# On Internal Hydraulics with Entrainment

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ABSTRACT

The hydraulics of a single layer flow with entrainment is examined with a reduced-gravity model. Expressions are derived for the local change of Froude number and layer thickness as a function of the entrainment velocity. It is shown that entrainment, like friction, acts to force the flow toward criticality, although the layer thickness can increase if the Froude number is smaller than 1/2. For certain Froude numbers the effects of friction and entrainment on the layer thickness and the hydraulic state of the flow are found to be of comparable magnitude.

## 1. Introduction

Stratified flow past topography is often well approximated by a layered representation. Therefore many studies of the dynamics of strait and sill flow have used the hydraulic theory of layered flow (e.g., Armi 1986). A further simplification is that in many such layered flows it is not uncommon for only one of the layers to participate actively. This is readily observed in exchange flows, as in the Strait of Gibraltar for example, where there might be multiple controls: only one layer participates actively in the neighborhood of each control (Bryden and Kinder 1991). In these and similar cases the local dynamics is well described by a single layer reduced-gravity flow.

The classical hydraulic analysis of such flows is inviscid and excludes the possibility of friction and fluid or momentum exchange between the layers. This works well in some straits, such as Gibraltar, where frictional and entrainment effects are relatively minor (Farmer and Armi 1988; Pratt 1986). On the other hand, friction and entrainment can play an important role in longer, shallower, or narrower straits. In the Bosphorus, for example, substantial changes occur in the density structure of the water masses as they move through the strait. Moreover, the interface between the layers has an appreciable slope, even within the central portion which is well away from the controls (Unluata et al. 1990). It seems likely under these circumstances that both mass and momentum flux between the layers, together with friction along the boundaries, contribute to the balance of forces within the strait.

Frictional effects in open channel flow have been studied by Henderson (1966), and further explored by Pratt (1986) and Bormans and Garrett (1989) for the case of single layer reduced-gravity flow. A key parameter is the internal Froude number $F$, where $F^2 = u^2/(g'h)$, with $u$ the flow speed of the layer, $h$ its thickness, and $g'$ reduced gravity. Expressions for the Froude number $F$ (or the derivative $dF^2/dx$) and for the change of the layer thickness $dh/dx$ permit the comparison of effects of friction with those of a variation in channel topography. The principal conclusion of the above papers is that friction tends to drive the flow toward a critical state and simultaneously decreases the layer thickness, in much the same way as a decrease in width or an increase in bottom elevation.

It is the purpose of the present note to extend the analysis by including entrainment. We explore the implications of entrainment on single layer reduced-gravity flows in the presence of gradually changing topography. In particular, we investigate effects of exchange of mass and momentum between the layers on the layer thickness and the Froude number, that is, on the hydraulic state of the flow.

The general problem to be considered is illustrated

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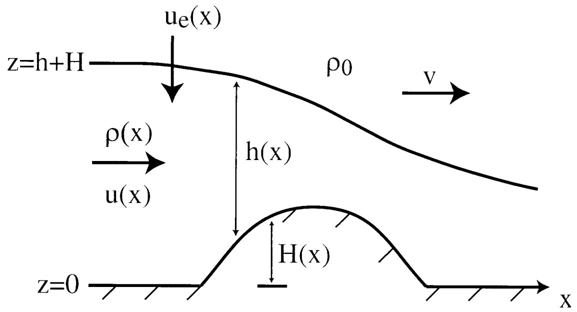


FIG. 1. Basic configuration. A single layer of fluid of depth $h(x)$ and density $\rho(x)$ flows at speed $u(x)$ over a sill of height $H(x)$. Fluid with density $\rho_0$ may be entrained with an entrainment velocity $u_e(x)$ from an overlying layer moving with a constant speed $v$.

