilon < 0$, so $\varepsilon_1 > \varepsilon_2 \Rightarrow |\varepsilon_1| < |\varepsilon_2|$).

12.5 Natural Monopoly

Natural monopoly is characterized by declining MC over the entire market range.

See Figure 12.4. The key implication of declining MC is that $AC > MC$. The "efficient" output is y^*, where $p = MC$, but this solution involves a loss to the supplying firm. If this solution is implemented through regulation or direct public provision then this loss must be covered by a subsidy. Raising the funds needed to finance the subsidy will generally create distortions elsewhere in the economy (because taxation is generally distortionary). Taking the efficiency cost of these distortions into account means that the "second-best" efficient output is less than y^*.

The maximum output not involving a loss is $\bar{y}$, where $p = AC$. Thus, the second-best output will lie somewhere between $\bar{y}$ and y^*.

The unregulated output is $\hat{y}$, where $MR = MC$. As drawn in figure 12.4, this involves positive profit; it need not. Thus, supply of *any* positive amount in natural monopoly may require public provision or subsidization. If the market is sufficiently small then the true optimum may be zero provision.

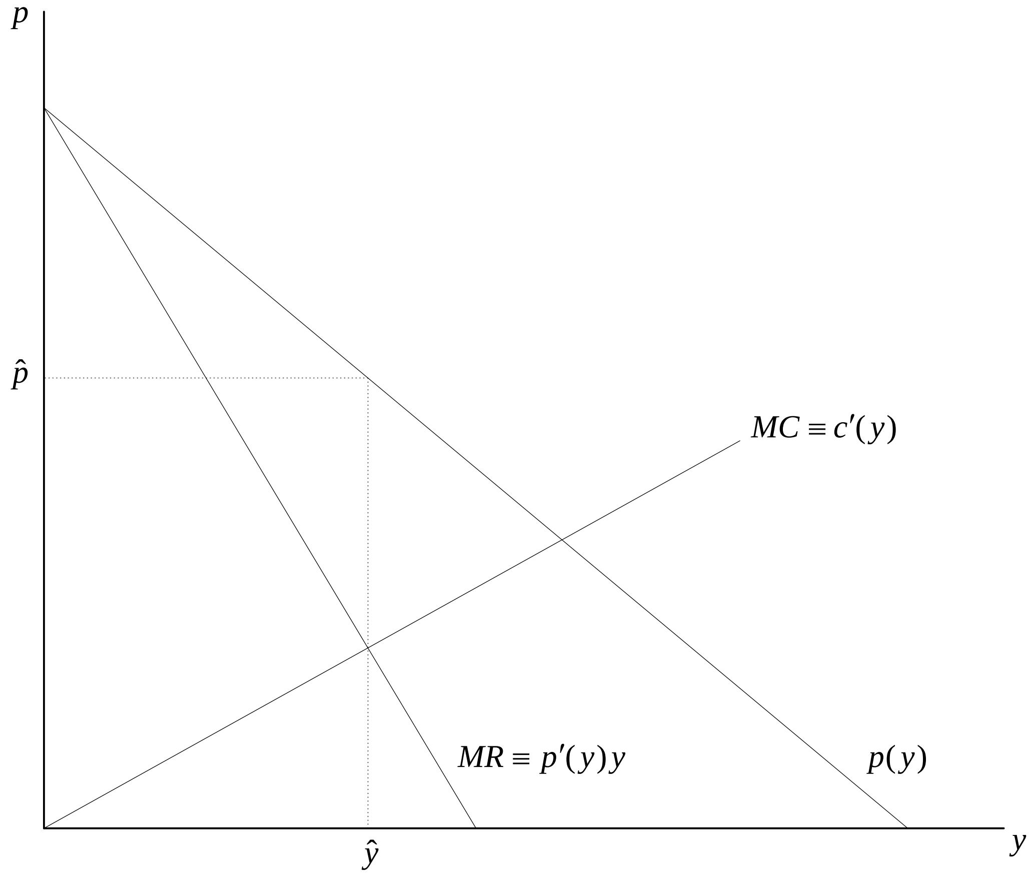


FIGURE 12.1

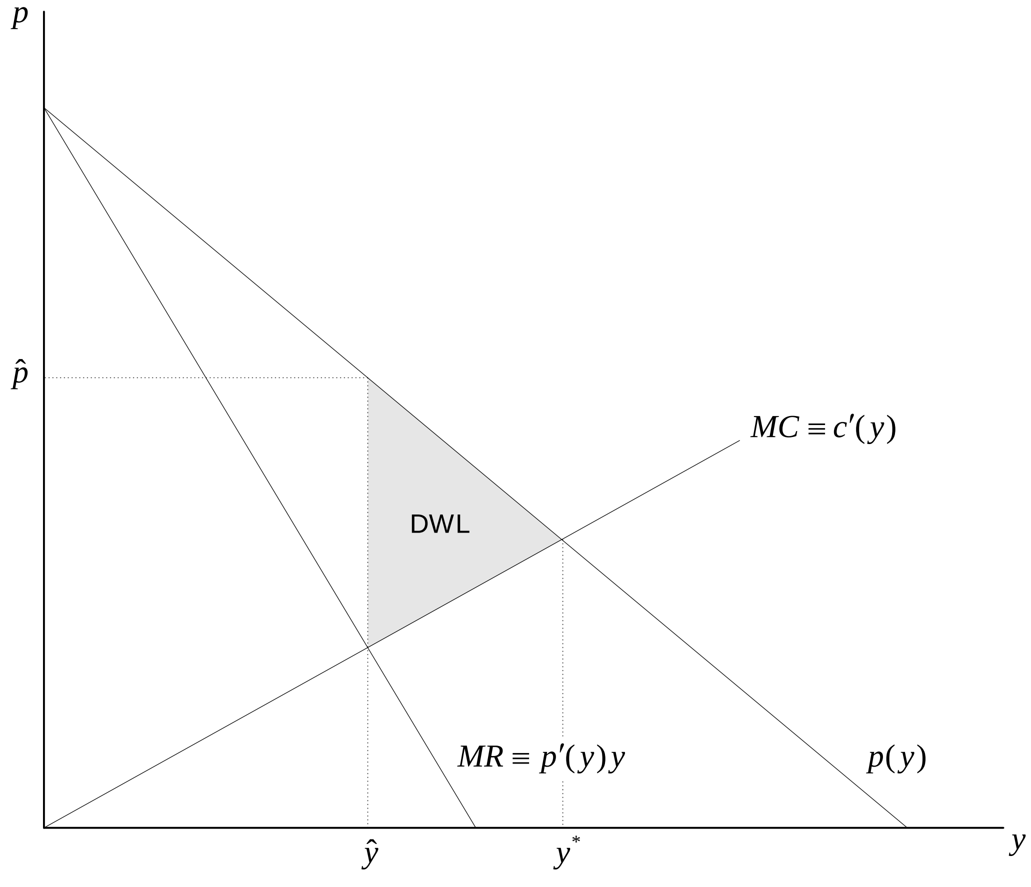


FIGURE 12.2

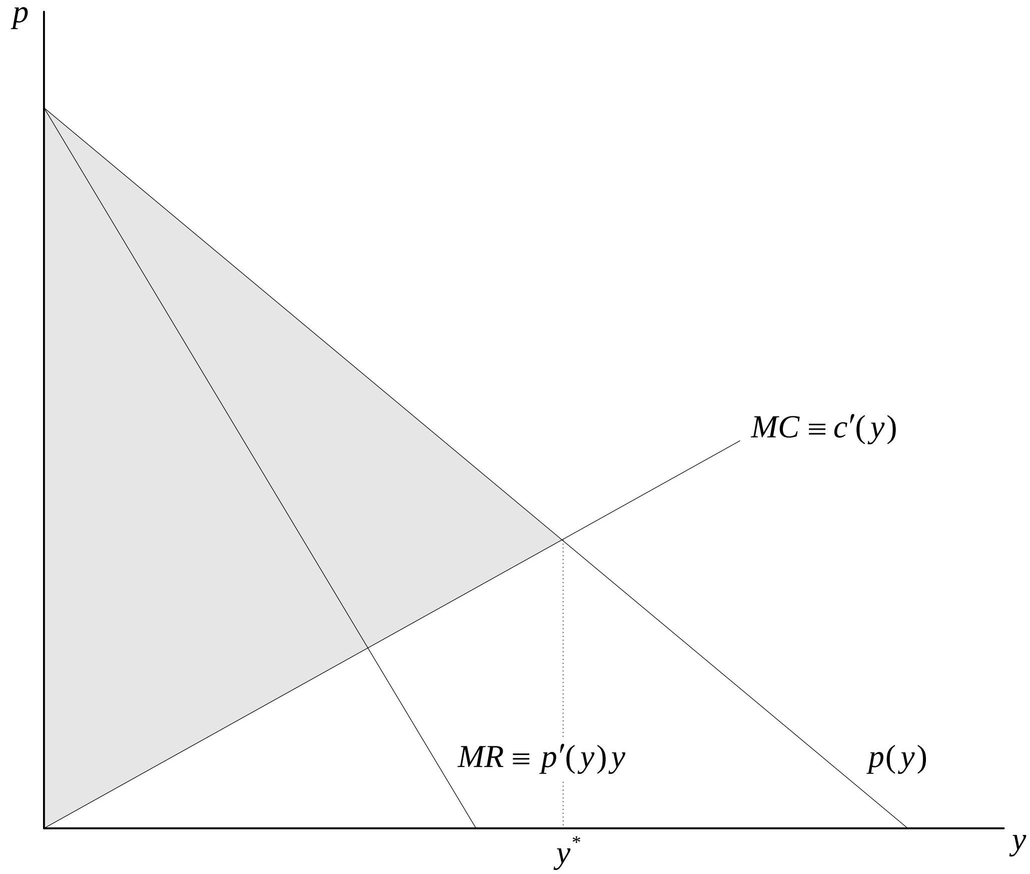


FIGURE 12.3

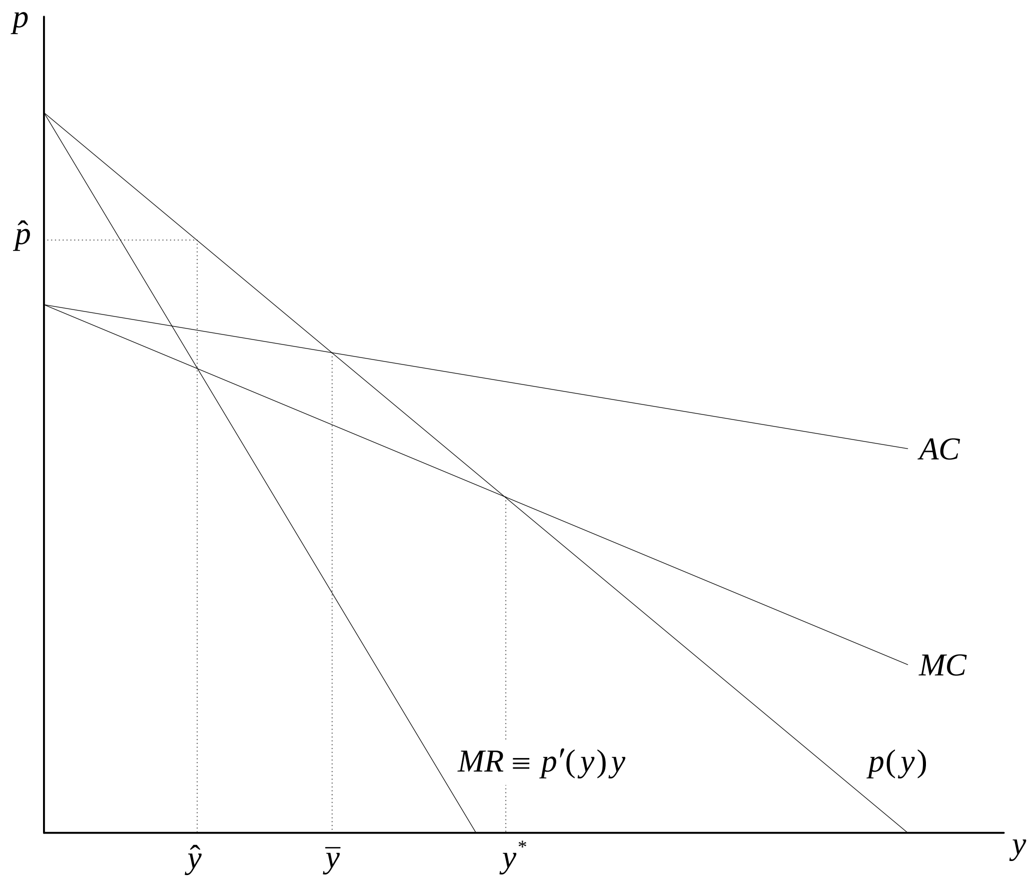


FIGURE 12.4

13. AN INTRODUCTION TO GAME THEORY AND OLIGOPOLY

13.1 Introduction

Game theory is the appropriate analytical framework in any setting with *strategic interaction*. Such a setting is where the actions of one agent affect the payoff (utility or profit) of another agent in a way that in turns affects the choice of best action by the affected agent.

All economic interaction can be examined using game theory. The price-taking behavior we examined earlier can be derived as a limiting case of more general environments involving strategic interaction. In Section 13.6 we will derive a perfectly competitive outcome as a limiting case of oligopoly.

13.2 Nash Equilibrium

Let s_i be the strategy of player *i*, and s_{-i} be the vector of strategies of all other players. Let $u_i(s_i, s_{-i})$ be the payoff to player *i*.

A *Nash equilibrium* is a vector $\{\hat{s}_i, \hat{s}_{-i}\}$ such that

$$u_i(\hat{s}_i, \hat{s}_{-i}) \geq u_i(s_i, \hat{s}_{-i}) \quad \forall s_i, \quad \forall i$$

That is, a NE is an outcome in which each player chooses her strategy to maximize her payoff, given the *equilibrium* strategies of all other players.

By definition, no player has an incentive to deviate from the Nash equilibrium.

13.3 A Normal Form Example

A "normal form" representation of a game is like a reduced form; the sequence of multiple moves that comprise the "extensive form" are subsumed into a single payoff matrix.

Consider the cartel problem illustrated in Figure 13.1. (This is the so-called "prisoners' dilemma" game). There are two firms (1 and 2) and two possible strategies: collude (C), or defect (D) from the collusive agreement. The first number in each cell of the patoff matrix is the payoff to the row player; the second number is the payoff to the column player. The Nash equilibrium in this game is {D,D}.

Note that {C,C} Pareto dominates the NE (from the perspective of the two firms). This inefficiency is a common, but not necessary, property of Nash equilibria. (The competitive general equilibrium is an instance in which the NE is Pareto efficient).

13.4 The Cournot Model of Oligopoly

There are *n* firms each selling an identical product on a market with inverse demand function $p(Y)$, where $Y = \sum_{j=1}^{n} y_j$ is aggregate output. Firm *i* has cost function $c_i(y_i)$, with $c_i''(y_i) \geq 0$. Firms choose output, and choices are made simultaneously.

The problem for firm *i* is

$$\max_{y_i} \quad p(\textstyle\sum_{j=1}^{n} y_j) y_i - c_i(y_i)$$

which can be rewritten as

$$\max_{y_i} \quad p(y_i + \textstyle\sum_{j \neq i}^{n} y_j) y_i - c_i(y_i)$$

Since decisions are made simultaneously, firm *i*'s choice cannot affect the choices of other firms. Thus, firm *i* perceives correctly that $\partial y_j / \partial y_i = 0 \quad \forall j \neq i$.

The best choice for firm *i* is given by

(13.1) $$p(y_i + \sum_{j \neq i}^{n} y_j) + y_i p' = c_i'(y_i)$$

This is just a MR = MC condition; but MR here is a function of aggregate output from other firms. Thus, there is a strategic interaction between firms.

Equation (13.1) can be interpreted as a *best response function* or reaction function for firm *i*. It specifies the best choice for firm *i* in response to (or in reaction to) the choices by other firms.

This terminology is somewhat misleading since firm *i* does not respond to the actions of other firms in a sequential sense (since all firms act simultaneously); firm *i* responds to what it expects other firms to do.

How are those expectations formed? Firm *i* expects all other firms to play the strategy (output choice) that is a best response to its choice. This is true for all firms. That is, each firm expects every other firm to behave rationally. Moreover, each firm knows that every other firm knows that it knows that every other firms know that ..… each firm will behave rationally. That is, there is *common knowledge* of rationality.

From common knowledge it follows that firm *i* expects every other firm to play its NE equilibrium strategy. Therefore, its own best response is to play its NE strategy.

The Cournot Nash equilibrium $\{\hat{y}\}$ is therefore characterized by

(13.2) $$p(\hat{y}_i + \sum_{j \neq i}^{n} \hat{y}_j) + \hat{y}_i p' = c_i'(\hat{y}_i) \quad \forall i$$

13.5 A Cournot Duopoly Example

Suppose there are two firms with constant marginal costs c_1 and c_2 respectively, and suppose demand is linear: $p(Y) = a - bY$.

The problem for firm 1 is

$$\max_{y_1} \; [a - b(y_1 + y_2)]y_1 - c_1 y_1$$

with the first-order condition given by

(13.3) $$[a - b(y_1 + y_2)] - by_1 = c_1$$

Rearrange this to obtain the best response function (BRF) in explicit form:

(13.4) $$y_1(y_2) = \frac{a - c_1 - by_2}{2b}$$

A similar BRF can be derived for firm 2:

(13.5) $$y_2(y_1) = \frac{a - c_2 - by_1}{2b}$$

<u>Nash Equilibrium</u>

The NE is $\{\hat{y}_1, \hat{y}_2\}$ such that $\hat{y}_1$ is a best response to $\hat{y}_2$, and $\hat{y}_2$ is a best response to $\hat{y}_1$. Thus, $\{\hat{y}_1, \hat{y}_2\}$ must solve (13.4) and (13.5) simultaneously. Solving by substitution yields

(13.6) $$\hat{y}_1 = \frac{a - 2c_1 + c_2}{3b}$$

(13.7) $$\hat{y}_2 = \frac{a - 2c_2 + c_1}{3b}$$

Note that $\hat{y}_1$ is decreasing in c_1 and increasing in c_2, as expected.

Profits in equilibrium are

(13.8) $$\hat{\pi}_1 = \frac{(a - 2c_1 + c_2)^2}{9b}$$

(13.9) $$\hat{\pi}_2 = \frac{(a - 2c_2 + c_1)}{9b}$$

Figure 13.2 provides a geometric representation of the game. The figure is drawn for the symmetric case, where $c_1 = c_2$. The intersection of the BRFs is the geometric interpretation of the simultaneous solution of (13.4) and (13.5). In the symmetric case the intersection lies on the 45^0 line.

Isoprofit Contours and the Pareto Frontier

To understand the properties of the equilibrium, first consider a set of *isoprofit contours* for firm 1, as depicted in Figure 13.3 . Each contour is a locus of pairs $\{y_1, y_2\}$ that yield a fixed level of profit for firm 1. That profit is highest at the monopoly output for firm 1, denoted y_1^M in Figure 13.3, where $y_2 = 0$. Profit decreases as we move away from the monopoly solution in any direction.

To understand the shape of a contour, start at a pair $\{y_1^+, 0\}$ where $y_1^+ > y_1^M$, as depicted in Figure 13.3. Profit at this output must be less than the monopoly profit because $y_1^+ > y_1^M$. Now suppose firm 1 reduces output towards y_1^M. If y_2 remains zero, profit for firm 1 would have to rise. Thus, for profit to remain constant, y_2 would also have to rise, thereby driving down price and offsetting the increase in profit for firm 1 that would otherwise occur. The same thought experiment can be conducted beginning at a pair $\{y_1^-, 0\}$ where $y_1^- < y_1^M$, as depicted in Figure 13.3.

The isoprofit contour becomes flat at the point where it crosses the BRF for firm 1 because that BRF by definition identifies the profit-maximizing output for firm 1 in response to any given output from firm 2. We can think of a point on the BRF for firm 1 as identifying the output that achieves the lowest possible isoprofit contour for any given value y_2. This implies a tangency condition, as depicted in Figure 13.3 at an arbitrarily chosen value $y_2 = \bar{y}_2$.

We can show this analytically. Totally differentiate profit for firm 1 to yield

$$d\pi_1 = (a - 2by_1 - y_2 - c_1)dy_1 - by_1 dy_2$$

By definition, $d\pi_1 = 0$ along any isoprofit contour for firm 1. Impose this requirement and solve for the slope of the contour in (y_1, y_2) space:

$$\left.\frac{dy_2}{dy_1}\right|_{d\pi_1 = 0} = \frac{a - 2by_1 - by_2 - c_1}{by_1}$$

It is straightforward to show that this slope is zero at the BRF by substituting $y_1 = y_1(y_2)$ from (13.4) above. Moreover, the slope of the contour is positive at any $y_1 < y_1(y_2)$, and negative at $y_1 > y_1(y_2)$, as depicted in Figure 13.3

.

Figure 13.4 depicts a set of isoprofit contours for firm 2, overlaid on the contours for firm 1 from Figure 13.3, all drawn for the symmetric case, where $c_1 = c_2$. There are two points to note from Figure 13-4. First, the contours for firm 2 have infinite slope where they cross the BRF for firm 2. Equivalently, they have zero slope in (y_2, y_1) space, reflecting the same interpretation that we gave earlier for the slope of the contours for firm 1.

Second, Figure 13.4 identifies a locus of tangencies between the isoprofit contours (in bold). Any point on this locus is the solution to a planning problem that maximizes profit for firm 1 subject to holding profit for firm 2 at some fixed level, *k*:

$$\max_{y_1, y_2} \; [a - b(y_1 + y_2)]y_1 - c_1 y_1 \quad \text{subject to} \quad [a - b(y_1 + y_2)]y_2 - c_2 y_2 = k$$

From the perspective of the two firms, this locus of tangencies is a *Pareto frontier*: the set of output pairs from which it is not possible to find an alternative pair that yields higher profit for one firm without reducing profit for the other firm.

It should be stressed that the output pairs on the Pareto frontier in this oligopoly game are Pareto-efficient only from the perspective of the two firms. We cannot say that they are Pareto efficient from a social perspective because we have not considered the welfare of consumers when deriving this frontier.

An analytical solution can be found for the Pareto frontier in this example. In particular, first derive the slope of the isoprofit contour for firm 2:

$$\left.\frac{dy_2}{dy_1}\right|_{d\pi_2 = 0} = \frac{by_2}{a - 2by_2 - by_1 - c_2}$$

Now set

$$\left.\frac{dy_2}{dy_1}\right|_{d\pi_2 = 0} = \left.\frac{dy_2}{dy_1}\right|_{d\pi_1 = 0}$$

and solve for y_2 to find the set of tangency pairs. In the symmetric case, where $c_1 = c_2 = c$, the solution reduces to a simple linear expression:

$$y_2^{PF}(y_1) = \frac{a-c}{2b} - y_1$$

This is the locus plotted in bold in Figure 13.4.

Figure 13.5 highlights the isoprofit contours passing through the NE, and identifies a shaded region called the *lens of mutual benefit*. The output pairs in the interior of this lens Pareto-dominate the NE from the perspective of the two firms; it would be to their mutual benefit if they could agree to move from the NE to a point inside the lens.

The segment of the Pareto frontier passing through the lens of mutual benefit – highlighted in bold in Figure 13.5 – is called the *core*. The core is the set of Pareto-efficient output pairs that Pareto-dominate the NE. Again we need to stress that this is all from the perspective of the two firms, not society as a whole.

<u>The Cartel Solution</u>

There is one output pair on the Pareto-frontier that has special status. It is the pair that maximizes the joint profits of the two firms:

$$\max_{y_1, y_2} \; [a - b(y_1 + y_2)]y_1 - c_1 y_1 + [a - b(y_1 + y_2)]y_2 - c_2 y_2$$

The solution to this problem is the *cartel solution* or the *collusive solution*. In our simple example where marginal costs are constant, the cartel solution is a corner solution: the firm with the lowest cost should produce its own monopoly output and the other firm should produce nothing. Thus, the cartel solution is

$$y_1^C = \frac{a - c_1}{2b} \quad \text{and} \quad y_2^C = 0 \quad \text{if} \quad c_1 < c_2$$

$$y_1^C = 0 \quad \text{and} \quad y_2^C = \frac{a - c_2}{2b} \quad \text{if} \quad c_1 > c_2$$

In the symmetric case, where $c_1 = c_2 = c$, any pair of outputs for which total output is

$$Y^C = \frac{a - c}{2b}$$

will maximize joint profits. Note that <u>all</u> points on the Pareto frontier satisfy this condition in the symmetric case.

