NOTES AND CORRESPONDENCE

What Fraction of a Kelvin Wave Incident on a Narrow Strait Is Transmitted?

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ABSTRACT

Parameters governing the fraction of a Kelvin wave transmitted through a narrow gap or channel include time dependence, nonlinearity, friction, and strait geometry, yet only limited regions of this parameter space have been explored. Linear inviscid models (which neglect advective and frictional terms in the momentum equations) predict that 100% of the volume flux of a low-frequency Kelvin wave or steady boundary current incident on a narrow strait is transmitted, even when the strait width becomes infinitesimally small. Here the nonlinear, inviscid, flat bottom problem is considered, and it is shown that, provided the geometry varies slowly, the quasi-steady solution can be found in the rotating-hydraulics literature. In the narrow channel limit the fraction transmitted can be approximated by a simple prediction based on nonrotating hydraulics. Unless an incoming Kelvin wave has a large amplitude in comparison with the background layer depth, the strait width must be considerably smaller than the deformation radius before it limits the volume flux passing through. Results also show that a Kelvin wave of given volume flux will squeeze through a narrower gap if it is pushed rather than pulled.

1. Introduction

The large-scale ocean circulation is characterized by the existence of intense narrow boundary currents that flow along topographic features. The dynamics of these boundary currents when they encounter narrow gaps and channels significantly affects circulation patterns, watermass properties, and heat and freshwater transports (e.g., Godfrey 1996), yet sea straits are generally poorly represented in ocean and climate models (e.g., Gent et al. 1998; Gordon et al. 2000).

In the Canadian Archipelago, for example, buoyant baroclinic boundary currents traveling around the Arctic basin encounter many gaps and channels of a width comparable to the local deformation radius (Melling 2000). Understanding what governs the flow through these straits is important in part because they deliver freshwater directly to the North Atlantic Ocean deepwater formation regions, though the impact of the Canadian Archipelago on the thermohaline overturning circulation in the Atlantic remains unclear (Goosse et al. 1997; Wadley and Bigg 2002; Komuro and Hasumi 2005). Similarly, the interaction of equatorial currents with the channels of the Indonesian Throughflow region has implications for the transfer of properties between the Indian and Pacific Oceans on seasonal (monsoon), interannual (ENSO) and longer time scales and is not yet well understood (e.g., Godfrey and Golding 1981; Godfrey 1996).

The understanding and prediction of time-dependent flow through these and other straits are also important because of the observational possibilities that straits offer, since they funnel large-scale flows through easily instrumented or surveyed regions, and because of the effect of strait flows on oceanographic conditions within the channels themselves.

Recent interest in flow through straits has focused largely on the problem of deep overflows, where the hope is that hydraulic theory can be used to link the volume flux of the overflow with some easily observed property in the upstream basin (e.g., Hansen et al. 2001). There is a large body of literature concerning

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rotating hydraulics [see Pratt (2004) for a review of recent advances]. Time-dependent studies are usually in the context of dam-break problems in which the flow is established by the propagation of waves initiated by the removal of an imaginary barrier at the sill or constriction. We concern ourselves instead with a low-frequency Kelvin wave or quasi-steady boundary current, propagating along a coast, which then encounters a gap or strait (i.e., the Kelvin wave amplitude upstream is prescribed and is not set by adjustment from the sill). What fraction of the volume flux associated with the Kelvin wave is transmitted through the strait, and what fraction continues unimpeded along the coast?

Here we show that the answer to the nonlinear, inviscid, quasi-steady problem can be found in the seminal paper on rotating hydraulics by Gill (1977). Gill presents a theory for the controlled flux through a strait of given slowly varying geometry, when the upstream conditions are specified. The theory can also be interpreted as giving the percentage of any given incident low-frequency Kelvin wave (or quasi-steady boundary current) transmitted through the strait. The purpose of this note is simply to demonstrate this useful, yet not commonly applied, interpretation of Gill's well-established rotating hydraulic control theory and to map out the relevant parameter space governing the transmission of a Kelvin wave through a narrow gap or channel. Important parameters are likely to include time dependence, friction, rotation, nonlinearity, and strait geometry, yet only limited regions of this parameter space have been explored. We outline some outstanding questions for the future.

The theory applies to both barotropic and reduced-gravity flows, where for the former g is the acceleration due to gravity, H is the undisturbed water depth, and $R = \sqrt{gH}/f$ is the external deformation radius and for the latter g is the reduced gravity, H is the undisturbed layer depth, and R is the internal deformation radius.

2. Linear inviscid theory

Consider the case of a low-frequency Kelvin wave, propagating along a coast, which then encounters a gap or strait. Since the pressure anomaly and velocity associated with the wave decay away from the boundary on a scale equal to the deformation radius (R), one might expect the wave to feel the opposite boundary of the strait for widths of magnitude R or smaller. For much larger widths the wave will simply propagate along one wall of the strait as if it were a curved coastline (provided that the radius of curvature is large compared with the inertial radius). We might therefore expect 100% of the Kelvin wave amplitude to be transmitted through the strait until the width $W \lesssim R$, after which some of the incident wave will be "reflected" from the strait and continue along the boundary in the upstream basin.