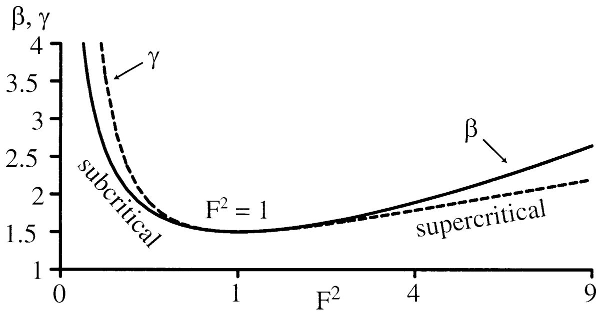


FIG. 2. Relationship between $\beta = 1/2F^{4/3} + F^{-2/3}$ and $F^2$ (solid line), and $\gamma = F^{2/3} + 1/2F^{-4/3}$ and $F^2$ (dashed line). Both curves have two branches that meet at $F^2 = 1$ and $\beta, \gamma = 3/2$.

in Fig. 1. A layer of fluid of density $\rho(x)$ flows steadily with a velocity $u(x)$ over a bottom profile of height $H(x)$ in a strait of rectangular cross section with width $W(x)$. Above this lies a deep layer of density $\rho_0$ moving with a constant velocity $v$.

The following discussion arbitrarily assumes the lower layer to be active, but the results apply equally well to an active upper and passive lower layer, recognizing that bottom topography lying within the passive layer then has no influence on the active layer. Although rotation is important in wide straits, it will be neglected here since our purpose is to determine the basic consequences of entrainment, and this is most readily achieved in the simplest environment.

## 2. Review of inviscid and frictional hydraulics

We set the scene by reviewing the essentials of hydraulic control of inviscid and frictional flow over a sill (Pratt 1986; Bormans and Garrett 1989) with no entrainment ($u_e = 0$).

Ignoring rotation and bottom friction, the governing shallow-water equations for the current $u$ and lower-layer depth $h$ are

$$u\frac{du}{dx} + g'\frac{d}{dx}(h + H) = 0, \qquad (1)$$

$$\frac{d}{dx}(uhW) = 0, \qquad (2)$$

where $g' \equiv g(\rho - \rho_0)/\rho$ is the reduced gravity, $W(x)$ is the width of the channel, $H(x)$ is the bottom elevation, and $Q = uhW$ is the transport. Integrating with respect to $x$ leads to equations for the conservation of energy and volume:

$$\frac{u^2}{2g'} + h + H = E_R, \qquad (3)$$

$$Q = uhW. \qquad (4)$$

If it is assumed that the flow originates in a large reservoir where $hW$ tends to $\infty$ and the velocity $u$ to 0 (such that $Q$ stays finite), then $E_R$ is the surface elevation of the reservoir. Eliminating $u$ yields an equation for the layer thickness $h$,

$$\frac{1}{2}\left(\frac{Q^2}{g'W^2}\right)h^{-2} + h = E_R - H, \qquad (5)$$

and, using $F^2 = Q^2/(g'W^2h^3)$, an equation for the Froude number (e.g., Armi 1989) is

$$\beta \equiv \frac{1}{2}F^{4/3} + F^{-2/3} = (E_R - H)\left(\frac{g'W^2}{Q^2}\right)^{1/3}. \qquad (6)$$

The left-hand side of (6) is shown in Fig. 2 as a function of $F^2$. For a given volume flux both an increase of bottom height $H$ and a decrease of the channel width $W$ reduce the value of the right-hand side of (6) and force the flow toward criticality. The flow is critical, with $F = 1$, if the right-hand side equals 3/2. At that point, $d\beta/dx = 0$ and

$$W^{-1}\frac{dW}{dx} - \left(\frac{g'W^2}{Q^2}\right)^{1/3}\frac{dH}{dx} = 0, \qquad (7)$$

showing that the control section is at the crest of a ridge if $W$ is constant, or at the narrowest point of the channel if the bottom is flat.