The cartel solution cannot be achieved if the game is played only one time (a "one-shot" game). In that setting, each firm has a strict incentive to cheat on any agreement to restrict output. (We know this must be true because the cartel solution is <u>not</u> a NE).

If instead the game is repeated then there may be scope for the firms to act as a cartel because cheating can be punished in a future stage of the game. However, the repetition must be <u>infinite</u> or else there is no future beyond the last period of the game, and cheating will occur in that period. A rational expectation of that last-period cheating then leads to cheating in every prior period.

13.6 An Example with Identical Firms

Suppose there are *n* identical firms each with marginal cost *c*, and suppose demand is linear: $p(Y) = a - bY$.

The problem for representative firm *i* is

$$\max_{y_i} \; [a - b\sum_{j=1}^{n} y_j] y_i - c y_i$$

The best response function (first-order condition) is

$$y_i = \frac{a - c - b\sum_{j \neq i}^{n} y_j}{2b} \quad \forall i \tag{13.10}$$

Since firms are identical, it is natural to look for a *symmetric* Nash equilibrium in which each firm chooses the same equilibrium output. (Symmetric equilibria do not always exist, and in some non-symmetric equilibria, identical agents may behave differently).

In symmetric equilibrium:

$$\hat{y}_i = \hat{y} \quad \forall i \quad \text{and} \quad \sum_{j \neq i}^{n} \hat{y}_j = (n-1)\hat{y}$$

Making these substitutions in (13.10) and solving yields

$$\hat{y} = \frac{a - c}{b(n+1)} \tag{13.11}$$

Note that setting $n = 2$ in (13.11) yields the same result as setting $c_1 = c_2 = c$ in (13.6).

The NE price is

$$\hat{p}(n) = a - bn\hat{y} = \frac{a + nc}{n+1} \tag{13.12}$$

<u>Special Cases</u>

1. Perfect competition:

$$\lim_{n \to \infty} \hat{p}(n) = c$$

2. Monopoly:

$$\hat{p}(n)\Big|_{n=1} = \frac{a + c}{2}$$

13.7 Takeovers and the Nash Bargaining Solution

Recall the duopoly model from Section 13.5. Suppose one of the firms can buy the other firm, and then act as a true monopoly. It is clear that the joint gains from this takeover are highest if the low-cost firm buys the high-cost firm and then closes down the high-cost firm entirely. At what price would the two firms agree to make this sale?

This is a type of *bargaining problem*, and there are two different approaches to solving it:

- a game-theoretic approach (find the equilibrium to a game of alternating offers)
- an axiomatic approach (find the Nash bargaining solution)

Under some circumstances the two approaches yield the same solution but in other instances they do not. Here we will consider only the axiomatic approach.

<u>The Nash Bargaining Solution</u>

Nash proposed that any bargaining solution should satisfy four requirements (axioms):

- Pareto efficiency (or else there would be room for renegotiation)
- symmetry (if the players are indistinguishable, then the agreement should not discriminate between them)
- invariance (monotonic transforms of the payoff functions should not change the outcome)
- independence of irrelevant alternatives (payoffs over non-feasible outcomes should have no effect on the bargaining solution)

Nash showed that there exists only one solution that meets these requirements, and it is now called the *Nash bargaining solution*.

Suppose two players bargain over how to share an amount *A*. The players submit bids $\{b_1, b_2\}$, and if $b_1 + b_2 \leq A$ then they each receive their bid. If $b_1 + b_2 > A$ then they each receive nothing. Each player has an outside option to which they revert if an agreement cannot be reached. The payoff to player *i* in her outside option is d_i, called the *disagreement payoff*.

The **Nash bargaining solution** solves

$$\max_{\{b\}} \left(u_1(b_1) - u_1(d_1)\right)\left(u_2(b_2) - u_2(d_2)\right)$$

where this maximand is called the *Nash product*.

Application to the Takeover Problem

Suppose $c_1 < c_2$, then firm1 will buy firm 2. As a monopoly, firm 1 will produce output

$$y_1^M = \frac{a - c_1}{2b}$$

and earn profit

$$\pi_1^M = \frac{(a - c_1)^2}{4b}$$

Firm 1 pays a fraction $s \in (0,1)$ of this profit to firm 2 in the takeover sale. Thus, in this setting

$$u_1(b_1) = (1 - s)\pi_1^M \quad \text{and} \quad u_2(b_2) = s\pi_1^M$$

If the firms cannot agree on a sale then they will continue to compete in a duopoly, with profits $\hat{\pi}_1$ and $\hat{\pi}_2$ for firms 1 and 2 respectively, as given in (13.8) and (13.9). These equilibrium profits are the disagreement payoffs in this setting. That is,

$$u_1(d_1) = \hat{\pi}_1 \quad \text{and} \quad u_2(d_2) = \hat{\pi}_2$$

We can now make the substitutions in the Nash product and maximize with respect to *s* to yield the Nash bargaining solution:

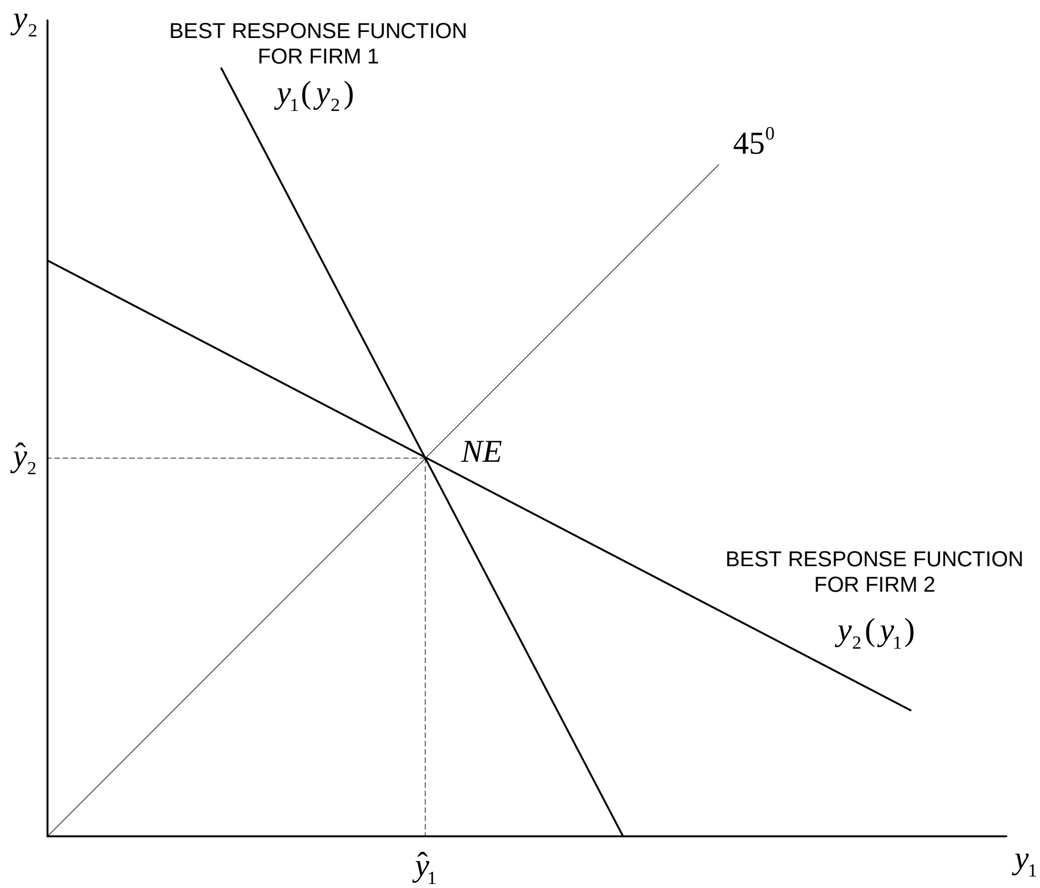
$$\tilde{s} = \frac{(3a - 2c_2)(a - 2c_2) + c_1(2a - c_1)}{6(a - c_1)^2}$$

This share is decreasing in c_2 because high cost reduces the bargaining power of firm 2. In the symmetric case, where $c_1 = c_2 = c$, $\tilde{s} = \frac{1}{2}$; the monopoly profit is split evenly.

< 2 >

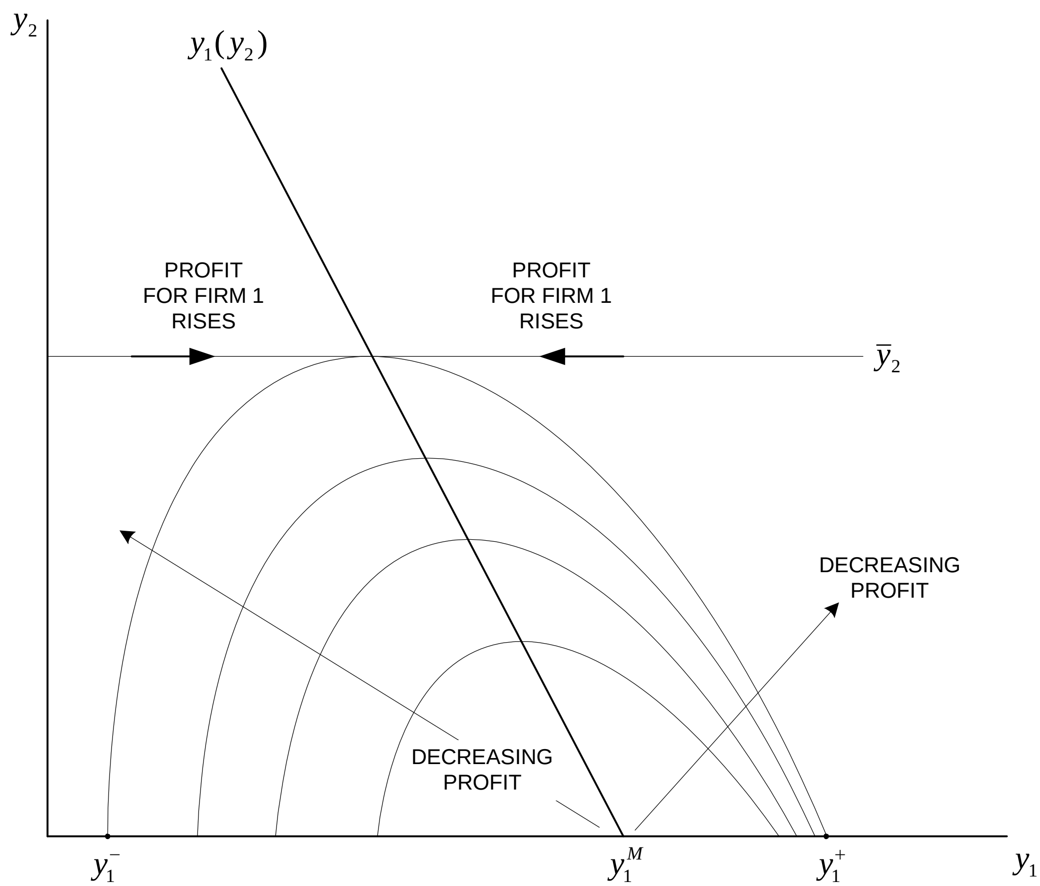
< 1 >		C	D
	C	* 4 , 4	2 , 5
	D	5 , 2	NE 3 , 3

FIGURE 13.1



y_2
BEST RESPONSE FUNCTION FOR FIRM 1
$y_1(y_2)$
45^0
$\hat{y}_2$
NE
BEST RESPONSE FUNCTION FOR FIRM 2
$y_2(y_1)$
$\hat{y}_1$
y_1

FIGURE 13.2



y_2
$y_1(y_2)$
PROFIT FOR FIRM 1 RISES
PROFIT FOR FIRM 1 RISES
$\overline{y}_2$
DECREASING PROFIT
DECREASING PROFIT
y_1^-
y_1^M
y_1^+
y_1

FIGURE 13.3

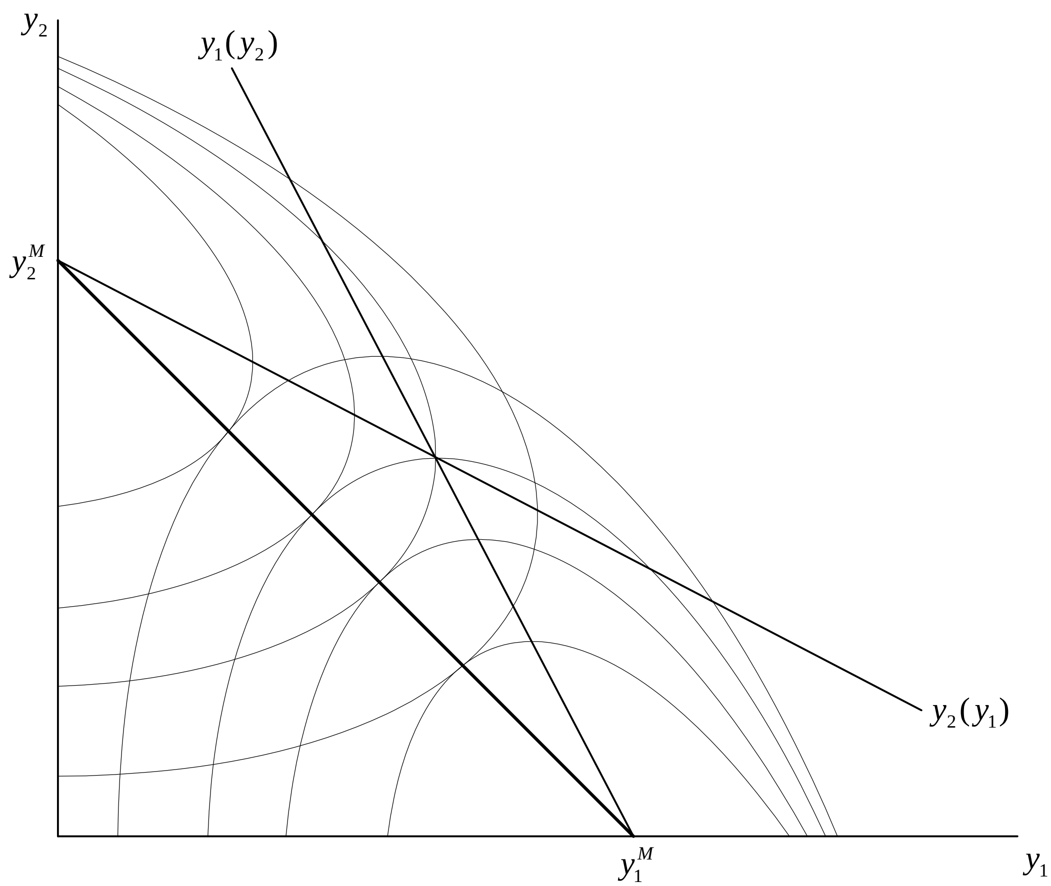


FIGURE 13.4

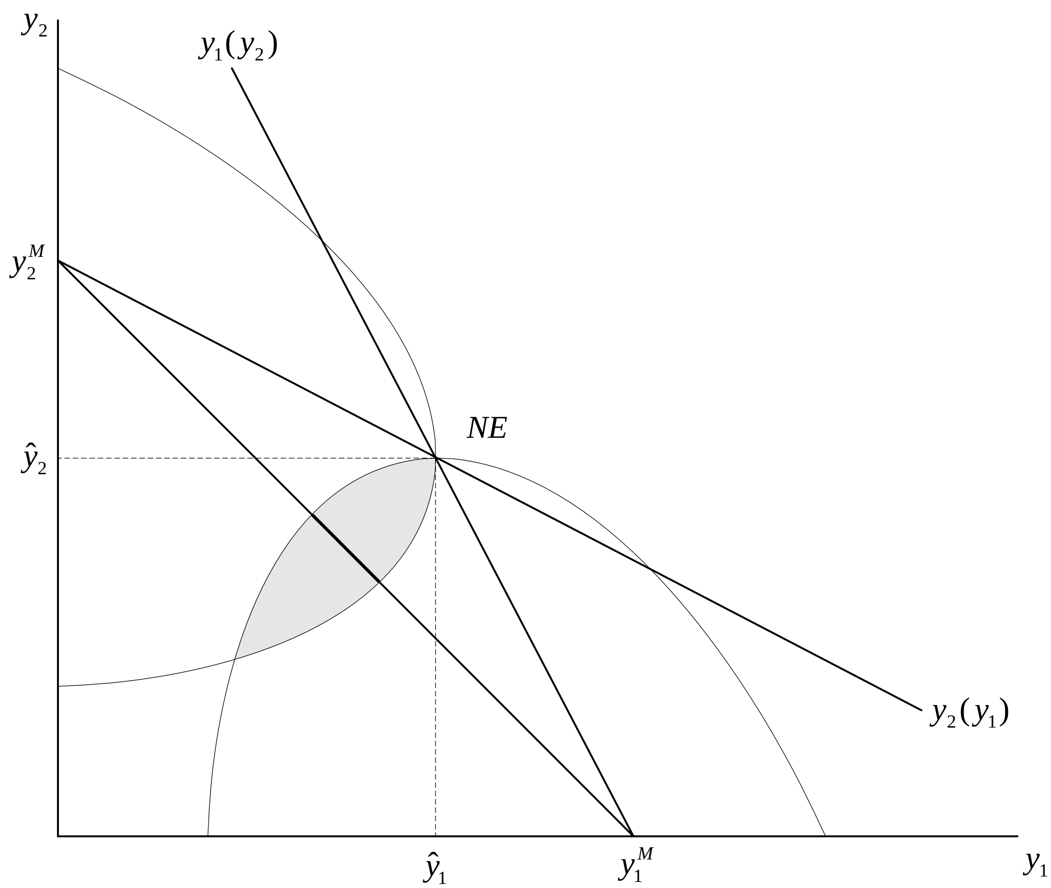


FIGURE 13.5

14. EXTERNALITIES AND PUBLIC GOODS

14.1 Introduction

An *externality* (or *external effect*) is a cost or benefit associated with an action that is external to the agent taking that action. Externalities can be positive (in which case there is an external benefit) or negative (in which case there is an external cost).

We will identify the agent taking the action as the *source agent*, and those external agents affected by the action as the *external agents*.

Externalities can be *unilateral*, where the external agents are passive players, or *reciprocal*, where at least some external agents are also source agents (as for example, with road congestion).

The key economic feature of an externality is the potential for inefficiency. In particular, if an action has an associated externality then the privately optimal action may be inefficient; it may be possible for the agent to take a different action that leaves no one worse-off and at least one agent better-off than at the private optimum.

Our plan in this chapter is as follows. We will begin with a graphical treatment of unilateral externalities to so as establish some basic intuition for the economic problem. We will then examine reciprocal externalities as a simple game between source agents, and consider the properties of the Nash equilibrium of that game. Finally, we will examine a special case of a reciprocal positive externality: *public goods*.

14.2 Unilateral Externalities: A Graphical Treatment

Consider a setting in which the external agents are passive recipients of an external effect from the action, z of a single source agent. Suppose z is continuous. (Imagine that z is the discharge of effluent into a river). We will first derive the private optimum for the source agent and then show that social surplus is not maximized at that private optimum.

<u>The Private Optimum</u>

Let *PB(z)* denote private benefit from the action and let *PC(z)* denote the private cost of that action. The source agent will choose the level of the action to maximize her private surplus (or private net benefit). That is, she will choose $\hat{z}$ such that

(14.1) $$[PB(\hat{z}) - PC(\hat{z})] \geq [PB(z) - PC(z)] \quad \forall z$$

Suppose *PB(z)* and *PC(z)* are increasing and twice continuously differentiable in *z,* and that $PB''(z) < 0$ and $PC''(z) > 0$. Then $\hat{z}$ is given by

(14.2) $$PB'(\hat{z}) = PC'(\hat{z})$$

That is, $\hat{z}$ is chosen to equate marginal private benefit (*MPB*) with marginal private cost (*MPC*). This solution is illustrated in Figure 14.1.

<u>Social Surplus Maximization</u>

For generality in exposition, suppose the action *z* potentially imposes an external cost *D(z)* and an external benefit *G(z)*. Define the *social cost* of *z*:

(14.3) $$SC(z) = PC(z) + D(z)$$

and the *social benefit* of *z*:

(14.4) $$SB(z) = PB(z) + G(z)$$

Social surplus (or net social benefit) is maximized at z^* such that

(14.5) $$[SB(z^*) - SC(z^*)] \geq [SB(z) - SC(z)] \quad \forall z$$

If *D(z)* and *G(z)* are increasing and twice continuously differentiable in *z,* with $D''(z) \geq 0$ and $G''(z) \leq 0$, then z^* is defined by

(14.6) $$SB'(z^*) = SC'(z^*)$$

That is, z^* equates marginal social benefit (*MSB*) with marginal social cost (*MSC*).

A Positive Externality

Figure 14.2 illustrates a positive externality. It is drawn for the case where $D(z) = 0$ (that is, there is no external cost) and $G''(z) < 0$ (which means that *MSB* and *MPB* diverge decreasingly).

The vertical difference between *MSB* and *MPB* in Figure 14.2 is *marginal external benefit* (*MEB*). Social surplus at z^* is the area between *MSB* and *MSC* from zero to z^*.

The presence of the external benefit means $z^* > \hat{z}$. Intuition: the agent does not take into account the external benefit she bestows on others when she chooses her action, and so her chosen level of the action is too low from a social perspective.

Note that a reallocation from $\hat{z}$ to z^* would make the external agents better off:

(14.7) gain to external agents

$$= G(z^*) - G(\hat{z})$$

$$= area(abcd)$$

but the source agent would be made worse off:

(14.8) loss to source agent (foregone private surplus)

$$= [PB(\hat{z}) - PC(\hat{z})] - [PB(z^*) - PC(z^*)]$$

$$= area(acd)$$

Thus, the move from $\hat{z}$ to z^* would *not* be a Pareto improvement. However, it would be a *potential Pareto improvement*: the external agents could in principle compensate the source agent for her loss in moving from $\hat{z}$ to z^* and still be better off. That is, the gain to the external agents is greater than the loss to the source agent. The move from $\hat{z}$ to z^* would raise social surplus by *area(abc)* in Figure 14.2.

A Negative Externality

Figure 14.3 illustrates a negative externality. It is drawn for the case where $G(z) = 0$ (that is, there is no external benefit) and $D''(z) > 0$ (which means that *MSC* and *MPC* diverge increasingly).

The vertical difference between *MSC* and *MPC* in Figure 14.3 is *marginal external cost* (*MEC*). Social surplus at z^* is the area between *MSB* and *MSC* from zero to z^*.

The presence of the external cost means $z^* < \hat{z}$. Intuition: the agent does not take into account the external cost she imposes on others when she chooses her action, and so her chosen level of the action is too high from a social perspective.

Note that a reallocation from $\hat{z}$ to z^* would make the external agents better off:

(14.9) gain to external agents (reduced external cost)

$$= D(\hat{z}) - D(z^*)$$

$$= area(abcd)$$

but the source agent would be made worse off:

(14.10) loss to source agent (foregone private surplus)

$$= [PB(\hat{z}) - PC(\hat{z})] - [PB(z^*) - PC(z^*)]$$

$$= area(abd)$$

Thus, the move from $\hat{z}$ to z^* would *not* be a Pareto improvement. However, it would be a potential Pareto improvement: the external agents could in principle compensate the source agent for her loss in moving from $\hat{z}$ to z^* and still be better off. That is, the gain to the external agents is greater than the loss to the source agent. The move from $\hat{z}$ to z^* would raise social surplus by *area(bcd)* in Figure 14.3.