Durland and Qiu (2003) have recently demonstrated, however, that for the linear inviscid problem (i.e., when advective and frictional terms are neglected) of a 1½-layer subinertial Kelvin wave incident on an ideal rectangular strait, the energy transmission through the strait approaches 100% at low frequencies, regardless of strait width and length. All of the Kelvin wave is transmitted through the strait, even when the width is infinitesimally small! They arrive at this conclusion by considering a wave-interference problem involving matching solutions at the entrance and exit of a strait. Their result can be anticipated, however, by considering the geostrophic limit for flow through a strait (linear, inviscid, frequency $\omega = 0$) identified by Garrett and Toulany (1982).

Garrett and Toulany (1982) assume geostrophic balance in the across-strait direction and an along-strait momentum balance between acceleration, pressure gradient, and friction. To close the problem they also assume that the sea level anomaly on the right-hand side of each strait entrance (looking into the strait) is the same as that in the neighboring basin, consistent with the direction of Kelvin wave propagation into the strait. The volume flux through a strait connecting basins 1 and 2 is then

$$Q = WHu = \frac{gH(h_1' - h_2')/f}{1 + L(fW)^{-1}(i\omega + r)}, \tag{1}$$

where h' is the sea level anomaly in each basin, H is the undisturbed water (or layer) depth, L is the length of the strait, W is its width, f is the Coriolis parameter, and r is the simplest possible linear friction coefficient. The numerator in this expression is simply the volume flux of a Kelvin wave whose amplitude is equal to the sea level difference between basins. When friction is negligible and the frequency $\omega \ll Wf/L$ the denominator becomes equal to 1, and the flux through the strait is simply given by the volume flux of this Kelvin wave (i.e., 100% transmission).

In deriving Eq. (1) Garrett and Toulany (1982) assume a constant depth H in both the strait and neighboring basins. Toulany and Garrett (1984) show that this geostrophically controlled limit agrees with the precise linear diffraction theory for a narrow gap (Buchwald and Miles 1974) if the strait length is replaced by an effective length or "end correction."

Rocha and Clarke (1987) extend these results to a strait that has a depth different from the depths of the two basins it connects. They find that it is the depth of the basins that governs the flux through the strait, rather than the depth of the strait itself.

Provided the two basins are of equal depth, Eq. (1) can be generalized to allow for variable strait depth using the recent approach of Garrett and Cummins (2005). The along-strait momentum balance can be expressed as

$$\frac{\partial}{\partial t}\left(\frac{Q}{A}\right) = -g\frac{\partial h}{\partial x} - \frac{rQ}{A}, \tag{2}$$

where the velocity u has been written in terms of the volume flux $Q(t)$, which only varies in time, provided that the strait length is much less than a wavelength, and the cross-sectional area $A(x)$, which we can assume only varies with along-strait distance x. Integrating along the length of the strait L we find that

$$c\frac{\partial Q}{\partial t} = g\Delta h - QF, \tag{3}$$

where Δh is the sea level drop in the along-strait direction,

$$c = \int_0^L \frac{1}{A}\,dx \quad \text{and} \quad F = \int_0^L \frac{r}{A}\,dx. \tag{4}$$

The geometrical factor c allows for variable bottom topography and is insensitive to the exact locations of the ends of the strait (at $x = 0$, L), which allows for a less subjective representation of real straits than simply using the length L and width W. Similarly, the frictional integral F is insensitive to the exact location of drag forces within the channel. Note that if r is constant then $F = rc$, but Eq. (4) allows for a friction coefficient that varies along the channel, as would be the case if the friction experienced by low-frequency flows is linearized about the tides.

Equation (1) now becomes

$$Q = \frac{gH(h_1' - h_2')/f}{1 + Hf^{-1}(i\omega c + F)}. \tag{5}$$

Comparison of Eqs. (1) and (5) shows that the effective length L_{eff} of a strait with variable bottom depth is

$$L_{\text{eff}} = HWc = \int_0^L \frac{HW}{H_S W_S}\,dx, \tag{6}$$

where H is the ocean depth outside the strait, W refers to the width at the strait entrance, and $H_S(x)$ and $W_S(x)$ are in-strait values; L_{eff} is always greater than the measured length L. Physically, this simply reflects the fact that a constriction increases the flow speed, which means greater acceleration and requires a larger gradient in sea level for a given flux.

It is not clear whether there is a similar simple generalization of the results of Garrett and Toulany (1982) and Rocha and Clarke (1987) in the case in which the two basins have different depths. In what follows, however, we assume a strait of constant depth equal to that of the basins at either end (or a reduced-gravity situation in which the strait depth is not important).