We include bottom friction in the problem by adding the term $-C_d u^2/h$ to the right-hand side of (1), where $C_d$ is the nondimensional bottom drag coefficient (Pratt 1986). Then

$$\frac{d}{dx}\left(\frac{1}{2}\frac{u^2}{g'} + h + H\right) = -C_d\frac{u^2}{g'h}. \qquad (8)$$

Using $F^2 = u^2/(g'h)$ as before we obtain

$$\frac{d}{dx}\left[\left(\frac{Q^2}{g'W^2}\right)^{1/3}\beta + H\right] = -C_dF^2, \qquad (9)$$

and hence

$$\frac{d\beta}{dx} = \frac{2}{3}\frac{\beta}{W}\frac{dW}{dx} - \left(\frac{g'W^2}{Q^2}\right)^{1/3}\left(\frac{dH}{dx} + C_dF^2\right). \qquad (10)$$

This shows that friction acts to force the flow toward criticality regardless of whether it is sub- or supercritical. At the control point $\beta = 3/2$, $d\beta/dx = 0$, and

$$W^{-1}\frac{dW}{dx} - \left(\frac{g'W^2}{Q^2}\right)^{1/3}\frac{dH}{dx} = C_d\left(\frac{g'W^2}{Q^2}\right)^{1/3}. \quad (11)$$

The control section is shifted downstream from its location in the inviscid case, as first pointed out by Henderson (1966).

The tendency of the Froude number is easily obtained from (10) using $\beta$ from (6). The slope $dh/dx$ may then be obtained after writing the term in parentheses on the left-hand side of (8) as ($\frac{1}{2}F^2h + h + H$), whence

$$\frac{dh}{dx} = \frac{F^2}{1 - F^2}\left(\frac{h}{W}\frac{dW}{dx} - F^{-2}\frac{dH}{dx} - C_d\right). \quad (12)$$

For a subcritical flow a negative interface slope is caused by bottom friction as well as by a decrease in width and an increase in bottom height.

## 3. Solutions with entrainment

We now proceed to the situation where fluid from the passive layer is entrained into the active layer with an entrainment velocity $u_e$. This would appear to be the most usual case, although entrainment from the active to the passive layer can dominate in rapidly accelerating flows over sills (Farmer and Armi 1999). We will consider several increasingly complex situations. For the sake of simplicity we initially ignore friction and assume a constant channel cross section. More general results are presented subsequently.

### *a. Entrainment of water of the same density and zero horizontal momentum*

The first problem we consider is with no upper layer (so that $\rho_0 = 0$) but with the entrained water having the same density as the lower layer. In this case $\rho$ remains constant. The problem is somewhat artificial but describes a single layer receiving water from a sprinkler system.

The momentum equation is best derived by considering the momentum flux per unit width, given by

$$\int_0^h (\rho u^2 + p)\, dz = \rho\left(\frac{1}{2}g'h^2 + hu^2\right), \quad (13)$$

where $p = \rho g'(h - z)$. (Really $g' = g$ for this problem, but we continue to write $g'$ for easy comparison with other examples.) The added water has zero horizontal momentum, so the momentum flux remains constant, giving

$$\frac{1}{2}g'h^2 + hu^2 = \frac{1}{2}g'h_R^2. \quad (14)$$

Here $h_R$ is the layer thickness in an upstream reservoir where $u$ approaches 0 (though this cannot actually happen with the present assumption, to be dropped later, of a constant channel width as well as flat bottom). Using $Q = uh$ and $F^2 = Q^2/(g'h^3)$ we obtain an algebraic expression relating $F$ and $Q$:

$$\gamma \equiv F^{2/3} + \frac{1}{2}F^{-4/3} = \frac{1}{2}\left(\frac{g'h_R^3}{Q^2}\right)^{2/3}. \quad (15)$$

Although $\gamma$ differs from $\beta$ (see Fig. 2), it also has a minimum of 2/3 at $F = 1$. This means that the entrainment of fluid, which causes $Q$ to increase, pushes the system toward criticality in a way that is similar to the effect of friction, as discussed above.