14.3 Reciprocal Externalities

Let us begin with a simple setting in which there are just two agents. Each agent derives some benefit from the activity but that activity imposes a cost on the other agent. To fix ideas, imagine two countries engaged in industrial activity *z* where the combined activity causes global environmental damage, which affects them both.

The private benefit of activity z_i to agent *i* is $b_i(z_i)$, with $b_i'(z_i) > 0$ and $b_i''(z_i) < 0$.

The cost (in terms of environmental damage) to agent *i* due to the combined activity is $c_i(Z)$, where $Z = z_1 + z_2$ is the aggregate activity. Assume that $c_i'(Z) > 0$ and $c_i''(Z) > 0$.

Note that we have allowed damage from *Z* to be different for the two agents. Thus, the externality is reciprocal but it is not necessarily symmetric in its impact.

We will examine the simplest possible formulation of this game, where the agents act simultaneously, and the game is played just once.

Nash Equilibrium

The choice problem for each agent 1 is

(14.11) $$\max_{z_1} \quad b_1(z_1) - c_1(z_1 + z_2)$$

In choosing z_1, agent 1 takes z_2 as given. The first-order condition for a maximum is

(14.12) $$b_1'(z_1) = c_1'(Z)$$

This condition is sufficient for a maximum given our assumptions on $b_i(z_i)$ and $c_i(Z)$.

Condition (14.12) states that agent 1 will set her activity level to equate her marginal private benefit with her marginal private cost, given the level of activity from agent 2.

Note that condition (14.12) can be interpreted as the best-response function (BRF) for agent 1, which we denote $z_1(z_2)$. This specifies the privately optimal activity for agent 1 for any given level of activity from agent 2.

An analogous BRF can be derived for agent 2. It is denoted $z_2(z_1)$ and defined by

(14.13) $$b_2'(z_2) = c_2'(Z)$$

The Nash equilibrium is $\{\hat{z}_1, \hat{z}_2\}$ such that $\hat{z}_1$ is a best response to $\hat{z}_2$, and $\hat{z}_2$ is a best response to $\hat{z}_1$. Thus, $\{\hat{z}_1, \hat{z}_2\}$ must solve (14.12) and (14.13) simultaneously.[1]

[1] Note the similarity between this problem and the Cournot duopoly problem from Chapter 13.

Figure 14.4 illustrates the BRFs and the Nash equilibrium in the symmetric case, in which the two agents have identical benefit functions and identical cost functions. The points labeled z_1^0 and z_2^0 identify the "sole-agent optima" for agents 1 and 2 respectively. That is, z_i^0 is the level of *z* agent *i* would undertake if she were the sole agent in this economy, and was thus unaffected by the actions of the other agent.[2]

14.4 An Example with Two Identical Agents

Suppose there are two identical agents and the payoff to agent *i* is

$$u_i = \theta \log(z_i) - \delta(z_1 + z_2)^2 \text{ for } i \in \{1,2\}$$

The first order condition for agent 1 is

(14.14) $$\frac{\theta}{z_1} = 2\delta(z_1 + z_2)$$

This can be interpreted as $MPB = MPC$, but in contrast to the unilateral externality case, the *MPC* for this agent now depends on the action taken by the other agent.

This first order condition is the BRF for agent 1. We can solve (14.14) for z_1 to obtain a closed form solution for this BRF:

(14.15) $$z_1(z_2) = \frac{(\delta^2 z_2^2 + 2\delta\theta)^{1/2} - \delta z_2}{2\delta}$$

An analogous BRF can be found for agent 2:

(14.16) $$z_2(z_1) = \frac{(\delta^2 z_1^2 + 2\delta\theta)^{1/2} - \delta z_1}{2\delta}$$

These BRFs are as illustrated in Figure 14.4. The "sole agent" optima, z_1^0 and z_2^0, can be calculated easily by setting $z_2 = 0$ in (14.15) and $z_1 = 0$ in (14.16) respectively to obtain

$$z_1^0 = \frac{1}{2}\left(\frac{2\theta}{\delta}\right)^{1/2} \quad \text{and} \quad z_2^0 = \frac{1}{2}\left(\frac{2\theta}{\delta}\right)^{1/2}$$

[2] This is akin to the monopoly output in Figure 13.3 from Chapter 13.

Nash Equilibrium

The Nash equilibrium, denoted $\{\hat{z}_1, \hat{z}_2\}$, must solve (14.15) and (14.16) simultaneously. Finding this solution involves lots of tedious algebra. Fortunately, this can be avoided by utilizing the symmetry of the problem. In the symmetric equilibrium, $\hat{z}_1 = \hat{z}_2 = \hat{z}$. Impose this restriction on (14.14) to obtain

(14.17) $$\frac{\theta}{\hat{z}} = 2\delta(\hat{z} + \hat{z})$$

which can be solved easily to yield

(14.18) $$\hat{z} = \frac{1}{2}\left(\frac{\theta}{\delta}\right)^{1/2}$$

This is the same solution that would obtain by solving (14.15) and (14.16) directly.

Isopayoff Contours

Figure 14.5 depicts a set of isopayoff contours for agent 1. The equation for an isopayoff contour for agent 1 can be found by setting $u_1 = u$ and solving for z_2 as a function of z_1:

$$z_2(z_1, u) = \frac{[\delta\theta \log(z_1) - \delta u]^{1/2} - \delta z_1}{\delta}$$

The different contours in Figure 14.5 correspond to different values of *u*. Note that utility increases as we move towards the sole-agent optimum. Note also that each isopayoff contour is flat at the point where it crosses the BRF because by definition the BRF identifies the utility-maximizing activity level for agent 1 in response to any given level of activity by agent 2.

The Pareto Frontier

There are a continuum of efficient allocations in this economy corresponding to different distributions of utility across the two agents. The Pareto frontier (the set of Pareto-efficient allocations) is found by maximizing the utility of one agent subject to maintaining a given level of utility for the other:

$$\max_{z_1, z_2} \ \theta \log(z_1) - \delta(z_1 + z_2)^2$$

$$\text{subject to } \theta \log(z_2) - \delta(z_1 + z_2)^2 = u$$

where *u* is a given level of utility for agent 2. The solution to this problem is[3]

(14.19) $$z_1^*(u) = \frac{1}{2}\left(\frac{2\theta}{\delta}\right)^{1/2} - \exp\left(\frac{u}{\theta} + \frac{1}{2}\right)$$

(14.20) $$z_2^*(u) = \exp\left(\frac{2u+\theta}{2\theta}\right)$$

These expressions identify efficient levels of z_1 and z_2 for a given value of *u*. The solution highlights the fact that there is a different efficient allocation for every different value of *u*, where *u* captures the distribution of utility across the agents. (A higher value of *u* means more utility for agent 2 and less utility for agent 1).

We can now use (14.19) and (14.20) to derive a closed-form solution for the Pareto frontier in (z_1, z_2) space. Rearrange (14.19) to obtain

(14.21) $$u = \theta \log\left(\frac{1}{2}\left(\frac{2\theta}{\delta}\right)^{1/2} - z_1^*\right) - \frac{\theta}{2}$$

Substitute this into (14.20), and simplify to obtain

(14.22) $$z_2^* = \frac{1}{2}\left(\frac{2\theta}{\delta}\right)^{1/2} - z_1^*$$

This frontier is linear with a slope of -1. It is depicted in Figure 14.6, labeled *PF*. The frontier is the locus of tangencies between the isopayoff contours.

<u>The Inefficiency of the Equilibrium</u>

Figure 14.7 highlights the isopayoff contours passing through the NE, and identifies the *lens of mutual benefit*. All points in the interior of this lens Pareto-dominate the NE; it would be to the mutual benefit of both agents if they could agree to move from the NE to a point inside the lens.

[3] The algebra for this problem gets messy. I used *Maple* to solve it.

Recall from Chapter 13 that the segment of the Pareto frontier passing through the lens of mutual benefit – highlighted in bold in Figure 14.7 – is called the *core*. The core is the set of Pareto-efficient output pairs that Pareto-dominate the NE.

It is clear from Figure 14.7 that the Nash equilibrium is inefficient; it does not lie on the Pareto frontier. Why? Each agent ignores the cost her activity imposes on the other agent precisely because that cost is external to her. This external cost is nonetheless part of the true social cost of the activity, and efficiency requires that it be taken into account.

Surplus Maximization

By definition, all points on the Pareto frontier are Pareto efficient. Conversely, all points not on the frontier are *inefficient* (including the NE). However, note from Figure 14.7 that there are many Pareto-efficient points – those points on the frontier but not in the core – that do not Pareto-dominate the NE. What can we say about these points?

Those points on the Pareto frontier that are not in the core are *potential Pareto improvements* over the NE. A move from the NE to any point on the frontier creates enough social surplus that the winner from that move could in principle compensate the loser and still be better off. (Of course, if the move is to a point in the core, then there are no losers).

Does a move from the NE to a point on the frontier create the same amount of social surplus, regardless of the point on the frontier to which we move?

In general, the answer is no. In most problems, there is a unique point on the Pareto frontier at which social surplus is maximized.[4] In a setting with identical agents, that surplus-maximizing point lies in the core on the 45^0 line.

[4] Analogously, recall from Topic 13.5 in Chapter 13 that the unique joint-profit-maximizing output is one of the two monopoly points.

We can solve for this allocation in our example by setting $z_1^* = z_2^* = z^*$ in (14.22) and solving for z^* to yield

(14.23) $$z^* = \frac{1}{2}\left(\frac{\theta}{2\delta}\right)^{1/2}$$

This solution corresponds to the point *S* in Figure 14.8. Note that $z^* < \hat{z}$: there is too much of this activity by both agents in equilibrium relative to the surplus-maximizing solution (SMS) because both agents fail to take account of the cost their activity imposes on the other agent.

Unilateral Externalities Revisited

In a setting in which agents are not identical, the SMS may not lie in the core. In that case, a move from the NE to the SMS is not a Pareto improvement. We can think of a unilateral externality as an extreme case of asymmetry in a reciprocal externality problem, where the external cost on one of the agents is vanishingly small. We have already seen (in Section 14.2) that the SMS is not a Pareto improvement over the private optimum in the unilateral problem. The private optimum in that problem is just a special case of the Nash equilibrium in which all agents have *dominant strategies*.

In general, any externality problem can – and should – be modeled as a game. The simple unilateral externality problem can then be derived as a limiting case of that more general framework.

14.5 Public Goods

Public goods are characterized by two features:

- joint consumption possibilities
- high exclusion costs

Joint consumption possibilities means that the benefits of the good can be enjoyed by more than one agent at the same time. For example, a lecture, a park, a lighthouse beam.

That is, consumption of public goods is “non-rivalrous” (in contrast to private goods like bread, cheese and wine).

High exclusion costs means that it is costly to prevent agents from consuming the good once it is provided (eg. it is costly to build a fence around a national park).

Public goods are a type of positive externality in the sense that provision of the good by one agent bestows a positive benefit on other agents (who can enjoy the public good without paying for it).

Note that public goods may or may not be provided by the public sector. Moreover, many goods provided by the public sector are not public goods. Thus, we need to keep a clear distinction between public goods and goods provided by the public sector; they are not the same thing.

Public goods are often classified according to the degree to which they are non-rivalrous and/or non-excludable. In particular:

- *pure public goods* are those that are perfectly non-rivalrous (eg. radio signals, a lighthouse beam, knowledge).
- *impure* (or *congestible*) *public goods* are subject to congestion; that is, the benefits of consumption declines as more agents use the good (eg. roads, the radio spectrum, a beach, a wilderness area).
- *club goods* are congestible public goods with relatively low exclusion costs (eg. a swimming pool, a restaurant).

This categorization is somewhat artificial since there are in fact a continuum of possibilities with respect to congestibility and exclusion costs.

14.6 Efficient Provision of a Public Good

Consider an economy with two agents, A and B, and two goods y and G, where y is a private good and G is a continuous public good. Production possibilities are represented by the transformation function $T(y,G) = 0$.

We can derive the set of Pareto-efficient allocations as the solution to the following planning problem:

$$\max_{y,y^A,G} \quad u^A(y^A, G)$$

$$\text{subject to} \qquad u^B(y^B, G) = u$$

$$y^A + y^B = y$$

$$T(y,G) = 0$$

That is, we are looking for a point on the production possibility frontier, and a division of the associated output of the private good, such that it is not possible to make agent A better-off without making agent B worse-off.

The associated Lagrangean is

$$L = u^A(y^A, G) + \lambda[u^B(y - y^A, G) - u] + \phi T(y,G)$$

and the first-order conditions for a maximum are

(14.24) $\qquad \lambda u_y^B + \phi T_y = 0$

(14.25) $\qquad u_y^A - \lambda u_y^B = 0$

(14.26) $\qquad u_G^A + \lambda u_G^B + \phi T_G = 0$

Divide (14.26) by u_y^A to obtain

(14.27) $\qquad \left[u_G^A / u_y^A\right] + \left[\lambda u_G^B / u_y^A\right] + \left[\phi T_G / u_y^A\right] = 0$

Then use (14.24) and (14.25) in (14.27) to obtain

(14.28) $\qquad \left[u_G^A / u_y^A\right] + \left[u_G^B / u_y^B\right] = T_G / T_y$

Equation (14.28) has the following interpretation:

$$MRS_{Gy}^{A} + MRS_{Gy}^{B} = MRT_{Gy}$$

This condition is known as the *Samuelson Condition* for efficient public good provision. Intuition: the *sum* of the marginal private valuations of the public good should be equated to the marginal cost of provision (in terms of private good foregone). Why the sum of marginal private valuations? Because G can be consumed jointly by all agents.

Contrast this condition with efficient provision of private goods x and y:

$$MRS_{xy}^{A} = MRS_{xy}^{B} = MRT_{xy}$$

Only one agent can consume a unit of the private good; efficiency dictates that it should be allocated to the agent with the highest valuation of that good until marginal valuations are just equated across agents, and these are in turn equated to marginal cost.

See Figure 14.9 for a geometric representation of the Samuelson condition. Note that for every different value of u (corresponding to a distribution of utility in the economy) there is a different corresponding efficient level of G. That is, the efficient level of G is not unique.

14.7 Voluntary Private Provision of the Public Good

Private provision of the public good is subject to a *free-rider problem*: each agent has an incentive to free-ride on the contributions to the public good made by other agents. This can lead to inefficiency in private provision.

Consider the Nash equilibrium provision in our example economy. To simplify matters, assume the transformation function is linear with slope α. (That is, it takes α units of the private good to create one unit of the public good).

Suppose we start from the point where $G = 0$ and $y = \bar{y}$, as illustrated in figure 14.9 Suppose further that the initial allocation of $\bar{y}$ between the agents is $\{m^A, m^B\}$ such that $m^A + m^B = \bar{y}$. Let g^i be the voluntary contribution to the public good by agent *i*. Then

$$G = g^A + g^B$$

The choice problem for agent A is

$$\max_{g^A} \; u^A(m^A - \alpha g^A, g^A + g^B)$$

The first-order condition defines the BRF for agent A:

(14.29) $$-\alpha u_y^A + u_G^A = 0$$

Totally differentiate with respect to g^A and g^B to obtain the slope of this best response function:

(14.30) $$dg^A / dg^B = \left[\alpha u_{yG}^A - u_{GG}^A\right] / \left[\alpha^2 u_{yy}^A - \alpha u_{yG}^A + u_{GG}^A\right]$$

This slope is *negative* if the agent has convex preferences. Intuition: if agent B contributes more to the public good, agent A will tend to free ride on the higher contribution, and so *reduce* on her own contribution.

A similar BRF can be derived for agent B. The two BRF, together with the associated Nash equilibrium, are illustrated in Figure 14.10 for the case of identical agents.

The Inefficiency of the Equilibrium

The Nash equilibrium is inefficient. This property is illustrated in Figure 14.10, which depicts indifference curves for the two agents. The shape of these indifference curves reflects the fact that utility is *increasing* in the contributions from other agents because *G* is a public good (in contrast to the negative externality problem form Section 14.4).

The Pareto frontier is the locus of tangencies of the indifference curves, identified in Figure 14.10 as *PF*. The lens of mutual benefit is the shaded region in the figure, and the core is the bold portion of the *PF* passing through the lens. All points in the core Pareto-dominate the NE.

The tangency requirement underlying the Pareto frontier follows directly from the Samuelson condition. To see this, note that the indifference curve for agent A in (g^A, g^B) space is found by totally differentiating

$$u^A(m^A - \alpha g^A, g^A + g^B) = \bar{u}$$

with respect to g^A and g^B to obtain

(14.31) $$\left[-\alpha u_y^A + u_G^A\right] dg^A + u_G^A dg^B = 0$$

The slope of this indifference curve in (g^A, g^B) space is

(14.32) $$\left(dg^B \big/ dg^A\right)\Big|_A = \left[\alpha u_y^A - u_G^A\right] \Big/ u_G^A$$

We can similarly derive the slope of the indifference curve for agent B:

(14.33) $$\left(dg^B \big/ dg^A\right)\Big|_B = u_G^B \Big/ \left[\alpha u_y^B - u_G^B\right]$$

Tangency occurs where these slopes are equated:

(14.34) $$\left[\alpha u_y^A - u_G^A\right] \Big/ u_G^A = u_G^B \Big/ \left[\alpha u_y^B - u_G^B\right]$$

Rearranging this expression yields

(14.35) $$\left[u_G^A \big/ u_y^A\right] + \left[u_G^B \big/ u_y^B\right] = \alpha$$

This is the Samuelson condition (where $MRT = \alpha$).

14.8 A Symmetric Example

Consider an economy in which *n* identical agents each have the following utility function

$$u(y, G) = y + \log G$$

where *y* is a private good and *G* is a continuous public good. Each agent has income *m* (in terms of the private good) which she divides between consumption of the private good and a contribution *g* to the provision of the public good, such that

$$G = \sum_{i=1}^{n} g_i$$

Thus, in this economy, $MRT = 1$.

Nash Equilibrium in Voluntary Contributions

Agent *i* solves

$$\max_{g_i} \ (m - g_i) + \log(g_i + G_{-i})$$

where G_{-i} is the total contribution from agents other than agent *i*. The first-order condition is

$$-1 + \frac{1}{g_i + G_{-i}} = 0$$

Simplifying yields

$$g_i = 1 - G_{-i}$$

This represents the best response function for agent *i*. Note that it is downward-sloping; the more agent *i* expects others to provide, the less she will provide herself. This reflects the free-rider problem.

In a symmetric Nash equilibrium, $g_i = g \ \ \forall i$ and so $G_{-i} = (n-1)g$. Thus, in equilibrium

$$\hat{g} = \frac{1}{n}$$

The aggregate contribution is

$$\hat{G} = n\hat{g} = 1$$

The Inefficiency of the Equilibrium

The Pareto frontier in a setting with *n* agents is a surface in *n*-dimensional space. We will focus on just one point on that surface, at which all agents have equal utility. This point is the surplus-maximizing solution (SMS) for this problem because agents are identical.

The most straightforward way to solve for the SMS when agents are identical is to maximize the utility of a representative agent subject to the resource constraints of the economy:

$$\max_{G,y} \ \ y + \log G \quad \text{subject to} \quad ny + G = nm$$

where *nm* is the total amount of the private good available in the economy for allocation between direct consumption and transformation into the public good.

The first-order condition is

$$-1+\frac{n}{G}=0$$

which solves for

$$G^{*}=n$$

Comparing this with $\hat{G}$ reveals that the Nash equilibrium level of *G* is inefficiently low. Note too that in this example,

$$\frac{\partial\left(G^{*}/\hat{G}\right)}{\partial n}>0$$

This reflects the so-called "Mancur Olsen conjecture": the distortion associated with free-riding gets worse as the number of agents rises.