Toulany and Garrett (1984), Pratt (1991), and Durland and Qiu (2003) all recognize that these results (100% transmission as the frequency tends to zero) only hold as long as the dynamics remains linear, which becomes less likely at low frequencies when the particle excursion is comparable to or greater than the strait length and advection becomes important. Nevertheless, there is some evidence in support of this result from numerical models. Lombok Strait, for example, has a width that is about 1/3 of the local first-mode baroclinic deformation radius, yet the nonlinear modeling study by Qiu et al. (1999) demonstrates almost 100% transmission of variability in the intraseasonal band. Coastal Kelvin waves generated by remote wind forcing in the central equatorial Indian Ocean propagate southward along the coast of Indonesia until they reach Lombok Strait, through which all of the signal is transmitted. No significant signal continues along the coast.

From an observational point of view it is interesting to note that the dimensionless parameter

$$\frac{H}{f}(i\omega c + F) \tag{7}$$

or, for an ideal strait of length L and width W,

$$\frac{(i\omega + r)L}{fW} \tag{8}$$

is equal to the ratio of the along-strait sea level difference to the across-strait difference when the flow is linear. With relatively little data the combined importance of time dependence and friction for any given strait may thereby be assessed.

3. Allowing for nonlinearity

For the nonlinear but frictionless case one simple hypothesis, the natural extension of what would happen in the nonrotating case (in which a current approaching a constriction will be partially transmitted and partially reflected such that a critical flow is set up), is that a Kelvin wave approaching a strait will all squeeze through up to the point at which the flow becomes hydraulically controlled. For water originating in a basin of depth H nonrotating hydraulics tells us that the controlled flow through the strait will have flux Q, where

$$H = \frac{3}{2}\left(\frac{Q^2}{gW^2}\right)^{1/3} \tag{9}$$

and W is the width at the constriction. If this flux is associated with a Kelvin wave of amplitude h' ($\ll H$) at the wall, it is given to first approximation by $Q = gHh'/f$, which implies that all of the Kelvin wave will be transmitted through straits wider than W_c, where

$$W_c = R\frac{h'}{H}\left(\frac{3}{2}\right)^{3/2}. \tag{10}$$

This equation suggests that straits must be significantly narrower than the deformation radius to inhibit the passage of a Kelvin wave. We have assumed, however, that $h' \ll H$ and have ignored, for now, the fact that the relationship between H and Q will differ slightly for rotating hydraulics.

a. When does nonlinearity become important?

Seven dimensional quantities can be identified that may play a role in governing the flux through a frictionless strait of constant depth. These can be chosen as ω, f, W, R, strait length L (strictly here the scale over which the strait width changes by 100%), wavenumber $k = 2\pi/\lambda$, and particle excursion along the strait ξ. (The prescribed parameter is really the particle excursion in the incident Kelvin wave; we take ξ to be R/W times this.) With dimensions of only length and time, we therefore expect five dimensionless numbers, which could be ξk, ξ/L, W/R, ω/f, and kR. The last of these is, in fact, redundant since $kR = \omega/f$, leaving four independent nondimensional numbers.

The degree of nonlinearity, or the ratio of the advective term to the local acceleration, is the largest of the first two parameters: ξ/L and ξk. If either of these two parameters is large, the Kelvin wave has a finite amplitude, and it is possible that transmission through the strait will be limited by hydraulic control.

The relative size of these two parameters also determines whether the incoming wave will be partially reflected by the converging channel walls. Even small discontinuities in a narrow channel will partially reflect a linear Kelvin wave (Buchwald 1990). Lighthill (1978) shows that such reflection is unavoidable unless $L \gg 1/k$; that is, the scale over which the strait width changes by 100% is far longer than the wavelength of the incoming wave.

To avoid reflection by the converging channel walls, then, requires that $\xi/L \ll \xi k$. This has implications for the frequency of Kelvin wave that can pass unreflected through a strait of any given length scale. Since $k = \omega/c$, where $c = \sqrt{gH}$ is the gravity wave speed, for there to be no reflection

$$\frac{\omega}{f} \gg \frac{R}{L}. \tag{11}$$

Partial reflection is avoided only at high frequencies or in long slowly varying straits.

On the other hand, in the linear case we can get a second estimate from Eq. (1) of the effect of frequency in limiting the flow. Unless

$$\frac{\omega}{f} \ll \frac{W}{L}, \tag{12}$$

some of the wave's energy is reflected. Thus, for a frictionless, linear strait to have no effect on transmission

$$\xi k \ll \frac{W}{R}\frac{\xi}{L}. \tag{13}$$

This suggests that, even when $\xi/L \ll \xi k$, some reflection occurs if $W/R \lesssim 1$. Also, when $\xi/L \gg \xi k$, if $W > R$, 100% of the volume flux may be transmitted even though the strait is apparently short enough to cause reflection according to standard wave ideas.

As nonlinearity begins to play a role (ξ/L and ξk become larger), the requirement of a long slowly varying channel to avoid reflection by the converging channel geometry is unlikely to change significantly, but Eq. (1) becomes less relevant.