By differentiating (15) and (14) with respect to $x$ we obtain expressions for the rate of change of the squared Froude number and layer thickness, respectively:

$$\frac{dF^2}{dx} = \frac{4}{Q}\frac{F^2(F^2 + 1/2)}{1 - F^2}\frac{dQ}{dx}, \quad (16)$$

$$\frac{dh}{dx} = -\frac{2h}{Q}\frac{F^2}{1 - F^2}\frac{dQ}{dx}. \quad (17)$$

Here $dQ/dx$ is given by the entrainment velocity $u_e$. Note that the effect of entrainment is to decrease the layer thickness of a subcritical flow, notwithstanding the addition of entrained fluid.

### *b. Entrainment of lighter water with zero horizontal momentum*

A more realistic situation occurs when the upper layer is less dense than the deeper, active layer. We expect that the entrained fluid of lower density will lead to a horizontal density gradient in the lower layer, which will tend to accelerate the flow.

The conserved momentum flux is now

$$\int_0^{h_T} (\rho u^2 + p)\, dz = \rho h u^2 + \frac{1}{2}g(\rho - \rho_0)h^2 + \frac{1}{2}g\rho_0 h_T^2, \quad (18) \quad (19)$$

where $h_T$ is the height of a horizontal surface in the upper layer, with, consequently, no flux across it of horizontal momentum.

We combine $dQ/dx = u_e$ (conservation of volume) and $d(\rho Q)/dx = \rho_0 u_e$ (conservation of mass) to obtain $(\rho - \rho_0)Q =$ const. It is now convenient to define $g' = g(\rho - \rho_0)/\rho_0$ so that $g'Q = A^3$ (const), or

$$Q\frac{dg'}{dx} + g'\frac{dQ}{dx} = 0. \quad (20)$$

Replacing $\rho$ by $\rho_0$ in the first term of (19) (i.e., making the Boussinesq approximation) we now have

$$\frac{1}{2}g'h^2 + hu^2 = \frac{1}{2}g'_R h_R^2, \quad (21)$$

where $g'_R$, $h_R$ are the reduced gravity and layer thickness in a fictitious reservoir. This may be rewritten as

$$F^{2/3} + \frac{1}{2}F^{-4/3} = \frac{1}{2}g'_R h_R^2 (QA)^{-1}; \tag{22}$$

whence

$$\frac{dF^2}{dx} = \frac{3}{Q}\frac{F^2(F^2 + 1/2)}{1 - F^2}\frac{dQ}{dx}. \tag{23}$$

As before, entrainment acts to force the flow toward criticality, but at a slower rate than in (16) because of the presence of a density gradient.

The density gradient has a remarkable effect on the slope of the interface. In contrast to (17) we now obtain

$$\frac{dh}{dx} = -\frac{2h}{Q}\frac{F^2 - 1/4}{1 - F^2}\frac{dQ}{dx}. \tag{24}$$

For $F^2 < 1/4$ the layer thickness $h$ increases instead of decreasing, even though the system is forced toward criticality, whereas for $F^2 > 1/4$ the layer thickness decreases. This may be clarified by considering the particular case of $F^2 = 1/4$. In this case the pressure gradient associated with the density gradient is just great enough to accelerate the flow and thus transport the entrained water downstream, such that the layer thickness remains constant.

### *c. Entrainment of lighter water with nonzero horizontal momentum*

Finally, consider the case where the upper layer is moving with a constant velocity $\upsilon$. It can be shown that treating $\upsilon$ as constant is a good approximation provided that the upper layer is much thicker than the lower layer. The fluid entrained into the lower layer has nonzero horizontal momentum and the lower-layer momentum changes accordingly.