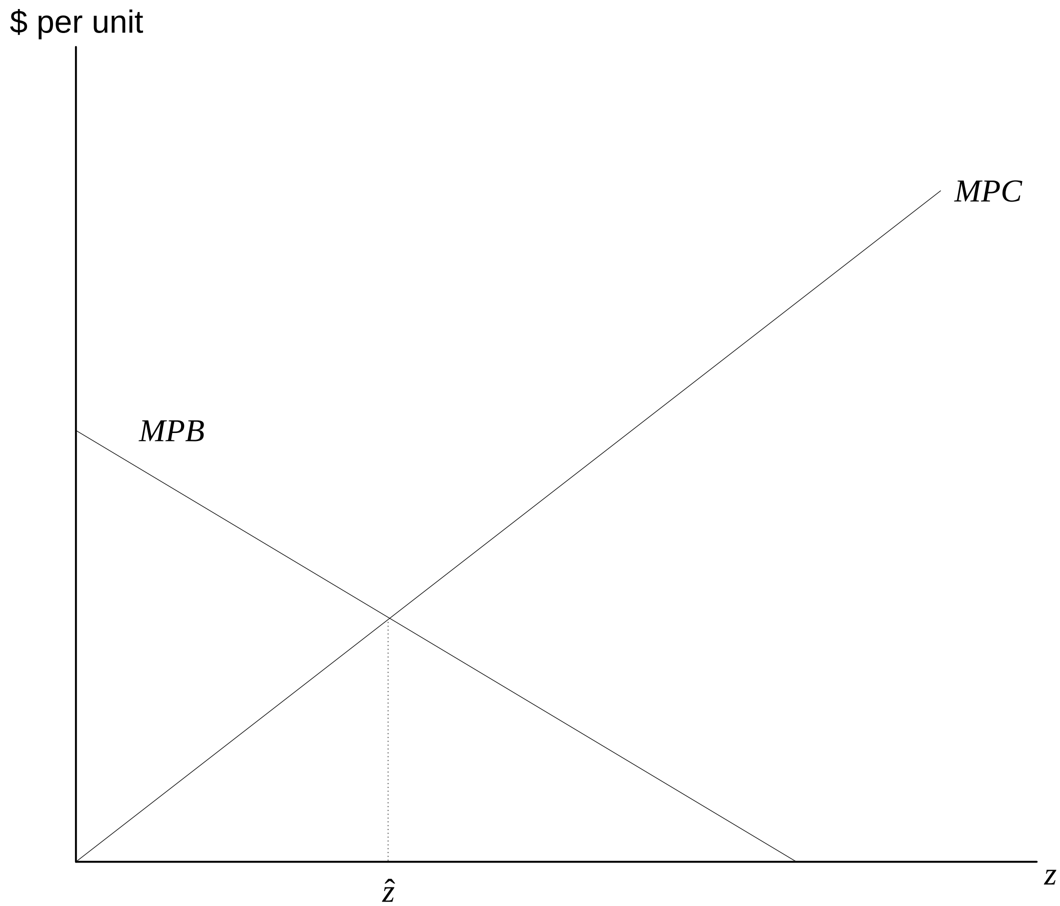


FIGURE 14.1

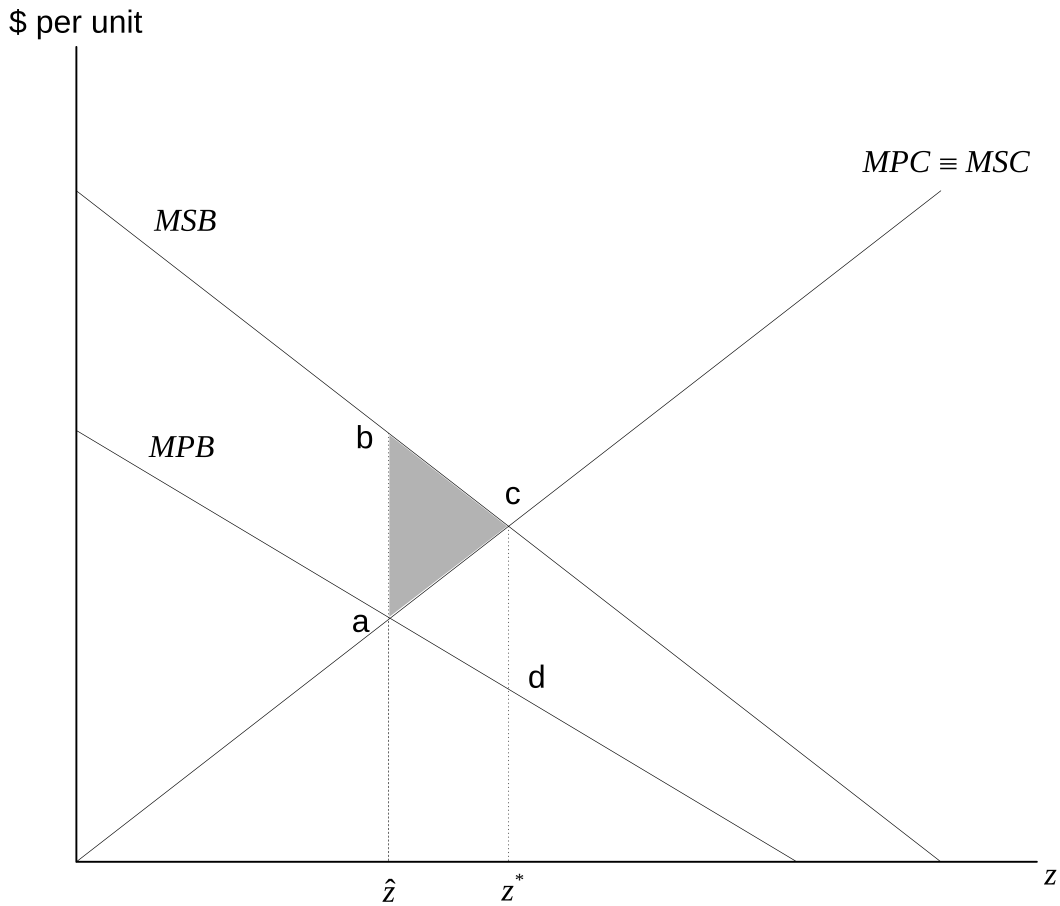


FIGURE 14.2

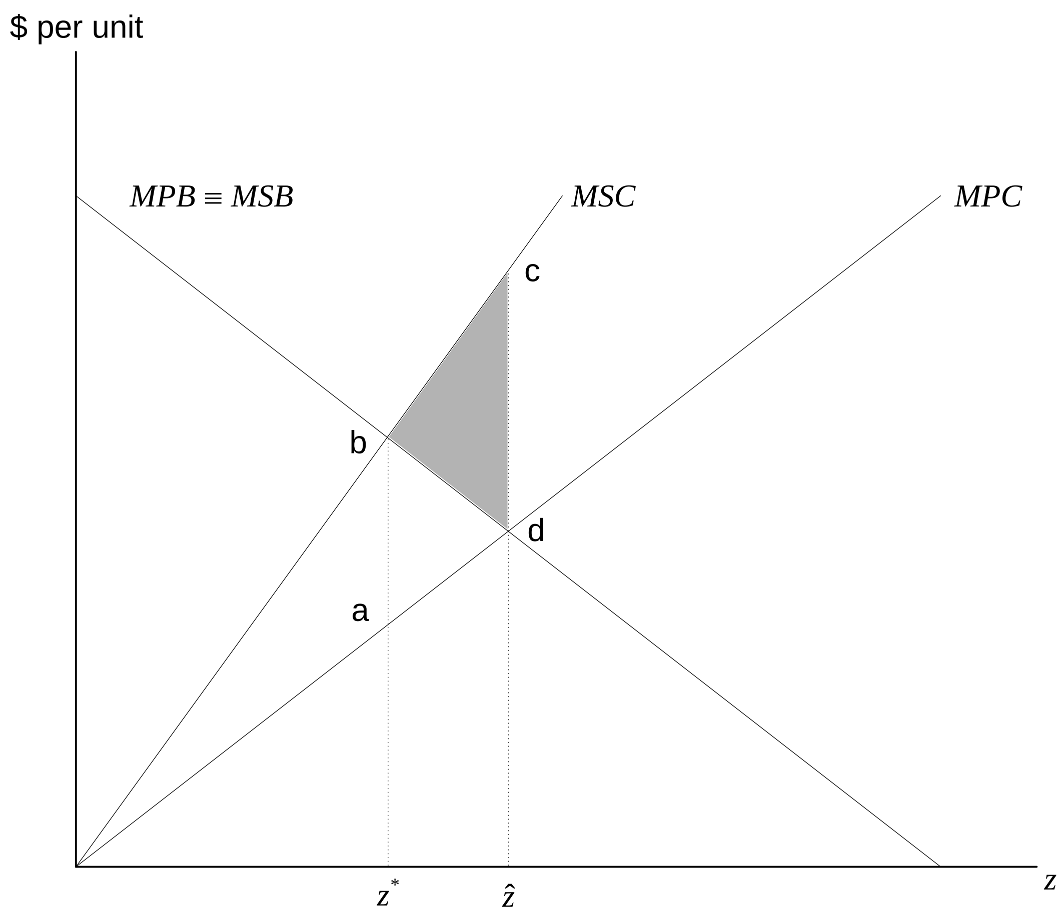


FIGURE 14.3

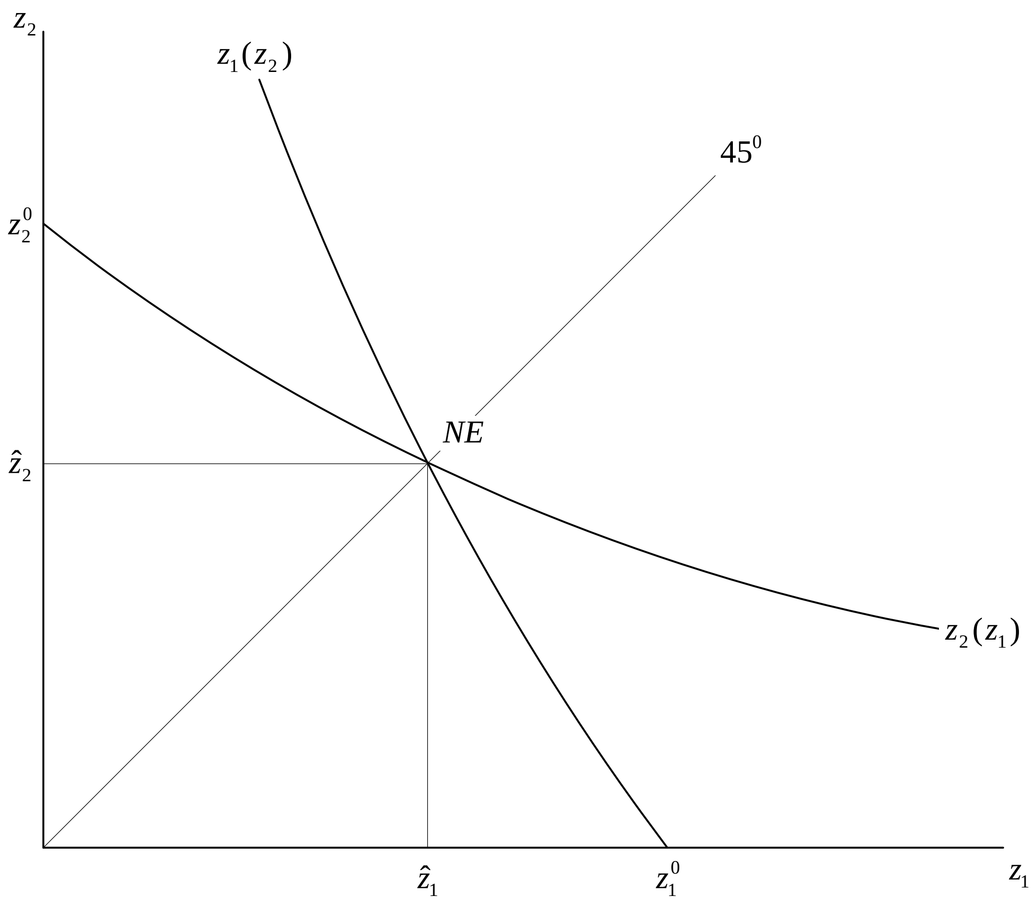


FIGURE 14.4

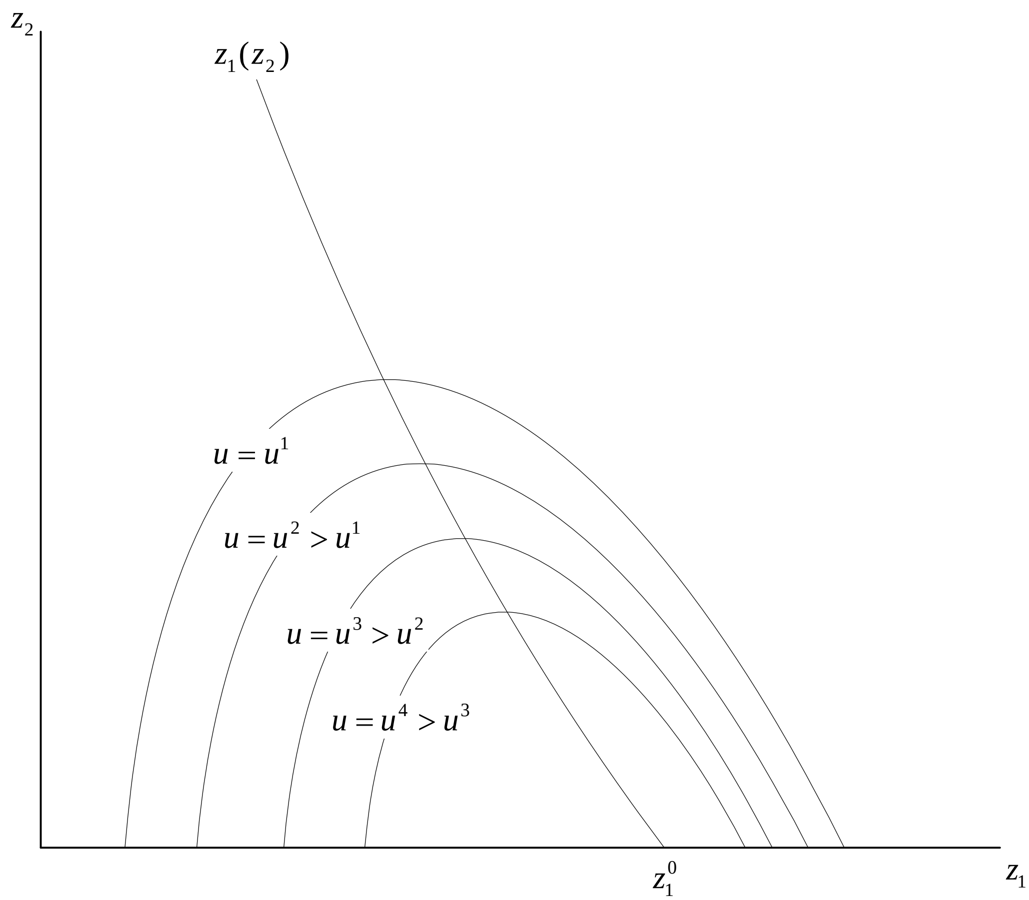


FIGURE 14.5

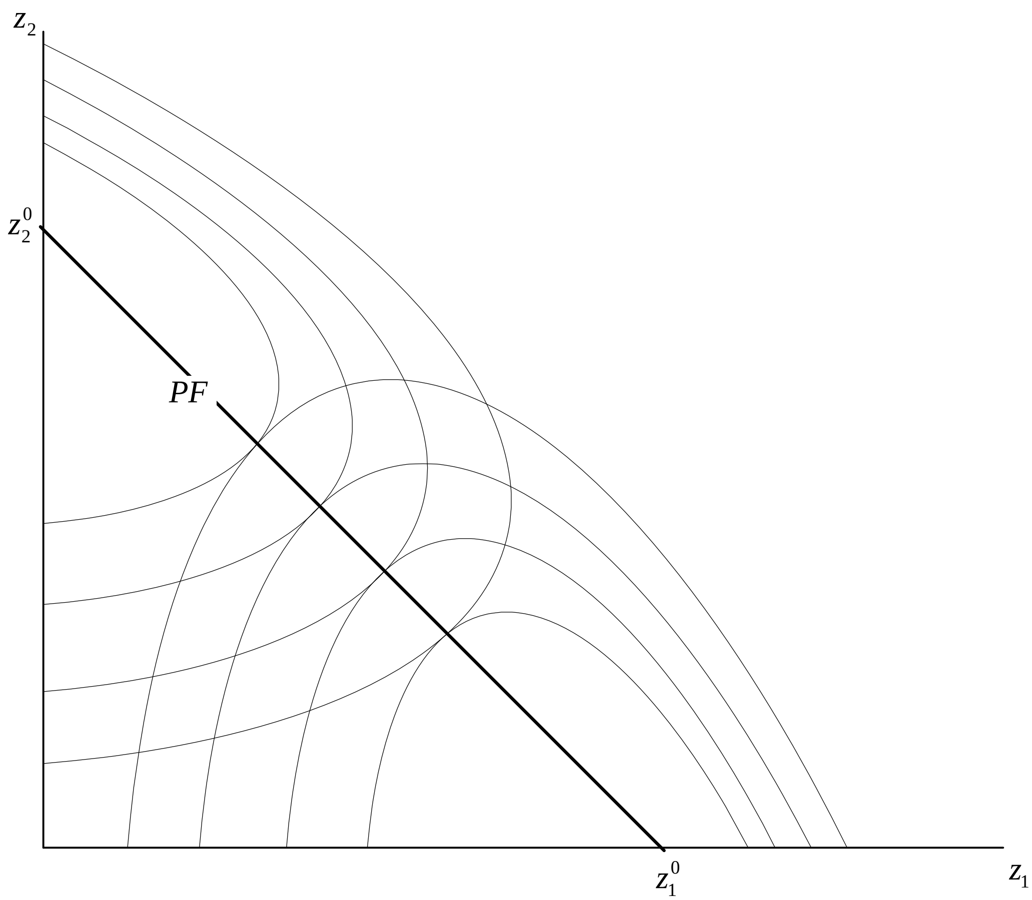


FIGURE 14.6

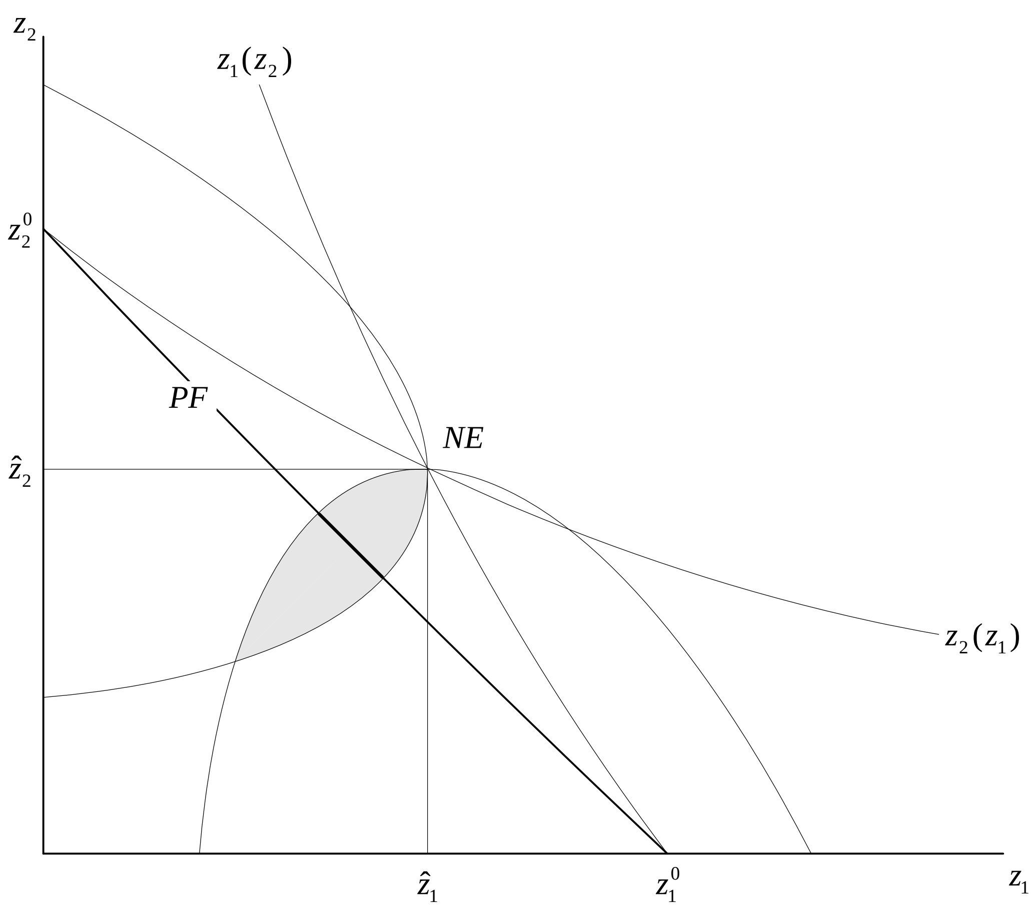


FIGURE 14.7

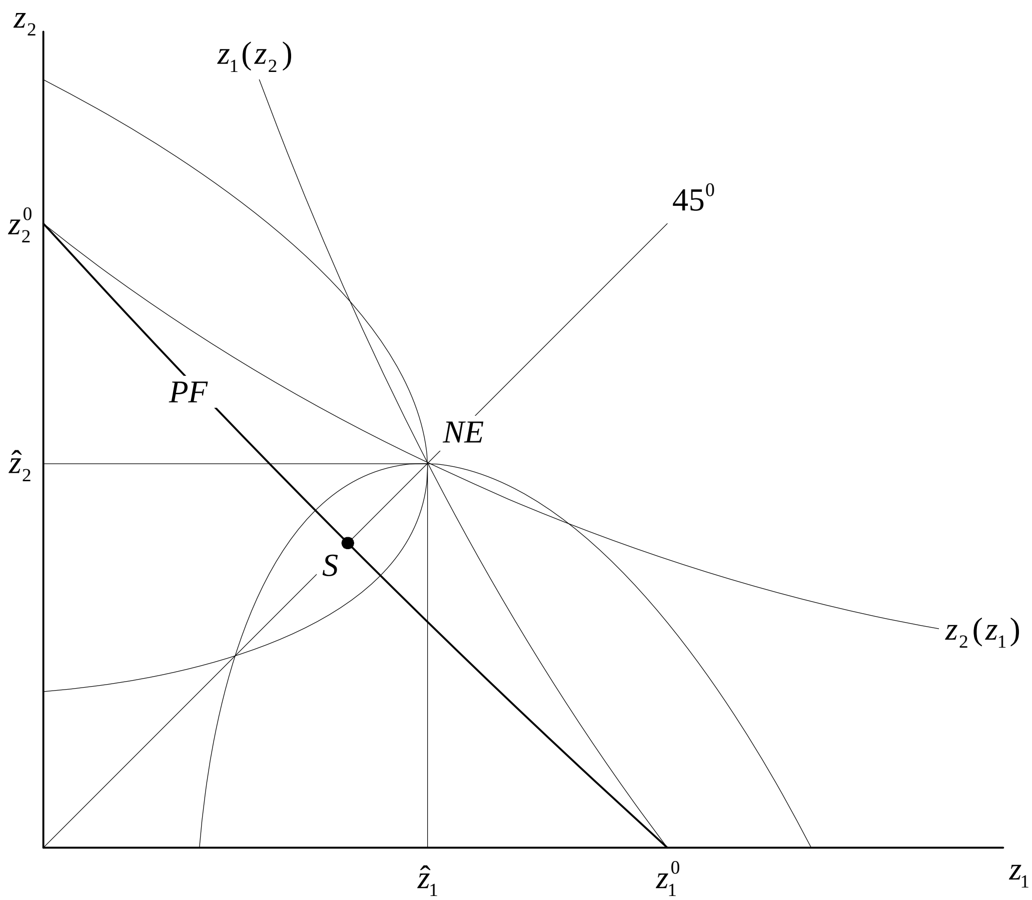


FIGURE 14.8

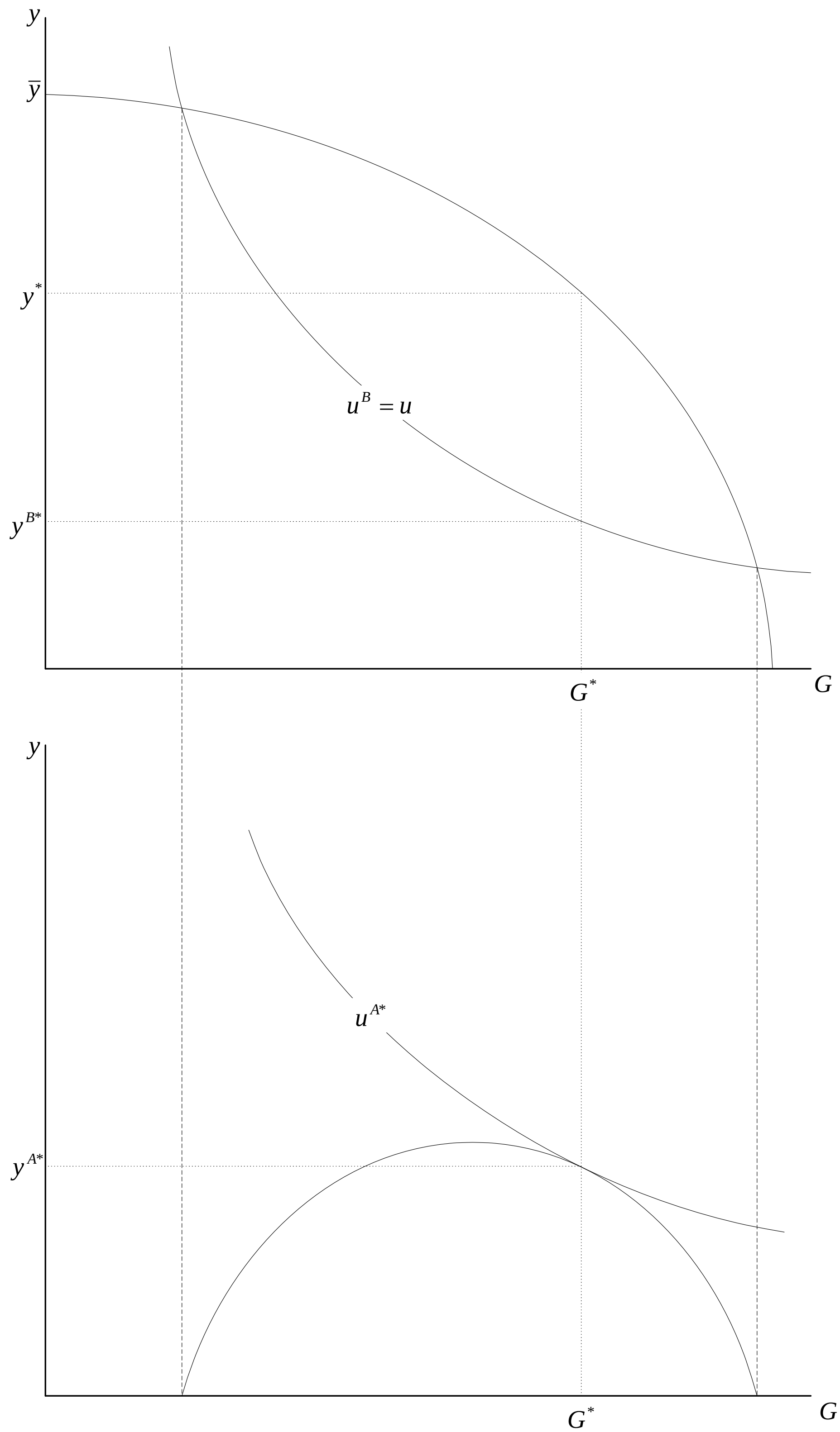


FIGURE 14.9

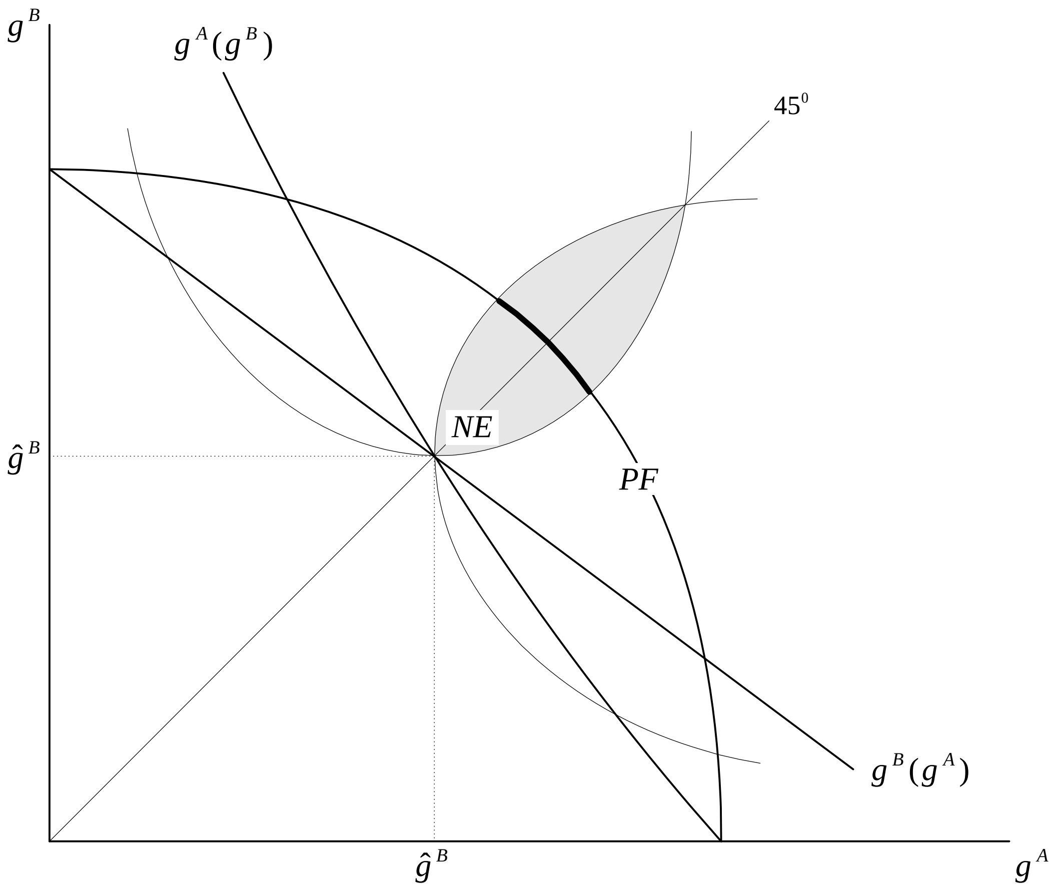


FIGURE 14.10

15. AN INTRODUCTION TO ASYMMETRIC INFORMATION

15.1 Introduction

Asymmetric information describes an economic environment in which one agent in a transaction has different information to the other agent in that transaction. There are two classes of asymmetric information problem: *adverse selection* and *moral hazard.*

(a) Adverse Selection

Consider a market for goods of variable quality where the seller of a particular good knows its quality but the buyer does not. The buyer will base her initial valuation of the good on the market-wide expected quality.