Figure 1 illustrates these variations in partial wave reflection across the parameter space mapped out by ξ/L and ξk. Note that $\xi k \lesssim 1$, unless the flow is supercritical, and that there will, of course, be some kind of gradual transition between regimes $\xi/L < 1$ and $\xi/L > 1$ and between $L < 1/k$ and $L > 1/k$, indicated by the dashed lines drawn in the figure. Regions A and B represent the linear regime, as in Garrett and Toulany (1982) or Durland and Qiu (2003), where both ξ/L and ξk are small. Reflection here depends also on the third nondimensional parameter W/R as discussed above. In region C the amplitude of the waves is still small, but the particle excursion is a significant fraction of the strait length and so advection within the strait cannot be neglected. Some reflection from the channel walls is expected. In regions D and E, the waves have finite amplitude and linear approximations are no longer valid. Reflection is expected in E but not in D (although some dependence on W/R as in regions B and A, respectively, is likely).

For the quasi-steady nonlinear flow that is the subject of the remainder of this note (region F), $\xi/L \gg 1$ and hydraulic theory (which neglects local acceleration) seems reasonable. Then, even though L may be less than $1/k$, provided the flow is essentially parallel to the central axis of the channel, convergence of the walls does not prove a problem. This limit will apply when $\omega L/u \ll 1$. For a velocity of order 0.1 m s^{-1} and a strait

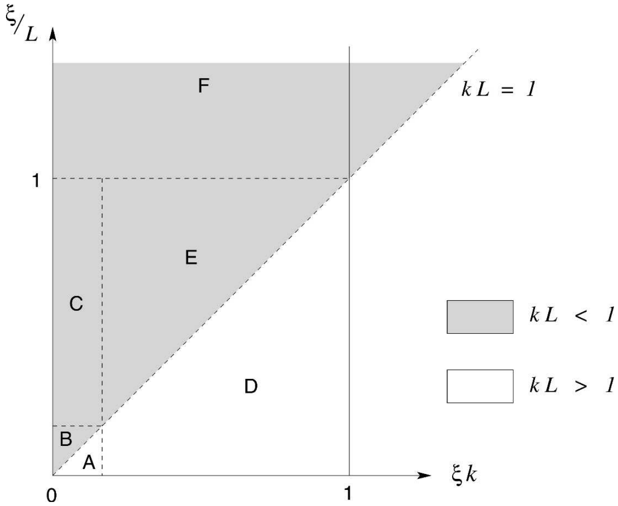

FIG. 1. Schematic diagram illustrating the regimes in which energy is reflected from the sides of a converging channel as a function of the two different measures of nonlinearity, ξ/L and ξk; ξ is the particle excursion along the strait, $k = 2\pi/\lambda$ is the wavenumber, and L is the strait length scale. Region A: small amplitude waves. Partial reflection from converging channel only expected if $W/R \lessapprox 1$. Region B: small amplitude waves. Partial reflection unless $\xi k \ll W/R\ \xi/L$. Region C: small amplitude waves. Cannot neglect advection in strait. Expect partial reflection. Region D: finite amplitude waves. No reflection. Region E: finite amplitude waves. Partial reflection. Region F: quasi-steady flow through constriction. Hydraulic theory applies and we expect no reflection from the converging geometry. This is the region of parameter space of interest in this paper.

of length 10 km this corresponds to frequencies less than 10^{-5} s^{-1}, or time scales longer than a few days.

The third nondimensional number W/R remains important in governing the percentage transmission in the quasi-steady nonlinear regime, as we will go on to demonstrate (see Fig. 4). Further analysis is required in order to establish the behavior of higher frequency finite-amplitude Kelvin waves incident on channels whose length is less than or similar to a wavelength, as is likely to be the case for most real straits if the incident flow is a typical first-mode subinertial wave.

b. Gill's solutions

Equation (10) gives a simple prediction for the strait width at which a quasi-steady Kelvin wave incident upon a strait becomes restricted, based on nonrotating hydraulics. The exact problem for the rotating system is, in fact, solved by Gill (1977). This seminal paper is most commonly cited for its general functional theory of rotating hydraulics, or with reference to its solution for the flow variable ($\overline{D}$) as a function of the two geometric parameters (the width and depth of the constriction) and the upstream parameters (the uniform potential vorticity in the upstream basin and the volume flux in each of two upstream boundary layers). It is our intention here simply to demonstrate that the solutions laid out in Gill (1977) also provide quite naturally the answer to the question of how much of a low-frequency Kelvin wave incident on a narrow strait is transmitted.