Starting with (18), as before, we obtain

$$\rho h u^2 + \frac{1}{2}g(\rho - \rho_0)h^2 - \rho_0 h u \upsilon = \text{const}, \tag{25}$$

in place of (19) so that, with the Boussinesq approximation as before,

$$hu^2 + \frac{1}{2}g'h^2 = \frac{1}{2}g'_R h_R^2 + Q\upsilon \tag{26}$$

in place of (21). The derivative of this is

$$\frac{d}{dx}\left(hu^2 + \frac{1}{2}g'h^2\right) = \upsilon\frac{dQ}{dx}, \tag{27}$$

showing how the momentum flux of the active layer is changed by the entrainment of moving water. The extension of (22) is

$$F^{2/3} + \frac{1}{2}F^{-4/3} = \frac{1}{2}g'_R h_R^2 (QA)^{-1} + \upsilon A^{-1}, \tag{28}$$

where $g'Q = A^3$ is still constant.

The extensions of (23) and (24) are

$$\frac{dF^2}{dx} = \left(\frac{3}{Q}\right)\frac{F^2\left(F^2 + \frac{1}{2} - \frac{\upsilon}{u}F^2\right)}{1 - F^2}\frac{dQ}{dx}, \tag{29}$$

$$\frac{dh}{dx} = -\left(\frac{2h}{Q}\right)\frac{F^2 - \frac{1}{4} - \frac{1}{2}\frac{\upsilon}{u}F^2}{1 - F^2}\frac{dQ}{dx}. \tag{30}$$

It follows from (29) and (30) that the results for exchange flows ($\upsilon < 0$) are qualitatively similar to those obtained for $\upsilon = 0$. The magnitude of both $dh/dx$ and $dF^2/dx$ will be larger for $\upsilon < 0$ as the entrainment of negative momentum has an effect similar to that of additional friction.

For unidirectional flows $\upsilon$ is positive. In principle, $dF^2/dx$ may change sign when $\upsilon$ exceeds a certain value:

$$\frac{\upsilon}{u} = 1 + \frac{1}{2}F^{-2}. \tag{31}$$

The ratio $\upsilon/u$ is 3/2 when the flow is critical. As $F^2 > 1$ increases, $\upsilon/u$ approaches unity but rapidly increases for $F^2 < 1$ decreasing. In most examples of subcritical flow, one might expect the upper-layer velocity to be not much larger than the lower-layer velocity so that $dF^2/dx$ is unlikely to become negative. This means that for most unidirectional flows entrainment acts to force the flow toward criticality.

For entrainment of zero momentum fluid we found that the interface slopes upward if $F^2 < 1/4$. For nonzero momentum of the upper layer the interface slopes upward if

$$F^2 < \frac{1}{4}\left(1 - \frac{1}{2}\frac{\upsilon}{u}\right)^{-1}$$

and is flat if

$$\frac{\upsilon}{u} = 2 - \frac{1}{2}F^{-2}. \tag{32}$$

Hence, for $\upsilon > 0$ the slope $dh/dx$ can be zero for Froude numbers larger than 0.5. The physical reason is that the entrained positive momentum supports the density gradient in accelerating the flow so that the required density gradient can be smaller and, correspondingly, the Froude number can be larger for conditions of zero slope.

## 4. Summary and discussion

This study was motivated by the question of how entrainment affects the hydraulic state and the layer thickness of a reduced-gravity flow. Different cases have been investigated under the general assumption that the upper layer is passive in the sense that its velocity is independent of $x$. As indicated earlier, the results can also be applied to an active upper and passive lower layer. In that case bottom topography is not directly relevant.

In all but one case it was found that entrainment acts to force the flow toward criticality in just the same way as friction. The exception is the unidirectional flow that occurs when the upper-layer velocity is larger than the lower-layer velocity by a certain, Froude number dependent, margin.

For the artificial example in which the entrained fluid has the same density as the active layer, the layer thickness decreases for subcritical flow and increases for supercritical flow. If less dense fluid is entrained, the layer thickness increases for subcritical flow with $F^2 < 1/4$ (for $\upsilon = 0$).

Results with entrainment have been presented for zero friction and constant channel cross section. However, friction and variable cross section can easily be included, thus combining the effects analyzed separately in sections 2 and 3.