The seller of a high quality good, not being able to credibly convince the buyer that it is high quality and thereby charge a high price, may decide to retain the good rather than sell it at an average-quality price. Conversely, the seller of a low quality good will be happy to sell it at an average-quality price.

Thus, the market *adversely selects* the lowest quality goods for sale, even though there may be buyers and sellers who would mutually benefit from the sale of the high quality goods. Adverse selection therefore leads to a loss of social surplus relative to a setting with symmetric information.

Adverse selection can potentially lead to the collapse of the market: buyers know that only low quality sellers will be willing to sell, so when they see a good for sale they revise downward their beliefs about the quality; this drives out still more sellers whose quality is above the "revised" average, and the downward spiral continues.

(b) Moral Hazard

Consider an insurance market where a risk averse agent, faced with some uncertainty (such as the possibility of a house fire), buys insurance from a firm.

If the agent buys full insurance (to completely cover all loss) and her actions are unobservable to the firm then she has no incentive to take precautionary action to prevent the loss, even if such action is not very costly. Full insurance can therefore lead to an inefficiently low level of precautionary action.

In response to this problem the firm offers only partial insurance and so the agent is exposed to some risk; she must therefore take precautionary action anyway and so incurs both the cost of the action and the cost of the remaining risk.

The agent would be better-off by taking precautionary action and getting full insurance but the moral hazard makes this impossible. Consequently, there is foregone social surplus relative to a setting with symmetric information..

The same problem arises more generally in any *principal-agent problem*, where the payoff to the principal depends in an uncertain way on the action of the agent contracted to perform that action, and the principal can only base payment for the agent's services on the observed outcome (because the action itself is not observable).

15.2 The Market for "Lemons": An Example of Adverse Selection

Consider a product of quality *s*. Suppose the seller values the product at $\theta_1 s$ and the potential buyer values it at $\theta_2 s$. Assume $\theta_2 > \theta_1$. Thus, Pareto efficiency requires trade (regardless of quality).

Suppose the seller knows *s*, but the buyer does not. Thus, there is asymmetric information. The buyer has prior beliefs about *s* represented by a uniform distribution over the interval $[0, \bar{s}]$. Thus, the *prior expectation* on quality is

$$\mu = [\bar{s} + 0]/2 = \bar{s}/2$$

To simplify the analysis, suppose the buyer knows θ_1.

Suppose the product is offered for sale at price *p*. What should the buyer infer about *s*? If the seller is willing to sell at price *p*, it must be that $\theta_1 s \leq p$. Thus, the buyer can infer that $s \in [0, p/\theta_1]$. That is, the buyer revises her beliefs about quality in response to the observation that the product is being offered for sale at price *p*.

Conditional expected quality – that is, expected quality conditional on the product being offered for sale at price *p* – is

$$\hat{s}(p) = [p/\theta_1 + 0]/2 = p/2\theta_1$$

since beliefs are uniform. Recall that the *prior* expected quality is $\mu = \bar{s}/2$. Thus, $\hat{s}(p) \leq \mu$ if $p \leq \theta_1 \bar{s}$, and $\hat{s}(p) \geq \mu$ if $p \geq \theta_1 \bar{s}$.

The buyer will buy at price *p* if and only if

$$p \leq \theta_2 \hat{s}(p)$$

or, equivalently, if and only if $\theta_2 \geq 2\theta_1$. Thus, if valuations are not sufficiently different (that is, if $\theta_2 < 2\theta_1$), then there exists no price at which trade occurs, even though trade is always Pareto efficient.

15.3 The Spence Signaling Model: A Labour Market Example

There are three main mechanisms through which the market can potentially deal with problems of adverse selection: *warranties*, *reputation effects* (in a repeated interaction context such as repeat sales or word-of-mouth communication), and *signaling*. In this section we focus on signaling.

The Basic Model

Consider a situation where a worker obtains education level *e* and demands wage *w* from the employer. The firm accepts or refuses the demanded wage. Assume that education has no productivity effect (unlike a degree in economics).

The worker is one of two types: high productivity (*H*) or low productivity (*L*). The firm knows the true population distribution of workers. In particular, a fraction α of workers are of type *L*, and a fraction $1-\alpha$ are of type *H*.

For the worker, the cost of obtaining education level *e* is correlated with her productivity. In particular, effort cost of education level *e* is e/t, where $t = L$ or $t = H$. Thus, the net payoff to a worker of type *t* who obtains education level *e* and receives wage *w* is

$$u = w - \frac{e}{t}$$

A worker will undertake education only if $u \geq 0$. This means that a type *L* worker requires a higher wage to compensate for a given education level than does a type *H* worker. We will see below that this asymmetry between the *H* type and the *L* type creates the potential for the worker to signal her type via education level.

The firm accepts the wage demanded if and only if the wage does not exceed expected productivity; that is, if and only if $w \leq E(t/e)$. This means that any $w \leq L$ is always accepted, and any $w > H$ is always refused.

Key question of interest: can an *H* type ever convince the employer that she is an *H* type and so obtain $w = H$? She may be able to do so through her choice of *e*. That is, there may be an education level $\hat{e}$ that only an *H* type would be willing to undertake, which thereby signals that the worker must be of type *H*. To put this differently, there may be an education level $\hat{e}$ that allows an *H* type to *separate* herself from *L* types.

Separating Equilibria

We are looking for an education level $\hat{e}$ that convinces the employer that the worker is an *H* type, because the employer knows that only an *H* type would choose this education level. If there exists such a signal, then any worker who does not choose $\hat{e}$ will be viewed by the employer as an *L* type.

Thus, in a separating equilibrium (if one exists), the *H* type will choose $\hat{e}$ and receive wage $w = H$, and obtain net payoff

$$u^H = H - \frac{\hat{e}}{H}$$

and the L type will choose $e = 0$ and receive wage $w = L$, and obtain net payoff

$$u^L = L$$

If $\hat{e}$ is a separating equilibrium then it must be *incentive compatible* for both types:

- the *L* type must prefer her equilibrium strategy to any alternative strategy, including one where she mimics the *H* type; and
- the *H* type must prefer her equilibrium strategy to any alternative strategy, including one where she mimics the *L* type.

These incentive compatibility conditions are

(15.1) $$L \geq H - \frac{\hat{e}}{L} \quad \text{for the L type}$$

(15.2) $$H - \frac{\hat{e}}{H} \geq L \quad \text{for the H type}$$

Equation (15.1) requires $\hat{e} \geq L(H-L)$. Equation (15.2) requires $\hat{e} \leq H(H-L)$. These conditions can be mutually satisfied if and only if $H \geq L$. Since this condition holds, a separating equilibrium does exist in this example; education level can signal productivity.

15.4 Moral Hazard in Insurance

Suppose an agent has wealth w_1 in the good state, and wealth $w_2 < w_1$ if an accident occurs (the bad state). Let $\alpha(e)$ denote the probability of an accident, as a function of preventative effort *e*, where $\alpha'(e) < 0$.

Expected utility without insurance is

$$E^0u = \alpha(e)u(w_2 - e) + [1 - \alpha(e)]u(w_1 - e)$$

The agent chooses *e* to maximize this expected utility. The first-order condition is

$$\alpha'(e)u(w_2 - e) - \alpha(e)u'(w_2 - e) - \alpha'(e)u(w_1 - e) + [1 - \alpha(e)]u'(w_1 - e) = 0$$

This solves for the optimal preventative effort e^*. In general, $e^* > 0$.

Now suppose this agent can purchase full insurance for a total premium *p*. Then she has expected utility

$$E^1u = \alpha(e)u(w_1 - p - e) + [1 - \alpha(e)]u(w_1 - p - e) = u(w_1 - p - e)$$

She then chooses *e* to maximize this expected utility, with solution $\hat{e} = 0$. That is, having obtained full insurance, thereby eliminating all risk associated with an accident, she has no incentive to prevent an accident. This is the essence of the moral hazard problem.

A partial solution to this problem is *co-insurance*: a deductible of *x* is required in the event of a claim. Then her expected utility is

$$E^2u = \alpha(e)u(w_1 - p - e - x) + [1 - \alpha(e)]u(w_1 - p - e)$$

This restores some incentive to take preventative effort but the agent is now exposed to some risk. The first-best solution is e^* and full insurance, but this cannot be achieved in the face of the moral hazard problem.

PROBLEM SETS

PROBLEM SET 1

Coverage: Chapters 2 and 3

PS1 Question 1

A consumer has the following utility function:

$$u(x) = x_1^{1/2} + x_2^{1/2}$$

(a) Show that the expenditure function is given by

$$e(p,u) = \frac{p_1 p_2 u^2}{(p_1 + p_2)}$$

(b) Verify Shephard's lemma for x_1.

(c) Find the indirect utility function by inverting the expenditure function, and use Roy's identity to find the Marshallian demands.

PS1 Question 2

A consumer has the following utility function:

$$u(x) = -\left[\frac{1}{x_1} + \frac{1}{x_2}\right]$$

(a) Find the Marshallian demand functions.

(b) Find the indirect utility function and verify Roy's identity for x_1.

(c) Find the Hicksian demand functions.

(d) Find the expenditure function using two different methods.

PS1 Question 3

A consumer has the following utility function:

$$u(x) = \sum_{i=1}^{n} \beta_i \log(x_i - \gamma_i)$$

where $\beta_i > 0$ and $\gamma_i > 0 \ \forall i$.

(a) Find the Marshallian demand function for good *i*. What restriction must be placed on income to make this expression sensible?

(b) Provide an interpretation of γ_i in this utility function.

PS1 Question 4

A consumer has the following utility function:

$$u(x) = \min[x_1, x_2]$$

(a) Derive the Hicksian demands and explain their properties. Derive the expenditure function.

(b) Derive the Marshallian demands and the indirect utility function.

(c) Derive the cross-price elasticity of x_1 with respect to p_2.

PS1 Question 5

A consumer has the following utility function:

$$u(x) = \min_i \left[a_i x_i\right]$$

Note that this is a generalization of the Leontief function with *n* goods.

(a) Find the Marshallian demand for good *i*.

(b) Find the indirect utility function and the expenditure function.

(c) Find the Hicksian demand for good *i*. Explain why this is *not* a function of p_i.

PS1 Question 6

A consumer has the following expenditure function

$$e(p,u) = 2u(p_1 p_2)^{1/2}$$

(a) Show that the Hicksian demands are

$$h_1(p,u) = u\left(\frac{p_2}{p_1}\right)^{1/2} \quad \text{and} \quad h_2(p,u) = u\left(\frac{p_1}{p_2}\right)^{1/2}$$

(b) Derive the indirect utility function and use Roy's identity to show that the Marshallian demands are

$$x_1(p,m) = \frac{m}{2p_1} \quad \text{and} \quad x_2(p,m) = \frac{m}{2p_2}$$

(c) Verify the own-price Slutsky equation for these preferences.

PS1 Question 7

A consumer has the following expenditure function:

$$e(p,u) = u\sum_{i=1}^{n} p_i$$

(a) Derive the Hicksian demand for x_j. What sort of preferences underlie this expenditure function? Explain your answer.

(b) Derive the indirect utility function and use Roy's identity to derive the Marshallian demand for good x_j.

(c) Suppose $n = 2$. Draw an appropriate diagram to illustrate the substitution and income effects for a price rise for x_1, and explain the relationship between the substitution effect and the Hicksian demand.

PS1 Question 8

A consumer has the following utility function:

$$u(x) = \log x_1 + \log x_2$$

(a) Derive the Hicksian demands and explain their properties.

(b) Derive the expenditure function. Is it convex in prices?

(c) Are x_1 and x_2 complements or substitutes? Explain your answer.

PS1 Question 9

A consumer has preferences over three goods. Show that no more than two of these can be inferior.

PS1 Question 10

Show that if the income elasticities for a consumer are all equal and constant then they must all be equal to one.

PS1 Question 11

Show that Marshallian cross-price effects are equal for homothetic preferences.

SOLUTIONS TO PROBLEM SET 1

Answer to PS1 Question 1

(a) $min \;\; p_1 x_1 + p_2 x_2 \;\; st \;\; u = u(x)$

FOC: $p_i = \phi \dfrac{x_i^{-1/2}}{2}$

$$\Rightarrow \qquad \frac{x_2}{x_1} = \left(\frac{p_1}{p_2}\right)^2$$

$$\therefore \qquad u = x_1^{1/2} + \left(x_1\left(\frac{p_1}{p_2}\right)^2\right)^{1/2}$$

$$\Rightarrow \qquad x_1(p,u) = u^2\left(\frac{p_2}{p_1 + p_2}\right)^2 \text{ and } x_2(p,u) = u^2\left(\frac{p_1}{p_1 + p_2}\right)^2$$

$$e(p,u) = p_1 x_1(p,u) + p_2 x_2(p,u)$$

$$\therefore \qquad e(p,u) = \frac{p_1 p_2 u^2}{(p_1 + p_2)}$$

(b) Shephard's lemma:

$$x_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

$$\therefore \qquad x_1(p,u) = u^2\left(\frac{p_2}{p_1 + p_2}\right)^2$$

(c) Set $e(p,u) = m$ and solve for *u*:

$$\Rightarrow \qquad u \equiv v(p,m) = \left(\frac{(p_1 + p_2)m}{p_1 p_2}\right)^{1/2}$$

Roy's identity:

$$x_i(p,m) = \frac{-\partial v / \partial p_i}{\partial v / \partial m}$$

$$\Rightarrow \qquad x_i(p,m) = \frac{mp_j}{p_i(p_i + p_j)}$$

$$\therefore \qquad x_1(p,m) = \frac{mp_2}{p_1(p_1 + p_2)} \text{ and } x_2(p,m) = \frac{mp_1}{p_2(p_2 + p_1)}$$

Answer to PS1 Question 2

(a) *max u(x) st* $p_1 x_1 + p_2 x_2 = m$

$$\text{FOC:} \quad \frac{1}{x_i^2} = \lambda p_i$$

$$\Rightarrow \qquad \frac{x_1^2}{x_2^2} = \frac{p_2}{p_1}$$

$$\Rightarrow \qquad x_1 = x_2 \left(\frac{p_2}{p_1}\right)^{1/2}$$

$$\therefore \qquad m = p_1 x_2 \left(\frac{p_2}{p_1}\right)^{1/2} + p_2 x_2$$

$$\Rightarrow \qquad x_2(p,m) = \frac{m}{p_2 + (p_1 p_2)^{1/2}} \text{ and } x_1(p,m) = \frac{m}{p_1 + (p_2 p_1)^{1/2}}$$

(b) $v(p,m) = u(x(p,m))$

$$\therefore \qquad v(p,m) = -\left[\frac{1}{x_1(p,m)} + \frac{1}{x_2(p,m)}\right]$$

$$\Rightarrow \qquad v(p,m) = \frac{-\left[2(p_1 p_2)^{1/2} + p_1 + p_2\right]}{m}$$

Roy's identity:

$$x_i(p,m) = \frac{-\partial v / \partial p_i}{\partial v / \partial m}$$

$$\Rightarrow \qquad x_1(p,m) = \frac{m}{p_1 + (p_2 p_1)^{1/2}}$$

(c) $min \;\; p_1 x_1 + p_2 x_2 \;\; st \;\; u = u(x)$

FOC: $p_i = \dfrac{\phi}{x_i^2}$

$$\Rightarrow \qquad \frac{p_1}{p_2} = \frac{x_2^2}{x_1^2}$$

$$\Rightarrow \qquad x_1 = \left(\frac{p_2}{p_1}\right)^{1/2} x_2$$

Substitute into *u(x)* and solve for x_2 and then x_1:

$$x_2(p,u) = \frac{-(p_1^{1/2} + p_2^{1/2})}{up_2^{1/2}} \quad \text{and} \quad x_1(p,u) = \frac{-(p_1^{1/2} + p_2^{1/2})}{up_1^{1/2}}$$

(d) <u>Method 1</u>: direct substitution for the Hicksian demands

$$e(p,u) = p_1 x_1(p,u) + p_2 x_2(p,u)$$

$$\therefore \qquad e(p,u) = \frac{-\left[2(p_1 p_2)^{1/2} + p_1 + p_2\right]}{u}$$

<u>Method 2</u>: invert the indirect utility function. Set $v(p,m) = u$ and solve for *m*:

$$m \equiv e(p,u) = \frac{-\left[2(p_1 p_2)^{1/2} + p_1 + p_2\right]}{u}$$

Answer to PS1 Question 3

(a) *max u(x) st* $p_1 x_1 + p_2 x_2 = m$

$$\text{FOC:} \quad \frac{\beta_i}{(x_i - \gamma_i)} = \lambda p_i$$

Rearrange and multiply by p_i to obtain:

$$p_i x_i = \left(\frac{\beta_i}{\lambda}\right) + p_i \gamma_i$$

Sum across *i* and set equal to *m*. Then solve for λ:

$$\lambda = \frac{\sum \beta_i}{(m - \sum p_i \gamma_i)}$$

Then use FOC to solve for a particular x_j:

$$x_j(p, m) = \gamma_j + \frac{\beta_j (m - \sum p_i \gamma_i)}{p_j \sum \beta_i}$$

Required restriction: $m \geq \sum p_i \gamma_i$.

(b) γ_j is the minimum amount of x_j that the consumer needs to survive. Any amount less than this yields $-\infty$ utility (that is, death).

Answer to PS1 Question 4

(a) At any prices, expenditure is minimized where $x_1 = x_2$. Thus, the Hicksian demands are simply given by

$$h_1(p,u) = u$$

$$h_2(p,u) = u$$

The key properties of the Hicksians are

(i) negativity

$$\frac{\partial h_1}{\partial p_1} = 0 \leq 0 \quad \text{and similarly for } h_2(p,u)$$

(ii) symmetry

$$\frac{\partial h_1}{\partial p_2} = 0 = \frac{\partial h_2}{\partial p_1}$$

(iii) homogeneity

$$h_1(tp,u) = u = t^0 h_1(p,u) \quad \text{and similarly for } h_2(p,u)$$

The expenditure function is

$$e(p,u) = p_1 h_1(p,u) + p_2 h_2(p,u) = u(p_1 + p_2)$$

Note that this is weakly concave in *p*.

(b) At any prices, utility is maximized where $x_1 = x_2$. The constraint is then used to solve for Marshallian demands:

$$x_1(p,u) = \frac{m}{p_1 + p_2}$$

$$x_2(p,u) = \frac{m}{p_1 + p_2}$$

The indirect utility function is

$$v(p,m) = \min[x_1(p,m), x_2(p.m)] = \frac{m}{p_1 + p_2}$$

Note that this is convex in *p*.

(c) $$\varepsilon_{12} = \frac{\partial x_1(p,u)}{\partial p_2}\frac{p_2}{x_1} = -\frac{p_2}{p_1 + p_2}$$

Answer to PS1 Question 5

(a) At the maximum

$$a_1 x_1 = a_2 x_2 = ... = a_n x_n$$

Then express $x_2, x_3, \dots, x_n$ all in terms of x_1. That is:

$$x_j = \frac{a_1 x_1}{a_j} \quad \text{for} \quad j = 2, \dots, n$$

Then substitute into the budget constraint

$$m = p_1 x_1 + \sum_{j=2}^{n} p_j x_j$$

to obtain

$$m = x_1\left(p_1 + \sum_{j=2}^{n} p_j\left(\frac{a_1}{a_j}\right)\right)$$

Solve for x_1:

$$x_1(p,m) = \frac{m}{\left(p_1 + \sum_{j=2}^{n} p_j\left(\frac{a_1}{a_j}\right)\right)}$$

In general:

$$x_i(p,m) = \frac{m}{\left(p_i + \sum_{j\neq i}^{n} p_j\left(\frac{a_i}{a_j}\right)\right)}$$

which can be written more succinctly as

$$x_i(p,m) = \frac{m}{\left(\sum_{j=1}^{n} p_j\left(\frac{a_i}{a_j}\right)\right)}$$

(b) $v(p,m) = u(x(p,m))$

$$\therefore \qquad v(p,m) = \min_i\left[a_i x_i(p,m)\right]$$

Since $a_i x_i = a_1 x_1 \;\; \forall\, i$, without loss of generality we can write

$$v(p,m) = \frac{a_1 m}{\left(\sum_{j=1}^{n} p_j\left(\frac{a_1}{a_j}\right)\right)} = \frac{m}{\sum_{j=1}^{n}\left(\frac{p_j}{a_j}\right)}$$

Derive the expenditure function by inverting $v(p,m)$:

$$e(p,u) = u\sum_{j=1}^{n}\left(\frac{p_j}{a_j}\right)$$

(c) Shephard's lemma:

$$x_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

$$\Rightarrow \qquad x_i(p,u) = \frac{u}{a_i}$$

This is independent of p_i because there is no substitution effect for Leontief preferences.