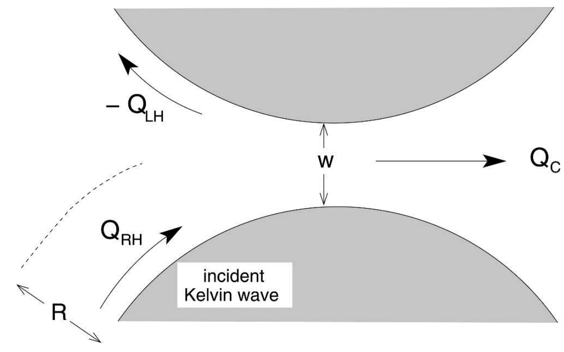

FIG. 2. A Kelvin wave with volume flux Q_{RH} in the Northern Hemisphere is incident upon the rhs (looking downstream) of a strait. Volume flux Q_C is transmitted through the strait, and Q_{LH} is reflected, continuing along the boundary in the upstream basin. Flux $Q_C = Q_{\mathrm{RH}}$ (100% transmission) when the flow is subcritical. If the flux through the strait is controlled, the fraction transmitted can be calculated by finding $\hat{\psi}_i$ as a function of Q_{RH} and strait width W from Gill's rotating hydraulic solutions.

Figure 2 illustrates the problem in Gill's terms. A quasi-steady Kelvin wave with volume flux Q_{RH} in the Northern Hemisphere is incident upon the right-hand side (looking downstream and toward the constriction) of a gap or channel composed of smoothly and slowly varying topography. (Note that the figure is highly schematic and exaggerates the sharpness of the strait geometry.) The exponential structure of a Kelvin wave is consistent with the uniform potential vorticity (PV) assumption essential to Gill's theory. Volume flux Q_C is transmitted through the strait, and Q_{LH} is reflected, continuing along the boundary in the original upstream basin as a reversed flow on the left-hand side of the strait. All volume fluxes can be nondimensionalized by a flow rate $Q_S = gH^2/f$ based on depth scale H, width scale equal to the deformation radius $\sqrt{gH}/f$ based on H, and velocity scale $\sqrt{gH}$, where H is the upstream depth in the stagnant interior away from the boundary layers (and hence sets the potential vorticity).[1] Since the volume flux of a Kelvin wave is

$$\frac{gHh'}{f}\left(1 + \frac{1}{2}\frac{h'}{H}\right),$$

[1] Note that Gill actually uses a factor of 2 in his nondimensionalization, such that the width scale and volume flux scale are 2 times those given above. The results presented in Figs. 3 and 4 are shown without this factor included.

the nondimensional incident volume flux is

$$\frac{Q_{\mathrm{RH}}}{Q_S} = \frac{h'}{H}\left(1 + \frac{1}{2}\frac{h'}{H}\right),$$

which is approximately equal to h'/H when $h'/H \ll 1$.

The fraction of the Kelvin wave volume flux transmitted through the strait is intimately linked to the streamfunction in the upstream basin since this gives the relative magnitude and direction of flows along the two upstream walls; that is, it tells us how much flow is approaching the strait along one boundary and how much continues along the other boundary without passing through the strait. To solve our problem we therefore need to extract from Gill's solutions the relationship between this streamfunction and the incident volume flux in the Kelvin wave. Gill nondimensionalizes the streamfunction using the volume flux through the strait, and characterizes the upstream flow by $\hat{\psi}_i$, the nondimensional streamfunction in the interior of the upstream basin, away from the influence of the two boundaries. This means that the ratio of the volume flux on the left bank (looking downstream and toward the constriction) to that on the right bank (Q_{LH}:Q_{RH}) is

$$\left(\frac{1}{2} + \hat{\psi}_i\right):\left(\frac{1}{2} - \hat{\psi}_i\right), \tag{14}$$

where the sign convention is such that both flows are positive when directed toward the strait.

Gill (1977) solves for the controlled flow through the strait when $\hat{\psi}_i$ is given. In particular, he focuses on the case $\hat{\psi}_i = 1/2$, where the upstream flow is all contained in a boundary layer on the left-hand wall (looking toward the strait and downstream) since this is the solution that would be obtained as the result of Kelvin wave adjustment in the Northern Hemisphere associated with removing a barrier in the constriction or at the sill. For the current problem, however, the relevant range is $\hat{\psi}_i \leq -1/2$, for which the flow in the right-hand wall is toward the strait and there may also be a flow away from the strait along the left-hand wall.

For this range of $\hat{\psi}_i$ the fraction of the volume flux transmitted (T) is given by

$$T = \frac{Q_C}{Q_{\mathrm{RH}}} = \frac{1}{1/2 - \hat{\psi}_i}. \tag{15}$$

In the limiting case $\hat{\psi}_i = -1/2$, $Q_{\mathrm{RH}} = Q_C$ and 100% of the incident flux is transmitted through the strait. The critical width at which this occurs can be found from the solution to Gill's functional equation for rotating hydraulics [Gill 1977, Eq. (5.13)] in the flat-bottom case:

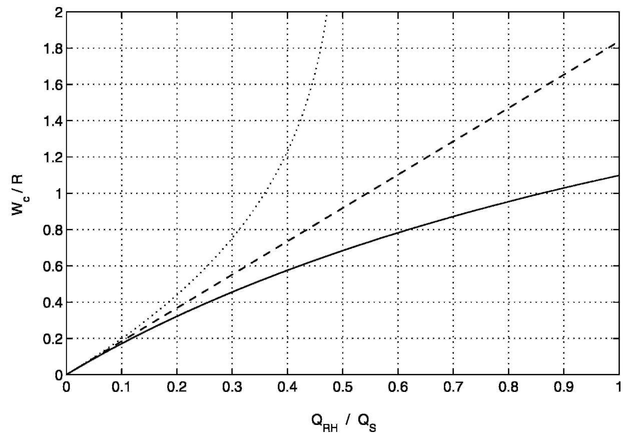

FIG. 3. Limiting strait width W_c, above which transmission is 100%, plotted as a function of incident volume flux when the direction of flow in the Kelvin wave is toward the strait (pushing: solid line) and away from the strait (pulling: dotted line). Volume flux is nondimensionalized by $Q_S = gH^2/f$ and strait width by $R = \sqrt{gH}/f$. The dashed line shows the simple prediction based on nonrotating hydraulics given in Eq. (10).

$$4\hat{\psi}_i + 2\hat{D}_\infty(\overline{D} - \hat{D}_\infty) + \frac{1}{(t\overline{D})^2} + t^2(\overline{D} - \hat{D}_\infty)^2 = 0 \tag{16}$$

together with its derivative [Gill 1977, Eq. (9.6)] with respect to $\overline{D}$

$$(1 - t^2)\hat{D}_\infty + t^2\overline{D} = \frac{1}{t^2\overline{D}^3}, \tag{17}$$

where $t = \tanh[W/(2R)]$, $\overline{D}$ is the flow variable (average of the nondimensional layer depth on the two walls) at the control section, and the parameter $\hat{D}_\infty = \sqrt{2Q_S/Q_C}$ determines the volume flux through the strait. When combined to eliminate $\overline{D}$, these two equations constitute an algebraic equation of order 8 for $\hat{D}_\infty$ in terms of t and $\hat{\psi}_i$. Since this is difficult to solve explicitly, the solution is obtained instead by interpolation in tables. Equation (16) gives $\hat{\psi}_i(t, \overline{D})$ [with the required values of $\hat{D}_\infty$ calculated from Eq. (17)]. Therefore, for a given t and $\hat{\psi}_i$, in particular for $\hat{\psi}_i = -1/2$ here, the value of $\overline{D}$ can be deduced. Equation (17) then provides the relevant $\hat{D}_\infty$ and, hence, the flux through the strait Q_C.

The case of interest here is considerably simpler than the full Gill solution since it involves only a horizontal constriction and no change in bottom depth. As such, it is equally valid for reduced gravity and barotropic solutions, where the deformation radius R, gravity g, and layer depth H would change between the two.

Figure 3 illustrates the result. The limiting strait width W_c, above which transmission through the strait

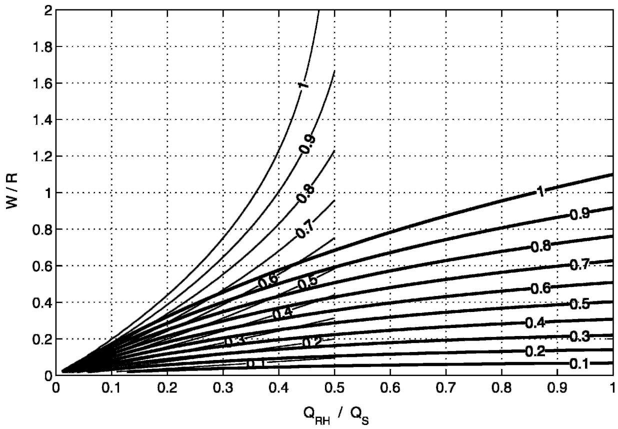

FIG. 4. Transmission through the strait as a function of incident volume flux and strait width. Volume flux is nondimensionalized by $Q_S = gH^2/f$ and strait width by $R = \sqrt{gH}/f$. Heavy lines indicate the pushing case, in which the volume flux is toward the strait, and the lighter lines apply when flow is pulled away from the strait.

is 100%, is plotted as a function of incident volume flux (solid line). The volume flux is nondimensionalized by $Q_S = gH^2/f$ and the width by R. It is clear that, unless an incoming quasi-steady Kelvin wave is of large amplitude (Q_{RH}/Q_S large and $h' \approx H$), the strait width must be considerably less than the deformation radius R before it begins to limit the volume flux passing through. For a nondimensional incident flux of 0.1 (i.e., $Q_{\mathrm{RH}}/Q_S = 0.1$, $h'/H \approx 0.1$), for example, the Kelvin wave transmission only falls below 100% for widths less than $0.17R$.