The rate of change of the Froude number is given by

$$\frac{dF^2}{dx} = \frac{F^2}{1 - F^2}\left[\frac{3}{h}\frac{u_e}{u}\left(F^2 + \frac{1}{2} - \frac{\upsilon}{u}F^2\right) - (2 + F^2)\frac{1}{W}\frac{dW}{dx} + \frac{3}{h}\frac{dH}{dx} + \frac{3}{h}F^2C_d\right], \quad (33)$$

where we used $Q = uhW$ and $dQ/dx = u_eW$. This equation is equivalent to (10) if $u_e = 0$ and to (29) if $W$ and $H$ are constant and $C_d = 0$. The interface slope is given by

$$\frac{dh}{dx} = \frac{F^2}{1 - F^2}\left[-2\frac{u_e}{u}\left(1 - \frac{1}{4}F^{-2} - \frac{1}{2}\frac{\upsilon}{u}\right) + \frac{h}{W}\frac{dW}{dx} - F^{-2}\frac{dH}{dx} - C_d\right], \quad (34)$$

which reduces to (12) and (30) in the appropriate limits. These results have assumed that the upper-layer speed $\upsilon$ is constant. This is really only possible if the channel width $W$ is also constant. Consequently, (33) and (34) with $dW/dx \neq 0$ are valid only if $\upsilon = 0$.

We may relate $u_e/u$ to the Froude number through an appropriate entrainment law. Christodoulou (1986) proposes

$$\frac{u_e}{u} = 0.002\mathrm{Ri}_b^{-1} \quad \text{for } 0.1 < \mathrm{Ri}_b < 10, \quad (35)$$

where $\mathrm{Ri}_b = g'h/(\Delta u)^2$ is the bulk Richardson number with $\Delta u$ the velocity difference between the layers. If $\upsilon = 0$, then $\mathrm{Ri}_b = F^{-2}$ so that $u_e/u = 0.002\,F^2$. The factor 0.002 is empirically determined from water tunnel data and is subject to considerable uncertainty. With $\upsilon = 0$ the ratio of the entrainment term and friction term is

$$\frac{0.002}{C_d}\left(F^2 + \frac{1}{2}\right) \quad \text{and} \quad \frac{0.004}{C_d}\left(F^2 - \frac{1}{4}\right)$$

in (33) and (34), respectively. For $C_d = 10^{-3}$ (Pratt 1986) this suggests that the effects of entrainment and friction on the flow can be of roughly equal importance.

It has been shown (Henderson 1966, and others) that bottom friction displaces the control downstream. The same is true for entrainment. Assuming constant width and a motionless upper layer, $dh/dx$ remains bounded for $F = 1$ only if

$$\frac{dH}{dx} = -C_d - \frac{3}{2}\frac{u_e}{u}. \quad (36)$$

Again, using $C_d = 10^{-3}$ and (35) we find that the control location is displaced farther than in the case of friction alone. A sill must then be sufficiently steep on its lee side for control to be possible. Hence, entrainment can totally inhibit control that would otherwise occur in an inviscid or purely frictional flow.

It has been pointed out by Pratt (1986) that friction can cause purely subcritical flows to take on an ‘‘overflow’’ character in which there is a net decrease in elevation across the sill in the manner of a sub- to supercritical transition. See, for example, curve A–A′ in Fig. 6 of Pratt (1986). This might be important for certain ocean overflows where hydraulic control is assumed based on the drawdown of the interface but where no verification has been made that the flow actually becomes critical. According to (34) this effect can also occur as the result of entrainment, at least if $1/4 < F^2 < 1$.

In the present note we are concerned only with the rate of change of layer thickness and Froude number. Evaluation of the layer properties as a function of the channel coordinate $x$ requires that (34) be evaluated simultaneously with $dQ/dx = u_eW$ subject to given initial conditions in the upstream reservoir. The two equations are coupled and need to be solved iteratively because the entrainment velocity $u_e$ [i.e., (35)] is a function of Froude number, which can be determined at each step given $h$, $Q$, $W$, and $g'$ from $g'Q = \text{const}$.

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