Answer to PS1 Question 6

(a) By Shephard's lemma,

$$h_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

(b) Invert the expenditure function to obtain the indirect utility function:

$$v(p,m) = \frac{m}{2(p_1 p_2)^{1/2}}$$

Then by Roy's identity,

$$x_i(p,m) = -\frac{\partial v}{\partial p_i} \Big/ \frac{\partial v}{\partial m}$$

(c) The own-price Slutsky equation is

$$\frac{\partial x_i}{\partial p_i} = \frac{\partial h_i}{\partial p_i} - x_i \frac{\partial x_i}{\partial m}$$

The LHS is

$$\frac{\partial x_i}{\partial p_i} = -\frac{m}{2p_i^2}$$

The RHS components are (i) and (ii) as follows:

(i) $$\frac{\partial h_i}{\partial p_i} = -\frac{u p_j^{1/2}}{2 p_i^{3/2}}$$

Since the Slutsky equation holds for all *u*, it must hold at $u = v(p,m)$. Make this substitution to obtain

$$\frac{\partial h_i}{\partial p_i} = -\left(\frac{m}{2(p_i p_j)^{1/2}}\right)\frac{p_j^{1/2}}{2p_i^{3/2}} = -\frac{m}{4p_i^2}$$

(ii) $$x_i \frac{\partial x_i}{\partial m} = \left(\frac{m}{2p_i}\right)\left(\frac{1}{2p_i}\right) = \frac{m}{4p_i^2}$$

Thus, we have for the RHS:

$$-\frac{m}{4p_i^2} - \frac{m}{4p_i^2} = -\frac{m}{2p_i^2}$$

Answer to PS1 Question 7

(a) By Shephard's lemma:

$$h_j(p,u) = \frac{\partial e(p,u)}{\partial p_j} = u$$

This Hicksian demand is independent of *p*; thus, there is no substitution effect. This is true for all goods; the preferences must therefore by Leontief.

(b) Set $e(p,u) = m$ and solve for *u*:

$$v(p,m) = \frac{m}{\sum_{i=1}^{n} p_i}$$

By Roy's identity:

$$x_j(p,m) = \frac{m}{\left(\sum_{i=1}^{n} p_i\right)^2} \Bigg/ \frac{1}{\sum_{i=1}^{n} p_i} = \frac{m}{\sum_{i=1}^{n} p_i}$$

(c) The Hicksian demand measures the substitution effect but in this case that effect is zero. See Figure P1.1

Answer to PS1 Question 8

(a) This is simpler to solve if $u(x)$ is first transformed to

$$u(x) = x_1 x_2$$

Set up the expenditure minimization problem:

$$\min_x \; p_1 x_1 + p_2 x_2 \quad \text{s.t.} \quad x_1 x_2 = u$$

The FOCs yield the tangency condition:

$$\frac{x_2}{x_1} = \frac{p_1}{p_2}$$

The constraint is then used to solve for Hicksian demands:

$$h_1(p,u) = \left(\frac{up_2}{p_1}\right)^{1/2}$$

$$h_2(p,u) = \left(\frac{up_1}{p_2}\right)^{1/2}$$

Note that if the problem is solved *without* transformation then the solutions are

$$h_1(p,u) = \left(\frac{\exp[u]p_2}{p_1}\right)^{1/2}$$

$$h_2(p,u) = \left(\frac{\exp[u]p_1}{p_2}\right)^{1/2}$$

These are economically equivalent to those derived from the transformed problem since u and $\exp[u]$ are simply monotonic transformations of each other, and utility has no cardinal meaning.

The key properties of the Hicksians are:

(i) negativity

$$\frac{\partial h_1}{\partial p_1} = -\frac{1}{2}\left(\frac{up_2}{p_1^3}\right)^{1/2} < 0 \quad \text{and similarly for } h_2(p,u)$$

(ii) symmetry

$$\frac{\partial h_1}{\partial p_2} = \frac{1}{2}\left(\frac{u}{p_1 p_2}\right)^{1/2} = \frac{\partial h_2}{\partial p_1}$$

(iii) homogeneity

$$h_1(tp,u) = \left(\frac{utp_2}{tp_1}\right)^{1/2} = t^0 h_1(p,u) \text{ and similarly for } h_2(p,u)$$

(b) The expenditure function is

$$e(p,u) = p_1 h_1(p,u) + p_2 h_2(p,u) = 2(up_1 p_2)^{1/2}$$

Note that this is <u>concave</u> in *p*. (The Hessian matrix is negative definite).

(c) They are neither substitutes nor complements; the <u>Marshallian</u> cross-price effects are zero.

Answer to PS1 Question 9

By Engel aggregation:

$$p_1\left(\frac{\partial x_1}{\partial m}\right) = 1 - \left[p_2\left(\frac{\partial x_2}{\partial m}\right) + p_3\left(\frac{\partial x_3}{\partial m}\right)\right]$$

If x_2 and x_3 are both inferior then the RHS must be positive. Hence, the LHS must also be positive, which means that x_1 must be normal.

Answer to PS1 Question 10

Express Engel aggregation in elasticity form:

$$\sum_{i=1}^{n} w_i \eta_i = 1$$

where $w_i = \frac{p_i x_i}{m}$ is the "expenditure share" for good *i*. Then let $\eta_i = \eta \ \ \forall i$. Thus,

$$\eta \sum_{i=1}^{n} w_i = 1$$

Since $\sum_{i=1}^{n} w_i = 1$, it follows that $\eta = 1$.

Answer to PS1 Question 11

If preferences are homothetic then $u(tx) = tu(x)$. It follows that $e(p,u) = a(p)u$. Then derive the indirect utility function by inverting the expenditure function, and writing it in the form

$$v(p,m) = b(p)m$$

Now invoke Roy's identity:

$$x_i(p,m) = -\frac{\partial v}{\partial p_i} \Big/ \frac{\partial v}{\partial m} = -m\frac{\partial b}{\partial p_i} \Big/ b(p)$$

and

$$x_j(p,m) = -\frac{\partial v}{\partial p_j} \Big/ \frac{\partial v}{\partial m} = -m\frac{\partial b}{\partial p_j} \Big/ b(p)$$

Then by the quotient rule for differentiation:

$$\frac{\partial x_i}{\partial p_j} = -\frac{b(p)m\dfrac{\partial^2 b}{\partial p_i \partial p_j} - m\dfrac{\partial b}{\partial p_i}\dfrac{\partial b}{\partial p_j}}{b(p)^2}$$

and

$$\frac{\partial x_j}{\partial p_i} = -\frac{b(p)m\dfrac{\partial^2 b}{\partial p_j \partial p_i} - m\dfrac{\partial b}{\partial p_j}\dfrac{\partial b}{\partial p_i}}{b(p)^2}$$

By Young's theorem, these two expressions are equal.

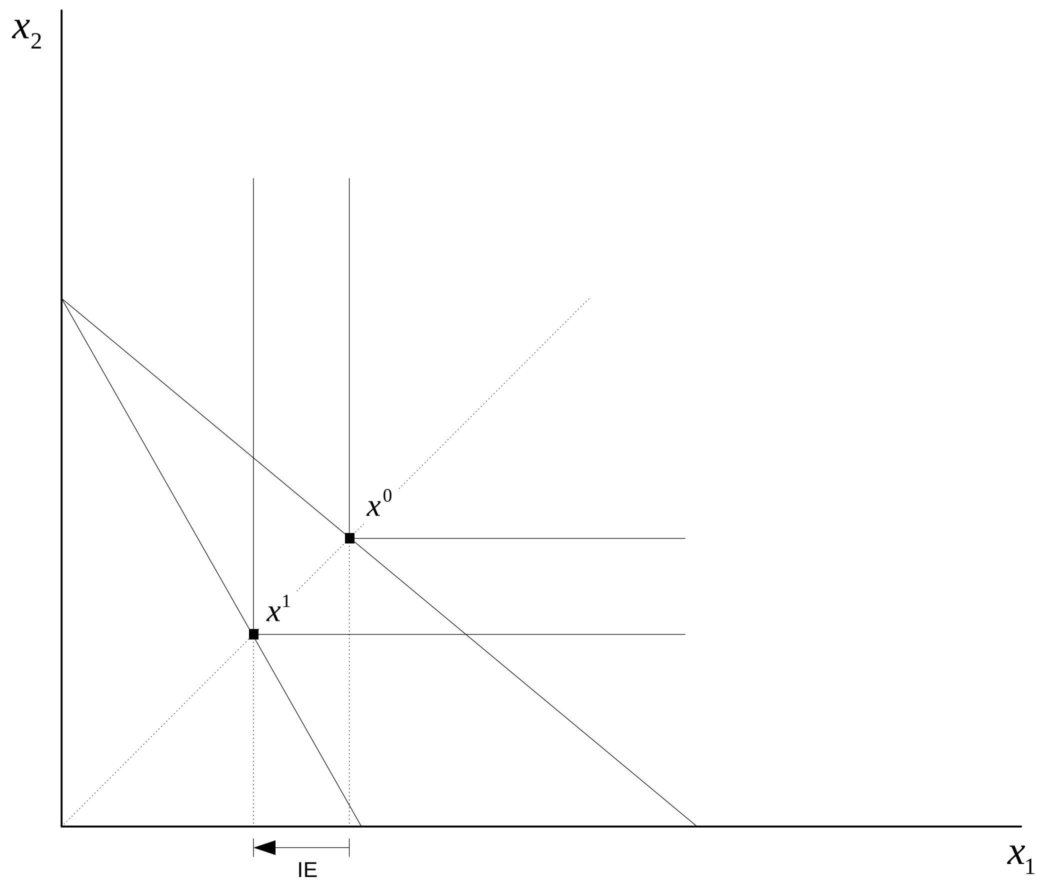


FIGURE P1.1

PROBLEM SET 2

Coverage: Chapters 4 – 6

PS2 Question 1

A consumer has the following utility function:

$$u(x) = \log x_1 + x_2$$

(a) Find the Marshallian demands and discuss the income elasticity of the demand for x_1.

Suppose she has income $m = 7$, the price of x_2 is $p_2 = 1$, and the price of x_1 falls from $p_1^0 = 1$ to $p_1^1 = 0.25$.

(b) Calculate the compensating and equivalent variations associated with the price change. Explain the relationship between your answers.

(c) Calculate the change in consumer surplus associated with the price change. Explain the relationship between this and your answers to part (b).

PS2 Question 2

A consumer has the following utility function:

$$u(x) = \min[a_1 x_1, a_2 x_2]$$

(a) Find the expenditure function.

(b) Suppose prices change from $\{p_{1,}^0 p_2^0\} = \{10{,}5\}$ to $\{p_{1,}^1 p_2^1\} = \{5{,}10\}$. Find the associated compensating and equivalent variations (as functions of *m*).

(c) Is the agent better-off or worse-off as a result of the price changes? Explain your answer with the aid of appropriate diagrams.

(d) Explain the relationship between the two measures of welfare change in this example.

PS2 Question 3

A consumer has the following indirect utility function:

$$v(p,m)=\left(\frac{(p_1+p_2)m}{p_1 p_2}\right)^{1/2}$$

(a) Find the Marshallian demand functions.

(b) Is x_1 a luxury good? Is the demand for x_1 price-inelastic? Explain your answers.

(c) Derive the expenditure function and find the Hicksian demand functions.

(d) Suppose prices change from $\{p_{1,}^0 p_2^0\}=\{2,2\}$ to $\{p_{1,}^1 p_2^1\}=\{1,3\}$. Find the associated compensating and equivalent variations (as functions of *m*). Is the agent better-off or worse-off as a result of the price changes?

PS2 Question 4

An agent has the following utility function:

$$u(x)=x_1^{1/2}+x_2^{1/2}$$

(a) Show that the indirect utility function is

$$v(p,m)=\left(\frac{(p_1+p_2)m}{p_1 p_2}\right)^{1/2}$$

Suppose $p_1=2$ and $p_2=2$. The agent's current income is $100,000 but there is a 5% chance that she will become unemployed in which case her income will fall to zero.

(b) Find the certainty-equivalent income level and the risk premium associated with this prospect.

(c) Suppose the government introduces an income insurance program that restores income to 64% of its previous level in the event of unemployment. Under this program each worker must pay a tax of $19 for every $100 of income earned while employed. Is the agent made better-off by the introduction of this program? (Assume that the introduction of the program does not change the probability of unemployment).

PS2 Question 5

An agent has the following intertemporal utility function:

$$u(c_1, c_2) = \log c_1 + \beta \log c_2$$

where c_1 is current consumption and c_2 is future consumption. The agent has income profile $\{y_1, y_2\}$.

(a) Suppose the agent can borrow and lend at interest rate *r*. Let

$$w \equiv y_1 + \frac{y_2}{1+r}$$

denote her lifetime wealth. Show that her consumption when young will be

$$c_1^* = \frac{w(1+\rho)}{(2+\rho)}$$

where $\frac{1}{1+\rho} = \beta$. Determine and explain the signs of $\partial c_1^* / \partial \rho$ and $\partial c_1^* / \partial r$.

(b) Show that she will be a lender when young if and only if

$$\frac{y_1}{y_2} > \frac{1+\rho}{1+r}$$

(c) Now suppose that interest earned on savings is subject to taxation at rate *t* but interest paid on borrowing is not tax deductible. Show that if the condition in part (b) holds then the tax causes her consumption when young to rise, but if the condition in part (b) does not hold then the tax has no effect on her consumption in either period. (Hint: draw a diagram and think about it carefully before doing any mathematics).

SOLUTIONS TO PROBLEM SET 2

Answer to PS2 Question 1

(a) $max \ \log x_1 + x_2 \quad st \quad p_1 x_1 + p_2 x_2 = m$

FOC: $\frac{1}{x_1} - \lambda p_1 = 0$

$1 - \lambda p_2 = 0$

$\Rightarrow \qquad x_1(p,m) = \frac{p_2}{p_1}$

$x_2(p,m) = \frac{m - p_2}{p_2}$ if $p_2 < m$, and zero otherwise

The demand for x_1 is income-neutral.

(b) Given the income and prices stated in the question we can restrict attention to the case where $p_2 < m$. So the indirect utility function is

$$v(p,m) = \log\left(\frac{p_2}{p_1}\right) + \left(\frac{m - p_2}{p_2}\right)$$

Invert this to find the expenditure function:

$$e(p,u) = p_2\left[u + 1 - \log\left(\frac{p_2}{p_1}\right)\right]$$

Compensating variation:

$$CV = m - e(p^1, u^0) = m - e(p^1, v(p^0, m))$$

where we use $v(p,m)$ evaluated at p^0 to find u^0. Thus,

$$CV = 7 - 1[(\log(1) + 6) + 1 - \log(4)] = \log(4)$$

Equivalent variation:

$$EV = e(p^0, u^1) - m = e(p^0, v(p^1, m)) - m$$

where we use $v(p,m)$ evaluated at p^1 to find u^1. Thus,

$$EV = 7 - 1[(\log(4) + 6) + 1 - \log(1)] = \log(4)$$

Note that *EV* and *CV* are equal here only because the good whose price has changed is income neutral.

(c) Change in consumer surplus:

$$\Delta CS = \int_{p_1^1}^{p_1^0} x_1(p,m)dp_1$$

In this case,

$$\Delta CS = \int_{p_1^1}^{p_1^0} \left(\frac{p_2}{p_1}\right) dp_1 = \left[\log(p_1)\right]_{0.25}^{1} = [\log(1) - \log(0.25)] = \log\left(\tfrac{1}{0.25}\right) = \log(4)$$

where the integral has been evaluated at $p_2 = 1$. $\Delta CS = CV = EV$ because the good whose price has changed in income neutral. Note that this equivalence would break down if the prices of both goods changed.

Answer to PS2 Question 2

(a) $min \quad p_1x_1 + p_2x_2 \quad st \quad \min[a_1x_1, a_2x_2] = u$

$$\Rightarrow \qquad x_1(p,u) = \frac{u}{a_1} \text{ and } x_2(p,u) = \frac{u}{a_2}$$

$$\Rightarrow \qquad e(p,u) = u\left[\frac{p_1}{a_1} + \frac{p_2}{a_2}\right] = u\left[\frac{a_1p_2 + a_2p_1}{a_1a_2}\right]$$

(b) By inversion, the indirect utility function is

$$v(p,m) = m\left[\frac{a_1a_2}{a_1p_2 + a_2p_1}\right]$$

Thus,

$$u^0 = v(p^0, m) = m\left[\frac{a_1a_2}{5a_1 + 10a_2}\right]$$

and

$$u^1 = v(p^1, m) = m\left[\frac{a_1a_2}{10a_1 + 5a_2}\right]$$

Compensating variation:

$$CV = m - e(p^1, u^0) = m - e(p^1, v(p^0, m))$$

where we use $v(p,m)$ evaluated at p^0 to find u^0. Thus,

$$CV = m - u^0\left[\frac{10a_1 + 5a_2}{a_1a_2}\right] = m\left[1 - \left(\frac{10a_1 + 5a_2}{5a_1 + 10a_2}\right)\right] = m\left[\frac{5(a_2 - a_1)}{5a_1 + 10a_2}\right]$$

Equivalent variation:

$$EV = e(p^0, u^1) - m = e(p^0, v(p^1, m))$$

where we use $v(p,m)$ evaluated at p^1 to find u^1. Thus,

$$EV = u^1\left[\frac{5a_1 + 10a_2}{a_1a_2}\right] - m = m\left[\left(\frac{5a_1 + 10a_2}{10a_1 + 5a_2}\right) - 1\right] = m\left[\frac{5(a_2 - a_1)}{10a_1 + 5a_2}\right]$$

(c) The agent is better-off if $a_2 > a_1$ and worse-off if $a_1 > a_2$. Why? If $a_2 > a_1$ then the consumption of x_1 (yes, x_1), is more limiting than the consumption of x_2 in the determination of utility. Thus, the reversal in the relative prices of the two goods, with x_1 becoming relatively less expensive, raises welfare. (See Figure P2.1). The converse is true if $a_1 > a_2$.(See Figure P2.2)

(d) $CV > EV$ if $a_1 > a_2$, and $EV > CV$ if $a_2 > a_1$. Why? Both goods are normal goods and one price has risen while the other has fallen, so there is ambiguity about the ranking of *EV* and *CV*. It depends on the relative magnitudes of a_1 and a_2. If $a_1 > a_2$ then the income effect is greater for x_2 (the good whose price has risen) than for x_1 (the good whose price has fallen) and so $CV > EV$. The converse is true if $a_2 > a_1$. Note that there are no substitution effects due to the nature of the preferences.

Answer to PS2 Question 3

(a) Roy's identity:

$$x_i(p,m) = \frac{-\partial v / \partial p_i}{\partial v / \partial m}$$

$$\Rightarrow \qquad x_i(p,m) = \frac{mp_j}{p_i(p_i + p_j)}$$

$$\therefore \qquad x_1(p,m) = \frac{mp_2}{p_1(p_1 + p_2)} \quad \text{and} \quad x_2(p,m) = \frac{mp_1}{p_2(p_2 + p_1)}$$

(b) A luxury good is one for which $\eta_i > 1$, where

$$\eta_i = \left[\frac{\partial x_i}{\partial m}\right]\left[\frac{m}{x_i}\right]$$

In this case, $\eta_1 = 1$. Thus, x_1 is not a luxury. The demand for x_1 is price-inelastic if $|\varepsilon_{11}| < 1$, where

$$\varepsilon_{11} = \left[\frac{\partial x_1}{\partial p_1}\right]\left[\frac{p_1}{x_1}\right]$$

In this case,

$$|\varepsilon_{11}| = \frac{2p_1 + p_2}{p_1 + p_2} > 1 \text{ at positive prices.}$$

(c) Invert the indirect utility function:

$$e(p,u) = \frac{p_1 p_2 u^2}{(p_1 + p_2)}$$

By Shephard's lemma:

$$x_i(p,u) = \frac{\partial e(p,u)}{\partial p_i}$$

$$\therefore \qquad x_1(p,u) = u^2\left(\frac{p_2}{p_1 + p_2}\right)^2$$

and

$$x_2(p,u) = u^2\left(\frac{p_1}{p_1 + p_2}\right)^2$$

(d) First find u^0 and u^1:

$$u^0 = v(p^0, m) = m^{1/2}$$

and

$$u^1 = v(p^1, m) = \left(\frac{4m}{3}\right)^{1/2}$$

Compensating variation:

$$CV = m - e(p^1, u^0)$$

$$\therefore \qquad CV = m - \left[\frac{3m}{4}\right] = \frac{m}{4}$$

Equivalent variation:

$$EV = e(p^0, u^1) - m$$

$$\therefore \qquad EV = \left(\frac{4m}{m}\right) - m = \frac{m}{3}$$

The agent is better-off, since $CV > 0$ (and *CV* and *EV* always have the same sign).

Answer to PS2 Question 4

(a) *max u(x) st* $p_1x_1 + p_2x_2 = m$

FOC: $\frac{1}{2}x_i^{-1/2} = \lambda p_i$

$$\Rightarrow \qquad x_1(p,m) = \frac{mp_2}{p_1(p_1 + p_2)} \quad \text{and} \quad x_2(p,m) = \frac{mp_1}{p_2(p_2 + p_1)}$$

Then substitution into $u(x)$ yields, after some manipulation,

$$v(p,m) = \left(\frac{(p_1 + p_2)m}{p_1p_2}\right)^{1/2}$$

(b) At the stated prices we have

$$v(p,m) = m^{1/2}$$

The expected utility is

$$Eu = 0.95(100{,}000)^{1/2} + 0.05(0)^{1/2} = 300.42$$

Certainty equivalent income is $\hat{m}$ such that $(\hat{m})^{1/2} = Eu$. Thus,

$$\hat{m} = (300.42)^2 = 90{,}250$$

Finally, the risk premium is

$$R = Em - \hat{m} = [0.95(100{,}000) + 0.05(0)] - 90{,}250 = 4{,}750$$

(c) Expected utility with the program in place is

$$Eu_P = 0.95(100{,}000 - 19000)^{1/2} + 0.05(64000)^{1/2} = 283.02$$

Expected utility without the program is 300.42 (from part (b)). Thus, the agent is worse-off under the program.

Answer to PS2 Question 5

(a) $max \ \log c_1 + \beta \log c_2 \ \ st \ \ c_1 + \dfrac{c_2}{1+r} = w$

FOC: $$\frac{1}{c_1} = \lambda$$

$$\frac{\beta}{c_2} = \frac{\lambda}{1+r}$$

Then taking the ratio yields the tangency condition (Euler equation):

$$\frac{c_2}{c_1} = \frac{1+r}{1+\rho}$$

Expressing c_2 in terms of c_1, and substituting into the wealth constraint yields

$$c_1^* = \frac{w(1+\rho)}{(2+\rho)}$$

Taking the derivative with respect to ρ yields

$$\partial c_1^* \big/ \partial \rho = \frac{w}{(2+\rho)^2} > 0$$

Intuition: a higher rate of time preference (ρ) means that the agent is relatively impatient to consume. Thus, consumption when young is increasing in ρ.

Taking the derivative with respect to *r* yields

$$\partial c_1^* \big/ \partial r = -\frac{(1+\rho)}{(2+\rho)}\left[\frac{y_2}{(1+r)^2}\right] < 0$$

Intuition: A higher interest rate means that lending when young is relatively more attractive than borrowing when young. Thus, consumption when young is decreasing in *r*.

(b) The agent will be a lender when young if and only if $c_1^* < y_1$.That is, iff

$$\frac{w(1+\rho)}{(2+\rho)} < y_1$$

$$\Rightarrow \qquad (1+\rho)\left[y_1 + \frac{y_2}{1+r}\right] < y_1(2+\rho)$$

$$\Rightarrow \qquad \frac{y_1}{y_2} > \frac{1+\rho}{1+r}$$

(c) Consider Figure P2.3. The dashed budget constraint is without the tax. The solid budget constraint is with the tax. Note that the budget constraint below the point (y_1, y_2) is not distorted by the tax since any point along that part of the budget constraint implies borrowing when young, and the interest cost of borrowing is not affected by the tax.

If the condition in part (b) does <u>not</u> hold then the agent would choose a point below (y_1, y_2) in the absence of the tax. Thus, this choice is unaffected by the tax.

If the condition in part (b) does hold then the agent would choose a point above (y_1, y_2) in the absence of the tax. Thus, the tax will affect her behavior. The easiest way to proceed is to recognize that the only difference between the solution with the tax and the solution without is that the effective interest rate is $r(1-t)$ rather than *r*. Thus, simply substitute $r(1-t)$ for *r* in the equation for c_1^* to obtain

$$c_1^*(t) = \frac{(1+\rho)}{(2+\rho)}\left[y_1 + \frac{y_2}{1+r(1-t)}\right]$$

This is clearly increasing in *t*. Thus, the tax causes consumption when young to rise.