Also plotted in Fig. 3 is the simple prediction for the limiting width, based on nonrotating hydraulic theory, discussed at the beginning of this section. This prediction diverges from the full solution but, as we might expect, it provides a good approximation in the narrow channel limit $W_c \ll R$, where h'/H for 100% transmission $\lesssim 0.1$.

For strait widths smaller than the limiting value shown in Fig. 3 at each Q_{RH} the flow through the strait is hydraulically controlled, transmission is less than 100%, and $Q_{\mathrm{LH}} < 0$. The fraction of the incident volume flux transmitted can again be calculated by interpolation in tables. Since tables of $\hat{D}_\infty(t, \overline{D})$ and $\hat{\psi}_i(t, \overline{D})$ based on Eqs. (16) and (17) are effectively tables of Q_C and the fraction of volume flux transmitted T [see the definition of parameter $\hat{D}_\infty$, and Eq. (15)], they can be used to construct $Q_{\mathrm{RH}}(t, \overline{D})$. By interpolation for a given t and Q_{RH} the transmission T can then be deduced, and is plotted in Fig. 4 (heavy lines). The contour labeled 1 in this figure represents 100% transmission through the strait and is identical to the solid line in Fig. 3.

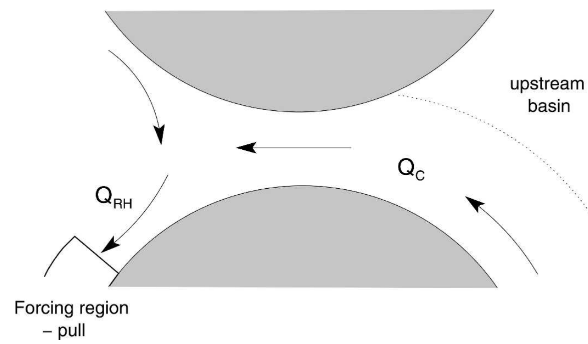

FIG. 5. Schematic illustrating the direction of flow in the pulling case. The controlled flux through the strait at any given width is now determined by $\hat{\psi}_i$ in the other, unforced basin, where it is equal to 1/2 since the flow is established on the left-hand wall (looking downstream).

c. Pushing versus pulling

So far, the volume flux in the Kelvin wave on the right bank has been assumed to be in the direction of the strait. What if the direction of flow in the incoming Kelvin wave is reversed as in Fig. 5? In this case the forcing is located in the downstream basin and consists of a "pulling" of fluid along the boundary. If the strait is wider than some critical width W_c, then this fluid will all be supplied through the strait. For narrower straits, however, the flow through is limited by hydraulic control, and some must also be drawn across the channel mouth in the forced, downstream basin to make up the difference. The controlled flux through the strait is determined by the properties of the other, upstream basin where $\hat{\psi}_i = 1/2$ since adjustment occurs via Kelvin wave propagation upstream from the constriction (as in the dam break problem). Assuming uniform PV in both basins and no background flow in the upstream basin, the fraction of the original Kelvin wave transmitted is given simply by Q_C/Q_{RH}, where $Q_C(W)$ is found from Gill's solution for $\hat{\psi}_i = 1/2$.

The dotted line in Fig. 3 shows the strait width above which 100% of the flux is supplied through the strait in this pulling case, as a function of the incident volume flux (or equivalently h'/H when this is small). It demonstrates that a low-frequency Kelvin wave of given volume flux will squeeze through a narrower gap if it is pushed rather than pulled. This is a result of the different surface elevation at the coast in each case. A Kelvin wave with velocity directed toward the strait has a positive sea surface elevation (or layer thickness in the reduced gravity problem), while a wave with velocity away from the strait has a negative surface elevation. As a result the mean depth is lower in a pulled wave, which therefore requires a wider channel to support the same volume flux at a given velocity.

In the pulling case the maximum flux through the strait is $|Q_C/Q_S| = 0.5$. This is because the difference in layer depth across the channel is then equal to 2 times the mean layer depth at the constriction and the flow begins to separate from the left-hand wall. In the limit $|Q_C/Q_S| = 0.5$ the critical width $W_c \to \infty$, indicating that in a channel of any width, once the flow becomes separated from the left-hand wall it is always controlled flow. The volume flux no longer depends on the width of the strait and is set by the value at the separation point. Therefore volume fluxes larger than 0.5 cannot be pulled through the strait. The maximum possible value of $|Q_{\mathrm{RH}}/Q_S|$ in the pulling case is also 0.5, corresponding to the largest possible anomaly in layer thickness in the forcing region $h' = -H$. For this reason the plot of percentage transmission as a function of width and incident Kelvin wave volume flux in Fig. 4 only shows values for $|Q_{\mathrm{RH}}/Q_S| < 0.5$ in the pulling case. In the pushing case the flow is attached to both side boundaries, for all the solutions discussed here, since separated solutions are not possible for the range $\hat{\psi}_i \leq -1/2$ that is relevant.