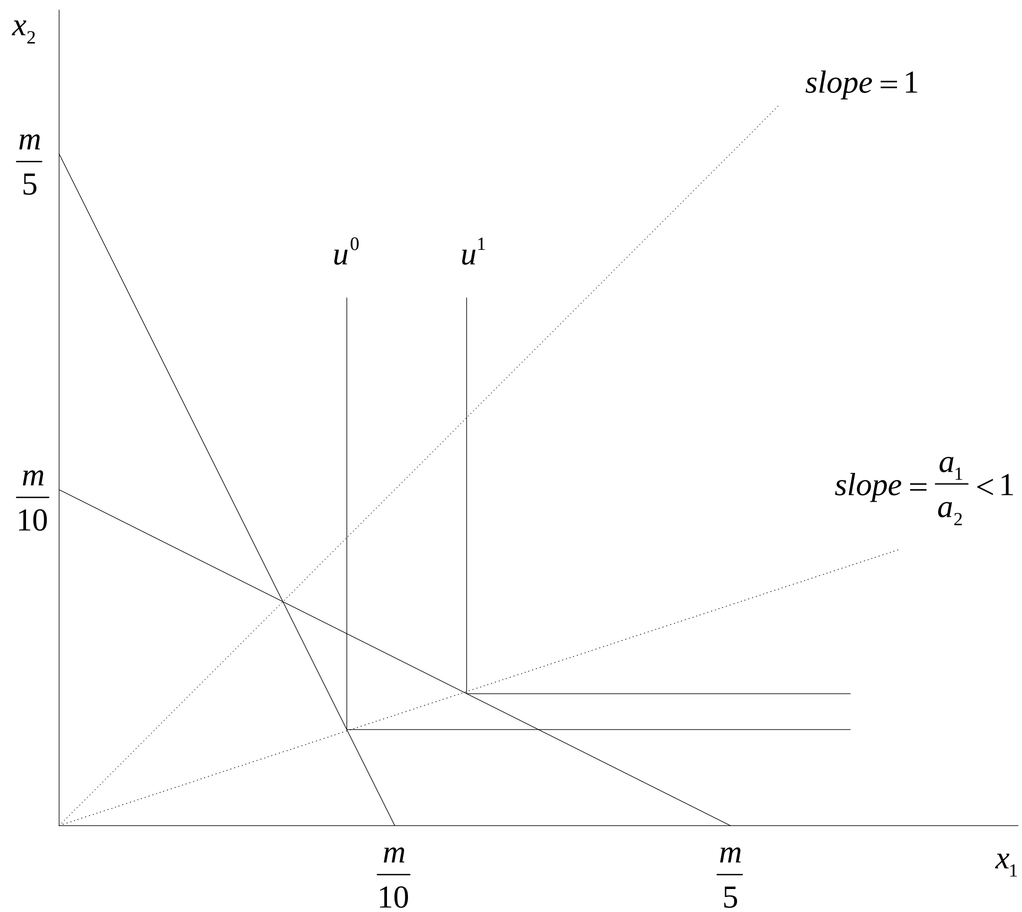
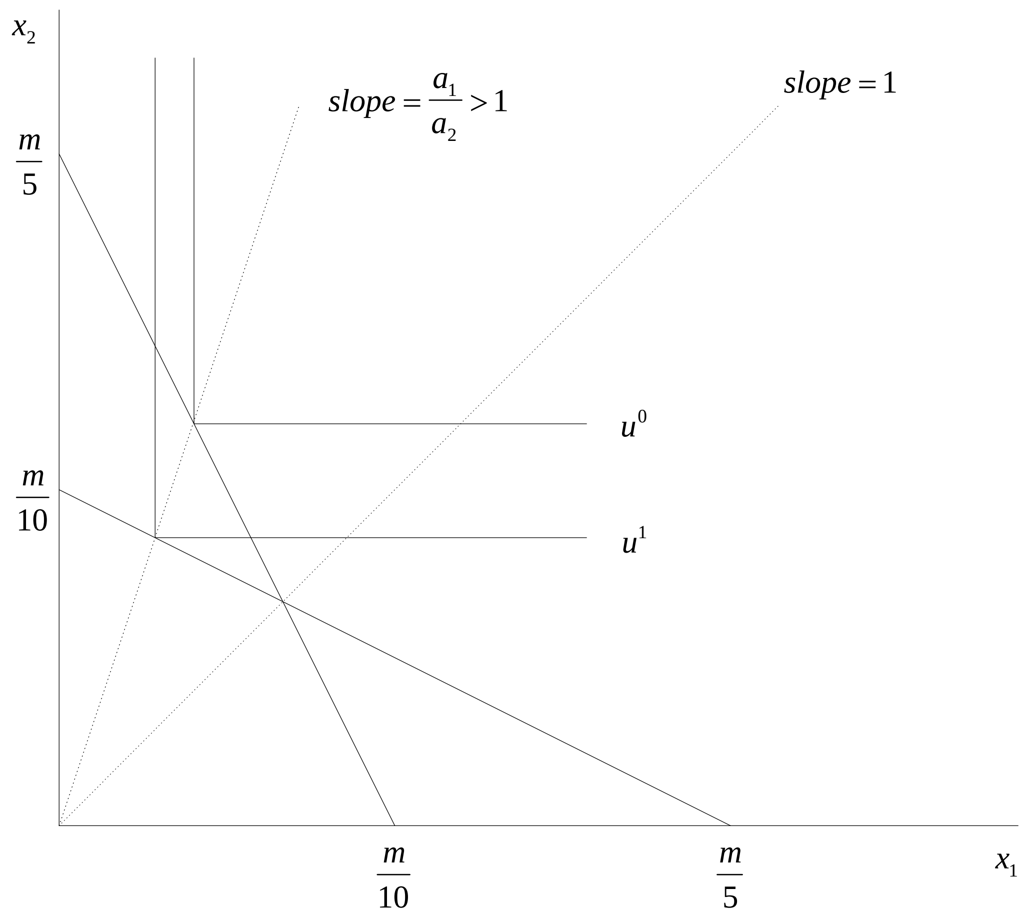

x_2
$slope = 1$
$\frac{m}{5}$
u^0
u^1
$\frac{m}{10}$
$slope = \frac{a_1}{a_2} < 1$
$\frac{m}{10}$
$\frac{m}{5}$
x_1

FIGURE P2.1



x_2
$slope = \frac{a_1}{a_2} > 1$
$slope = 1$
$\frac{m}{5}$
u^0
$\frac{m}{10}$
u^1
$\frac{m}{10}$
$\frac{m}{5}$
x_1

FIGURE P2.2

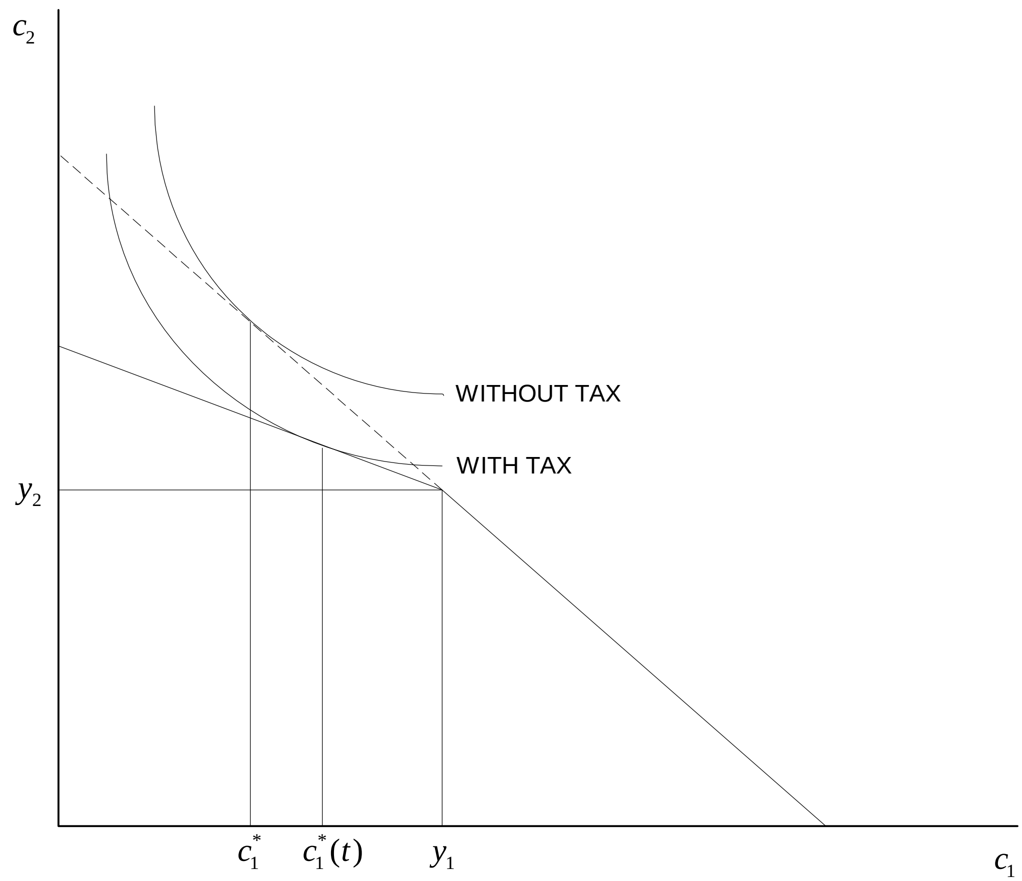


FIGURE P2.3

PROBLEM SET 3

Coverage: Chapters 7 – 10

PS3 Question 1

A price-taking firm has the following production function:

$$y = x^a$$

(a) Find the input demand $x(p,w)$ and the supply function $y(p,w)$.

(b) Find the conditional input demand $x(w,y)$ and the cost function $c(w,y)$.

(c) Show that

$$y(p,w) = \arg\max_y \quad py - c(w,y)$$

(d) Show that $x(w, y(p,w)) = x(p,w)$

PS3 Question 2

A price-taking firm has the following production function:

$$y = x_1^a x_2^b$$

Derive the cost function and verify that it is concave in *w*. Does this result depend on the value of *a* and *b*? Explain your answer.

PS3 Question 3

A price-taking firm has the following production function:

$$f(x) = x_1^{1/2} + x_2^{1/2}$$

Derive the supply function in two different ways.

PS3 Question 4

A firm has the following production function:

$$y = \min\left[x_1^{1/2}, x_2^{1/2}\right]$$

(a) Show that this production function exhibits decreasing returns to scale.

(b) Find the cost function and verify Shephard's lemma.

(c) All firms in the industry have the above technology and no firm can produce less than one unit. Market demand is given by

$$p = a - bX \ ,$$

and $w_1 = w_2 = 1$. Find the competitive industry equilibrium output and the equilibrium number of firms.

PS3 Question 5

Each firm in a competitive industry has the following production function

$$y = x_1^{1/4} x_2^{1/4}$$

(a) Find the cost function. Why is this cost function strictly convex in output?

(b) Let $w_1 = w_2 = 1$. Suppose aggregate demand is given by

$$X(p) = 1000 - 10p$$

Suppose each firm must pay a license fee of \$50 if it wants to produce. Otherwise entry is free. Find the equilibrium number of firms.

PS3 Question 6

A price-taking firm has the following production function:

$$y = (x_1^{1/2} + x_2^{1/2})x_3$$

where x_1 and x_2 are variable inputs, and $x_3 \in \{0,1\}$ is a quasi-fixed factor. That is, $x_3 = 1$ if the firm produces at all, and $x_3 = 0$ otherwise.

(a) Show that the cost function is given by

$$c(w, y) = y^2 \left[\frac{w_1 w_2}{w_1 + w_2} \right] + w_3 \quad \text{for} \quad y > 0$$

(b) Explain why average cost is "U-shaped".

(c) Let $w_1 = w_2 = 2$ and $w_3 = 1$. Find the supply function $y(p, w)$.

(d) Suppose market demand is given by

$$X(p) = 1000 - p$$

Suppose also that there is free entry and all firms have the above production function. Find the equilibrium price, the equilibrium aggregate output, and the equilibrium number of firms.

(e) Show that in general, the equilibrium number of firms is decreasing in w_3. Explain your answer.

PS3 Question 7

Show that the cost function $c(w, y)$ is concave in w. Explain how this result relates to the returns to scale of the underlying production function.

SOLUTIONS TO PROBLEM SET 3

Answer to PS3 Question 1

(a) $$\max_x \quad px^a - wx$$

FOC: $apx^{a-1} = w$

$$\Rightarrow \quad x(p,w) = \left[\frac{ap}{w}\right]^{\frac{1}{1-a}}$$

$$\therefore \quad y(p,w) = [x(p,w)]^a = \left[\frac{ap}{w}\right]^{\frac{a}{1-a}}$$

(b) The cost minimization problem is trivial since there is only one way produce a given level of output (unlike in instances with more than one input, where the relative mix of factors can be chosen). Thus,

$$x(w,y) = y^{1/a}$$

$$\therefore \quad c(w,y) = wy^{1/a}$$

(c) $$\max_y \quad py - wy^{1/a}$$

FOC: $p = \dfrac{wy^{\frac{1-a}{a}}}{a}$

$$\Rightarrow \quad y(p,w) = \left[\frac{ap}{w}\right]^{\frac{a}{1-a}}$$

(d) Substitute $y(p,w)$ for y in $x(w,y)$ to obtain

$$x(w,y(p,w)) = [y(p,w)]^{1/a} = \left[\frac{ap}{w}\right]^{\frac{1}{1-a}} = x(p,w)$$

Answer to PS3 Question 2

Set up the cost minimization problem:

$$\min_x \; w_1 x_1 + w_2 x_2 \quad \text{s.t.} \quad x_1^a x_2^b = y$$

The FOCs yield the tangency condition:

$$\frac{w_1}{w_2} = \frac{a x_2}{b x_1}$$

The constraint is then used to solve for conditional input demands:

$$x_1(p,y) = y^{\frac{1}{a+b}} \left(\frac{a w_2}{b w_1} \right)^{\frac{b}{a+b}}$$

$$x_2(p,y) = y^{\frac{1}{a+b}} \left(\frac{b w_1}{a w_2} \right)^{\frac{a}{a+b}}$$

The cost function is

$$c(y,w) = w_1 x_1(y,w) + w_2 x_2(y,w) = y^{\frac{1}{a+b}} \left(w_1 \left(\frac{a w_2}{b w_1} \right)^{\frac{b}{a+b}} + w_2 \left(\frac{b w_1}{a w_2} \right)^{\frac{a}{a+b}} \right)$$

Verification of concavity takes a bit of work. It requires showing that the Hessian for the cost function is negative semi-definite. The Hessian is

$$H = \begin{bmatrix} \dfrac{\partial^2 c}{\partial w_1^2} & \dfrac{\partial^2 c}{\partial w_1 \partial w_2} \\ \dfrac{\partial^2 c}{\partial w_2 \partial w_1} & \dfrac{\partial^2 c}{\partial w_2^2} \end{bmatrix}$$

where

$$\frac{\partial^2 c}{\partial w_1^2} = -\frac{y^{\frac{1}{a+b}} ab \left(w_1 \left(\frac{a w_2}{b w_1} \right)^{\frac{b}{a+b}} + w_2 \left(\frac{b w_1}{a w_2} \right)^{\frac{a}{a+b}} \right)}{(a+b)^2 w_1^2}$$

$$\frac{\partial^2 c}{\partial w_2^2} = -\frac{y^{\frac{1}{a+b}} ab\left(w_1\left(\frac{aw_2}{bw_1}\right)^{\frac{b}{a+b}} + w_2\left(\frac{bw_1}{aw_2}\right)^{\frac{a}{a+b}}\right)}{(a+b)^2 w_2^2}$$

$$\frac{\partial^2 c}{\partial w_1 \partial w_2} = \frac{\partial^2 c}{\partial w_2 \partial w_1} = \frac{y^{\frac{1}{a+b}} ab\left(w_1\left(\frac{aw_2}{bw_1}\right)^{\frac{b}{a+b}} + w_2\left(\frac{bw_1}{aw_2}\right)^{\frac{a}{a+b}}\right)}{(a+b)^2 w_1 w_2}$$

This Hessian is negative semi-definite iff $\frac{\partial^2 c}{\partial w_1^2} \leq 0$ and $|H| \geq 0$. The first condition clearly holds for any *a*,*b*. Evaluating $|H|$ yields $|H| = 0$ for any *a*,*b*. Thus, concavity is verified, and it does not depend on the values of *a* and *b*. Why? Recall that the concavity of the cost functions stems solely from its definition as a minimum value function.

Answer to PS3 Question 3

Method 1: Direct Profit Maximization

$$\max_{x_1, x_2} \; p(x_1^{1/2} + x_2^{1/2}) - w_1 x_1 - w_2 x_2$$

$$\Rightarrow \qquad x_1(p,w) = \left(\frac{p}{2w_1}\right)^2$$

$$x_2(p,w) = \left(\frac{p}{2w_2}\right)^2$$

Then

$$y(p,w) = f(x(p,w))$$

$$\therefore \qquad y(p,w) = \frac{p(w_1 + w_2)}{2w_1 w_2}$$

Method 2: Cost Minimization then Profit Maximization using the Cost Function

(i) Cost Minimization:

$$\min_{x_1,x_2} \; w_1x_1 + w_2x_2 \quad \text{st} \quad y = (x_1^{1/2} + x_2^{1/2})$$

FOC: $$w_1 = \frac{\lambda x_1^{-1/2}}{2}$$

$$w_2 = \frac{\lambda x_2^{-1/2}}{2}$$

Take the ratio to obtain the isocost-isoquant tangency condition:

$$\frac{w_1}{w_2} = \left[\frac{x_2}{x_1}\right]^{1/2}$$

Express x_2 in terms of x_1, substitute into the production function, and solve for x_1:

$$x_1(w,y) = y^2\left[\frac{w_2}{w_1 + w_2}\right]^2$$

Then use the tangency condition to solve for x_2:

$$x_2(w,y) = y^2\left[\frac{w_1}{w_1 + w_2}\right]^2$$

Then the cost function is

$$c(w,y) = w_1x_1(w,y) + w_2x_2(w,y)$$

Substituting for $x_1(w,y)$ and $x_2(w,y)$ yields

$$c(w,y) = y^2\left[\frac{w_1w_2^2 + w_2w_1^2}{(w_1 + w_2)^2}\right] = y^2\left[\frac{w_1w_2(w_1 + w_2)}{(w_1 + w_2)^2}\right]$$

Simplifying yields

$$c(w,y) = y^2\left[\frac{w_1w_2}{w_1 + w_2}\right]$$

(ii) Profit Maximization:

$$\max_{y} \; py - c(w,y)$$

$$\Rightarrow \qquad y(p,w) = \frac{p(w_1 + w_2)}{2w_1w_2}$$

Answer to PS3 Question 4

(a) $$f(tx_1, tx_2) = \min\left[(tx_1)^{1/2}, (tx_2)^{1/2}\right] = t^{1/2}\min\left[x_1^{1/2}, x_2^{1/2}\right] = t^{1/2} f(x_1, x_2)$$

(b) At the cost minimum:

$$x_1^{1/2} = x_2^{1/2}$$

$$\Rightarrow \quad x_1(w,y) = y^2 \quad \text{and} \quad x_2(w,y) = y^2$$

$$c(w,y) = w_1 x_1(w,y) + w_2 x_2(w,y) = [w_1 + w_2]y^2$$

Shephard's lemma:

$$\frac{\partial c(w,y)}{\partial w_i} = x_i(w,y)$$

In this case:

$$LHS = y^2 = x_i(w,y) = RHS \quad \forall i$$

(c) At $w_1 = w_2 = 1$:

$$c(w,y) = 2y^2$$

This cost function is strictly convex in *y*. Thus, for profit maximization, $p = MC$:

$$p = 4y$$

Therefore, the supply function is

$$y(p) = \frac{p}{4}$$

With *n* identical firms, aggregate supply is

$$Y(p) = \frac{np}{4}$$

In equilibrium, supply = demand:

$$\frac{np}{4} = \frac{a-p}{b}$$

$$\Rightarrow \quad p^*(n) = \frac{4a}{nb+4}$$

$$\Rightarrow \quad y(p^*(n)) = \frac{a}{nb+4}$$

The associated profit, as a function of *n*, is

$$\Pi(n) = p^*(n)y(p^*(n)) - c(y(p^*(n))) = \frac{2a^2}{(nb+4)^2}$$

$\Pi(n)$ is positive for any *n*. Thus, entry will continue until each firm is driven down to its minimum feasible level of production: $y^* = 1$. Therefore,

$$\frac{a}{nb+4} = 1$$

$$\Rightarrow \qquad n^* = \frac{a-4}{b} \quad \text{if} \quad a \geq 4$$

$$\Rightarrow \qquad Y^* = n^* y^* = \frac{a-4}{b} \quad \text{if} \quad a \geq 4$$

If $a < 4$ then demand is negative at $p = MC$ evaluated at $y = 1$. This cannot be an equilibrium. Thus, a competitive equilibrium can exist only if $a \geq 4$.

Answer to PS3 Question 5

(a) $$\min_x \; w_1 x_1 + w_2 x_2 \quad \text{st} \quad y = x_1^{1/4} x_2^{1/4}$$

FOC: $$w_1 = \frac{\lambda x_1^{-3/4} x_2^{1/4}}{4}$$

$$w_2 = \frac{\lambda x_2^{-3/4} x_1^{1/4}}{4}$$

Take the ratio to obtain the isocost-isoquant tangency condition:

$$\frac{w_1}{w_2} = \frac{x_2}{x_1}$$

Express x_2 in terms of x_1, substitute into the production function, and solve for x_1:

$$x_1(w,y) = y^2 \left[\frac{w_2}{w_1}\right]^{1/2}$$

Then use the tangency condition to solve for x_2:

$$x_2(w,y) = y^2 \left[\frac{w_1}{w_2}\right]^{1/2}$$

Then the cost function is

$$c(w,y) = w_1 x_1(w,y) + w_2 x_2(w,y) = 2y^2(w_1 w_2)^{1/2}$$

Note that for the general Cobb-Douglas production function:

$$y = \prod_{i=1}^{N} x_i^{a_i}$$

$$x_j(w,y) = \left[y\left(\frac{a_j}{w_j}\right)^A \prod_{i=1}^{N}\left(\frac{w_i}{a_i}\right)^{a_i} \right]^{\frac{1}{A}}$$

where $A = \sum_{i=1}^{N} a_i$

$$c(w,y) = \sum_{j=1}^{N}\left[y\left(\frac{a_j}{w_j}\right)^A \prod_{i=1}^{N}\left(\frac{w_i}{a_i}\right)^{a_i} \right]^{\frac{1}{A}} w_j$$

The strict convexity of the cost function in output stems directly from the decreasing returns to scale exhibited by the production function. To increase output, inputs must be increased more than proportionately, and so cost must increase more than proportionately, at given factor prices.

(b) At $w_1 = w_2 = 1$:

$$c(y) = 2y^2$$

Since this cost function is strictly convex in *y*, profit maximization occurs where $p = MC$:

$$p = 4y$$

Therefore, the supply function is

$$y(p) = \frac{p}{4}$$

With *n* identical firms, aggregate supply is

$$Y(p) = \frac{np}{4}$$

In equilibrium, supply = demand:

$$\frac{np}{4} = 1000 - 10p$$

$$\Rightarrow \qquad p^*(n) = \frac{4000}{n+40}$$

$$\Rightarrow \qquad y(p^*(n)) = \frac{1000}{n+40}$$

The associated profit, as a function of *n*, is

$$\Pi(n) = p^*(n)y(p^*(n)) - c(y(p^*(n))) = \frac{2{,}000{,}000}{(n+40)^2}$$

Entry will drive profit down to \$50. Solving $\Pi(n) = 50$ yields

$$n^* = 160$$

Answer to PS3 Question 6

(a) If the firm is to produce any $y > 0$ then it must set $x_3 = 1$. The other factors are then chosen to minimize cost:

$$\min_{x_1, x_2} \; w_1 x_1 + w_2 x_2 + w_3 \quad \text{st} \quad y = (x_1^{1/2} + x_2^{1/2})$$

$$\text{FOC:} \quad w_1 = \frac{\lambda x_1^{-1/2}}{2}$$

$$w_2 = \frac{\lambda x_2^{-1/2}}{2}$$

Take the ratio to obtain the isocost-isoquant tangency condition:

$$\frac{w_1}{w_2} = \left[\frac{x_2}{x_1}\right]^{1/2}$$

Express x_2 in terms of x_1, substitute into the production function, and solve for x_1:

$$x_1(w, y) = y^2 \left[\frac{w_2}{w_1 + w_2}\right]^2$$

Then use the tangency condition to solve for x_2:

$$x_2(w, y) = y^2 \left[\frac{w_1}{w_1 + w_2}\right]^2$$

Then the cost function is

$$c(w,y) = w_1 x_1(w,y) + w_2 x_2(w,y) + w_3 \quad \text{for } y > 0$$

Substituting for $x_1(w,y)$ and $x_2(w,y)$ yields

$$c(w,y) = y^2\left[\frac{w_1 w_2^2 + w_2 w_1^2}{(w_1 + w_2)^2}\right] + w_3 = y^2\left[\frac{w_1 w_2 (w_1 + w_2)}{(w_1 + w_2)^2}\right] + w_3$$

Simplifying yields

$$c(w,y) = y^2\left[\frac{w_1 w_2}{w_1 + w_2}\right] + w_3 \quad \text{for } y > 0$$

(b) Average cost is U-shaped because of the combination of the quasi-fixed cost w_3 (a non-variable cost that is only incurred if $y > 0$) and the increasing marginal cost. Spreading the quasi-fixed cost across more units of output initially causes AC to fall as output grows, but this effect is eventually offset by the increasing marginal cost.

(c) At $w_1 = w_2 = 2$ and $w_3 = 1$:

$$c(y) = y^2 + 1$$

Since this cost function is strictly convex in *y*, profit maximization occurs where $p = MC$:

$$p = 2y$$

Therefore, the supply function is

$$y(p) = \frac{p}{2}$$

With *n* identical firms, aggregate supply is

$$Y(p) = \frac{np}{2}$$

In equilibrium, supply = demand:

$$\frac{np}{2} = 1000 - p$$

$$\Rightarrow \qquad p^*(n) = \frac{2000}{n+2}$$

$$\Rightarrow \qquad y(p^*(n)) = \frac{1000}{n+2}$$

The associated profit, as a function of *n*, is

$$\Pi(n) = p^*(n)y(p^*(n)) - c(y(p^*(n))) = \frac{1{,}000{,}000}{(n+2)^2} - 1$$

Entry will drive profit down to zero. Solving $\Pi(n) = 0$ yields

$$n^* = 998$$

(e) Profit, as a function of *n* and w_3, is

$$\Pi(n, w_3) = \frac{1{,}000{,}000}{(n+2)^2} - w_3$$

Entry will drive profit down to zero. Solving $\Pi(n, w_3) = 0$ yields

$$w_3(n^* + 2)^2 = 1{,}000{,}000$$

Totally differentiating both sides yields:

$$\frac{dn^*}{dw_3} = \frac{-(n+2)}{2w_3} < 0$$

Intuition: A higher equilibrium price is needed to ensure non-negative profit at a higher quasi-fixed cost. Thus, the number of competing firms in equilibrium must be smaller.

Answer to PS3 Question 7

Let

$$w'' = tw^0 + (1-t)w'$$

where $0 < t < 1$. Then

$$c(w'', y) \equiv w''x(w'', y) = tw^0 x(w'', y) + (1-t)w'x(w'', y)$$

By definition of the cost function as a minimum value function,

$$w^0 x(w'', y) \geq c(w^0, y)$$

and

$$w'x(w'', y) \geq c(w', y)$$

Therefore,

$$tw^0 x(w'', y) + (1-t)w' x(w'', y) \geq tc(w^0, y) + (1-t)c(w', y)$$

Thus,

$$c(w'', y) \geq tc(w^0, y) + (1-t)c(w', y)$$

That is, $c(w, y)$ is concave in *w*.♣

This result has nothing to do with the returns to scale of the production function. (This is clear from the above proof, which makes no reference to the production function). *The cost function is concave in w for any production function*. The only link to the production function is the following. If the production function exhibits some substitutability between factors then the cost function will be strictly concave in *w*; if it exhibits no substitutability (such as a Leontief production function or a single input production function) then the cost function will be weakly concave in *w*.

The returns to scale of the production function is important for the properties of the cost function with respect to *y*. If the production function exhibits decreasing returns to scale then the cost function is strictly convex in *y*, for a given *w*. If the production function exhibits increasing returns to scale then the cost function is strictly concave in *y*, for a given *w*. If the production function exhibits constant returns to scale then the cost function is linear in *y*, for a given *w*.

PROBLEM SET 4

Coverage: Chapters 11 – 15

PS4 Question 1

Consider an economy with fixed endowments of two goods, "fat" and "lean". There are two agents in this economy, Jack Sprat and his friend. The economy has the following characteristic:

Jack Sprat could eat no fat,
His friend could eat no lean.
And so between them both you see,
They licked the platter clean.

There is only one Pareto efficient allocation in this economy. True or false?

PS4 Question 2

A monopolist sells its product in two distinct markets. The inverse demand in market $i \in \{1,2\}$ is

$$p_i = a - b_i X_i$$

The firm has cost function $c(y) = cy$. Determine how much the firm will sell in each market and at what prices. Explain your answer in terms of the demand elasticities in the two markets.

PS4 Question 3

Consider the following normal form game:

3,3	0,2
2,0	1,-1

What is the Nash equilibrium of this game? Is it Pareto efficient?

PS4 Question 4

Consider the following duopoly with differentiated products. Each firm has zero costs. The demand faced by firm *i* is

$$X_i = a - bp_i + cp_j$$

where p_j is the price firm *j* charges for its product. Thus, there is some substitutability between the two products from the perspective of consumers; an increase in the price of one product increases the demand for the other product, *ceteris paribus*.

Suppose each firm sets its price to maximize profit and both firms move simultaneously. Derive each firm's reaction function and find the symmetric Nash equilibrium price. Illustrate this equilibrium in a diagram in $\{p_1, p_2\}$ space.

PS4 Question 5

The inverse demand curve in an industry is

$$p = 100 - X$$

There are *n* identical firms each with cost function $c(y) = 2y^2$. Firms choose quantities and move simultaneously.

(a) Show that a representative firm's reaction function is given by

$$y_i = \frac{100 - \sum_{j \neq i} y_j}{6}$$

(b) Find the symmetric Nash equilibrium price and output for each firm, and find the aggregate industry output.

PS4 Question 6

Consider an economy in which *n* identical agents each have utility function

$$u(x, z, P) = xz - P$$

where *x* and *z* are both private goods with unit price, and *P* is pollution (a public bad). Each agent has income *m*. All agents know that pollution is caused by the consumption of *x* in the following way

$$P = \theta X$$

where $\theta > 0$ and X is the aggregate consumption of x in the economy.

(a) Find the Nash equilibrium consumption levels of x and z, and the Nash equilibrium level of P.

(b) Compare the Nash equilibrium with the symmetric efficient allocation. Explain your answer in terms of a free-rider problem.

PS4 Question 7

Consider a market in which a monopoly seller produces a product whose quality is known to the seller but unknown to potential buyers. If the product performs properly, it provides a service whose value to the buyer is $\theta > 1$. If the good malfunctions it provides no service. The product can be one of two types. A high quality product malfunctions with probability $1 - s_H$, where $s_H \in (0,1)$; a low quality product malfunctions with probability $1 - s_L$, where $s_L \in (0,1)$ and $s_L < s_H$.

The seller posts a take-it-or-leave-it price p for the product. If the buyer accepts the price then the product is produced and exchange takes place at the posted price. Buyers are risk neutral and their expected surplus from a purchase is the expected service from the product minus the purchase price.

A high quality product costs s_H to produce while a low quality product costs s_L to produce. Quality is determined by the firm's production technology and this is fixed.

The population fraction of high quality technologies is α and this is common knowledge. Production costs are also common knowledge.

(a) Describe in words a *pooling equilibrium* in this market. Is such an equilibrium Pareto efficient? Explain your answer.

(b) What is the market price in the pooling equilibrium (if one exists)?

(c) Derive a condition on α under which a pooling equilibrium does not exist. Interpret your result.

(d) Describe the equilibrium in this market when a pooling equilibrium does not exist. Is the equilibrium Pareto efficient? Explain your answer.

Now suppose that the seller is able to offer a credible product warranty with no transaction costs. The warranty is of the following form: if the product malfunctions then it is returned to the seller and an amount w is refunded to the buyer.

(e) Suppose w is equal to the purchase price (that is, the warranty provides a full refund). Describe the equilibrium under this warranty rule. Is this a *separating equilibrium*? Explain your answer.

(f) Now suppose that providing a refund requires a transaction fee k incurred by the seller. Derive a condition on k under which there exists a separating equilibrium in which a high quality seller offers a full-refund warranty but a low quality seller does not.

SOLUTIONS TO PROBLEM SET 4

Answer to PS4 Question 1

True. Jack derives no utility from fat; his friend derives no utility from lean. Thus, any allocation in which Jack has some fat and his friend has some lean is Pareto dominated by an allocation in which Jack has all the lean and his friend has all the fat. This allocation is the only Pareto efficient allocation.

Answer to PS4 Question 2

Profit maximization requires

$$MR_1 = MR_2 = MC$$

$$\Rightarrow \quad a - 2b_1 y_1 = a - 2b_2 y_2 = c$$

$$\therefore \quad \hat{y}_1 = \frac{a-c}{2b_1} \quad \text{and} \quad \hat{y}_2 = \frac{a-c}{2b_2}$$

$$\Rightarrow \quad \hat{p}_1 = \frac{a+c}{2} \quad \text{and} \quad \hat{p}_2 = \frac{a+c}{2}$$

Thus, price is the same in both markets. Elasticity of demand varies along a linear demand curve; outputs are chosen so that the demand elasticities are just equated in the two markets:

$$|\varepsilon_i| = \left(\frac{\partial X_i}{\partial p_i}\right)\left(\frac{p_i}{X_i}\right) = -\left(\frac{1}{b_i}\right)\left(\frac{a+c}{2}\bigg/\frac{a-c}{2b_i}\right) = \frac{a+c}{a-c} \quad \text{for} \quad i = 1,2$$

Answer to PS4 Question 3

A *Nash equilibrium* is a vector of strategies $\{\hat{s}_i, \hat{s}_{-i}\}$ such that

$$u_i(\hat{s}_i, \hat{s}_{-i}) \geq u_i(s_i, \hat{s}_{-i}) \quad \forall s_i, \quad \forall i$$

where s_i is the strategy of player *i*, s_{-i} is the vector of strategies of all other players, and $u_i(s_i, s_{-i})$ is the payoff to player *i*.

In the example, label the possible strategies for the row player U (up) and D (down), and the possible strategies for the column player L (left) and R (right). Then the problem for the row player is to choose between U and D. Suppose she expects the column player to play R, then

$$u_{row}(U,R) = 0 \quad \text{and} \quad u_{row}(D,R) = 1$$

Thus, the row player would play D. But if the column player expects the row player to play D, then she will not play R; she will play L instead since

$$u_{col}(L,D) = 0 \quad \text{and} \quad u_{col}(R,D) = -1$$

The row player knows that the column player will think this way, and so she will not expect the column player to play R. Thus, no Nash equilibrium could involve the column player playing R.

Suppose instead the row player expects the column player to play L. Then the row player will play U since

$$u_{row}(U,L) = 3 \quad \text{and} \quad u_{row}(D,L) = 2$$

And if the column player expects the row player to play U then she will play L since

$$u_{col}(L,U) = 3 \quad \text{and} \quad u_{col}(R,U) = 2$$

Thus, {U,L} is the unique Nash equilibrium. It is clearly Pareto efficient since it Pareto dominates all other possibilities.

Answer to PS4 Question 4

The profit maximization problem for firm *i* is

$$\max_{p_i} \quad p_i[a - bp_i + cp_j]$$

The best-response function (first-order condition) for firm *i* is

$$a - 2bp_i + cp_j = 0$$

which can be written in explicit form as

$$p_i(p_j) = \frac{a + cp_j}{2b}$$

In the symmetric Nash equilibrium, $p_i = p_j = p$, so we can simply substitute *p* for p_i and p_j in the best-response function for firm *i* and solve to obtain

$$\hat{p} = \frac{a}{2b - c}$$

See Figures P4.1 and P4.2.

Answer to PS4 Question 5

(a) The profit maximization problem for firm *i* is

$$\max_{y_i} \quad [100 - y_i - \sum_{j \neq i} y_j] y_i - 2y_i^2$$

The best-response function (first-order condition) for firm *i* is

$$y_i = \frac{100 - \sum_{j \neq i} y_j}{6}$$

(b) In symmetric Nash equilibrium

$$y_i = y \quad \forall i \quad \text{and} \quad \sum_{j \neq i} y_j = (n-1)y$$

Substitution into the best-reponse function for firm *i* yields

$$\hat{y} = \frac{100}{n+5}$$

$$\Rightarrow \qquad \hat{Y} = n\hat{y} = \frac{100n}{n+5}$$

$$\Rightarrow \qquad \hat{p} = 100 - \hat{Y} = \frac{500}{n+5}$$

Answer to PS4 Question 6

(a) The behavior of agent *i* is described by the solution to

$$\max_{x_i} \quad x_i (m - x_i) - (\theta x_i + P_{-i})$$

where a direct substitution of the budget constraint $x_i + z_i = m$ has been made, and where P_{-i} is the pollution associated with consumption of *x* by agents other than agent *i* (which agent *i* takes as given).

The best-response function (first order condition) is

$$x_i = \frac{m-\theta}{2}$$

Note that this is independent of P_{-i}. Thus, her best-response function describes a *dominant strategy*. That is, the privately optimal action for agent *i* does not depend on what she expects others to do; the dominant strategy for agent *i* dominates all other strategies regardless of what other players do. (This property stems from the quasi-linear nature of the utility function in this example). Thus, in the Nash equilibrium (which in this example is a *dominant strategy equilibrium*),

$$\hat{x} = \frac{m-\theta}{2} \quad \forall i$$

$$\therefore \qquad \hat{z} = m - \hat{x} = \frac{m+\theta}{2}$$

and

$$\hat{P} = \theta n \hat{x} = \frac{\theta n(m-\theta)}{2}$$

(b) In the symmetric efficient allocation, each agent has the maximum utility possible, subject to all agents being treated the same. This is characterized by maximizing the utility of a representative agent:

$$\max_x \quad x(m-x) - \theta n x$$

where the expression for *P* reflects the fact that if n agents are each allocated an amount *x* then $X = nx$. The first-order condition yields

$$x^* = \frac{m-n\theta}{2}$$

Note that $x^* < \hat{x}$ for $n > 1$. This inefficiency reflects the negative externality associated with pollution. No agent takes into account the damaging impact that her consumption of *x* has on the utility of other agents; thus, in equilibrium there is too much consumption. An equivalent interpretation is the following. A *reduction* in the level of pollution is a pure public good. Each agent tends to free-ride on the contributions of others to that public good (via reductions in the consumption of x) and so in equilibrium there is too little reduction in the level of pollution.

Answer to PS4 Question 7

(a) In a PE both types charge the same price and buyers cannot distinguish between the two types. Efficiency requires that a product is produced and exchanged if and only if the expected social surplus is positive. Expected social surplus from product i is $s_i\theta - s_i$. This is positive for both types since $\theta > 1$. Both types are produced and exchanged in the PE. Thus, the PE is efficient.

(b) Both types choose a price to extract the entire expected surplus from the buyer. Thus,

$$p^{PE} = \theta[\alpha s_H + (1-\alpha)s_L]$$

(c) The PE does not exist if either $p^{PE} < s_H$ or $p^{PE} < s_L$ since profit would be negative. The binding condition is the first of these. In terms of α :

$$\alpha < \frac{s_H - \theta s_L}{\theta(s_H - s_L)}$$

(d) The H seller withdraws from the market and only the L seller remains. It sets price equal to expected surplus:

$$\hat{p} = \theta s_L$$

It is not efficient since the H quality good should be produced but is not.

(e) Both firms sets price to extract the entire (certain) surplus, $p = \theta$. This is refunded in the event of a malfunction. It is not a separating equilibrium since the firms are not distinguished in equilibrium.

(f) In the SE, $p_H = \theta$ with $w = \theta$, and $p_L = s_L\theta$. We need to check incentive compatibility (IC) and participation conditions for both types.

For L type, IC requires that the SE payoff to be at least as great as the mimicking payoff. That is,

$$s_L\theta - s_L \geq \theta - s_L - (1-s_L)(\theta + k)$$

since a refund is made with probability $(1-s_L)$. This holds for any k since $s_L \leq 1$. Participation for the L type is trivial since $\theta > 1$.

For the H type, IC requires

$$\theta - s_H - (1-s_H)(\theta + k) \geq s_L\theta - s_H$$

which reduces to

$$k \leq \frac{\theta(s_H - s_L)}{1-s_H}$$

Participation for H type requires

$$\theta - s_H - (1-s_H)(\theta + k) \geq 0$$

which reduces to

$$k \leq \frac{s_H(\theta - 1)}{1-s_H}$$

Which of these conditions is more restrictive? If $s_L\theta > s_H$ then IC is the binding condition; otherwise, participation is the binding condition.

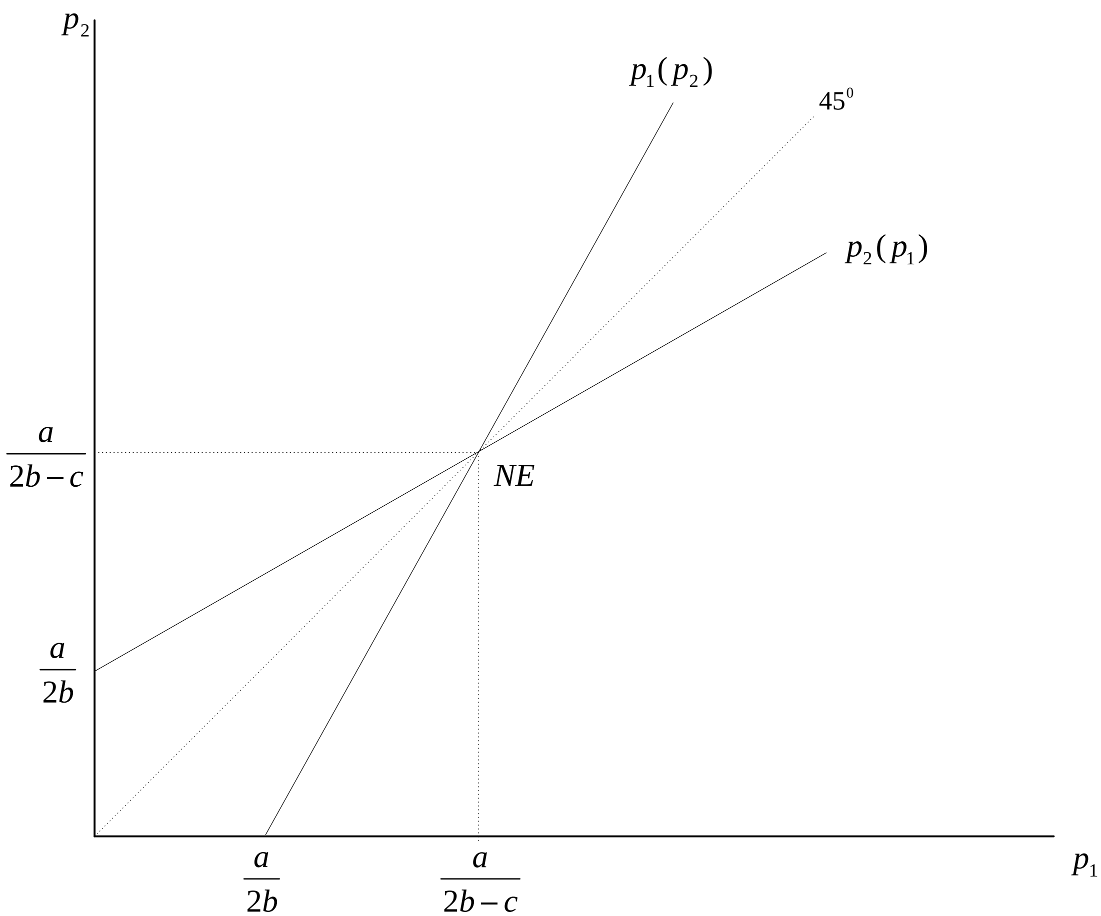


FIGURE P4.1

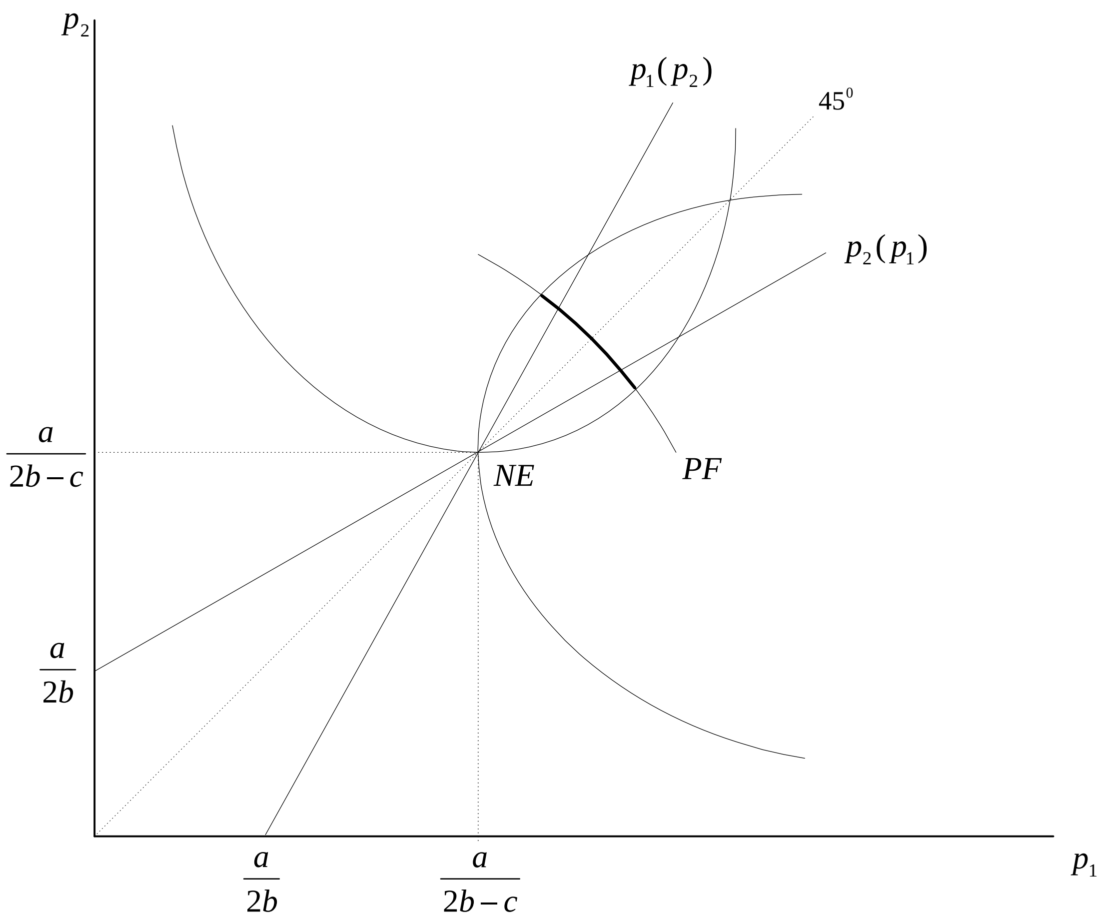


FIGURE P4.2