Helfrich and Pratt (2003) also consider the link between different upstream values of $\hat{\psi}_i$ and the controlled flux through a strait. However, as in other studies (e.g., Pratt and Llewellyn Smith 1997; Pratt 1997), their focus is on the coupling between the upstream basin and the strait, and the two upstream walls are assumed to interact. The problem addressed here is different in that it assumes an infinite upstream basin, although it does not make any a priori assumption that the fluid is stagnant at either of the two upstream walls. In a similar framework, Whitehead and Salzig (2001) also note the importance of forcing on the right-hand upstream wall.

Gill's (1977) uniform PV solutions are equivalent to the zero-PV solutions of Whitehead et al. (1974) in the narrow channel limit, where the width of the control section is small when compared with the deformation radius ($W \ll \sqrt{gH}/f$), which means that the potential vorticity $f/H \ll g/(fW^2)$.

4. Future questions

Linear models (which neglect the advective terms in the momentum equations) show 100% transmission of an inviscid low-frequency Kelvin wave (or steady boundary current) incident upon a narrow gap or channel. We have demonstrated here that the full nonlinear solution to the problem is embedded in Gill's (1977) rotating hydraulics equations and that, in the narrow channel limit, the fraction transmitted is given by a simple prediction based on nonrotating hydraulics. Unless it has a large amplitude compared with the background layer depth, a quasi-steady Kelvin wave propagates easily through all straits except those significantly narrower than the deformation radius. Kelvin waves in which the volume flux is away from the strait entrance in the forced basin become restricted more quickly due to the shape of their surface elevation.

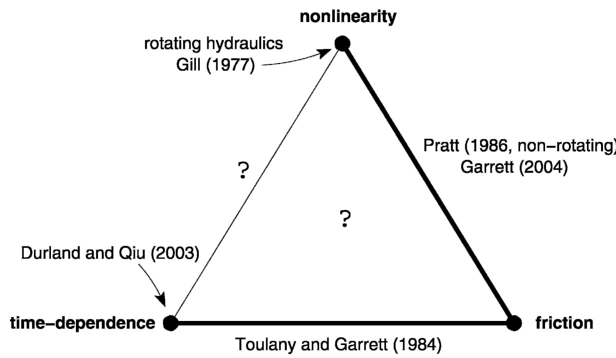

FIG. 6. The parameter space to be explored, represented as a triangle reminiscent of the ternary plots used in geology. Each apex represents one of the three fundamental parameters of interest.

As discussed in section 3a, other important parameters in the problem include time dependence and friction. Figure 6 shows this parameter space and indicates which regions have yet to be explored. Note that the left-hand side of the diagram corresponds to parts of Fig. 1: regions A and B for the bottom apex; C, D, and E for the left-hand side of the triangle, labeled with a question mark; and region F for the top apex.

Durland and Qiu (2003) addressed the time-dependent, linear, inviscid problem, demonstrating that the transmission decreases as the frequency increases above zero. For straits much shorter than a wavelength their result is a special case of the results of Toulany and Garrett (1984) who also included friction. For the nonlinear problem frictional effects on hydraulic control have been discussed without rotation by Pratt (1986); Garrett (2004) has partly extended the analysis to the rotating case. However, the combined effects of nonlinearity and time dependence (as well as friction) on the transmission of a Kelvin wave through a gap or strait have yet to be established.

This problem is perhaps best tackled initially with a simple numerical model, the first test of which should be the ability to reproduce the steady state solutions shown in Fig. 4. Preliminary experiments with a nonlinear 1½-layer reduced-gravity model show promising results, and suggest that the transmission decreases at higher frequencies in accordance with the predictions of Toulany and Garrett (1984) and Durland and Qiu (2003).

We have assumed throughout that the geometry is slowly varying to the extent that the flow can be considered parallel to the central axis of the channel. Section 3a discusses the problem of reflection from the converging channel walls when the flow is not quasi steady. Realistic channel geometries are rarely smooth and slowly varying and there are many outstanding questions about the effect of sharp topographic features on the flow through narrow channels. A coastline is considered to be sharp when the radius of curvature R_c is such that a flow following the coast will be subject to a centrifugal force comparable with the Coriolis force, that is, when $fu \lesssim u^2/R_c$, or $R_c/R_i \lesssim 1$, where $R_i = u/f$ is the inertial radius. In this regime topographic separation of boundary currents from the coast may occur (Bormans and Garrett 1989). Avicola and Huq (2003) demonstrate the role of channel outflow geometry in the formation of recirculating bulges. Results like these suggest that a Kelvin wave may bypass a gap or channel due to the development of a recirculation at the entrance region which spans the entire channel width. The laboratory results of Whitehead and Salzig (2001) and C. Cenedese (2004, personal communication) also suggest this possibility.

We have restricted our attention to the flat-bottom case (allowing us to discuss the reduced-gravity problem as well as the simplest barotropic case). Clearly bottom topography will play an important role (e.g., Whitehead 2003), and the results here could be extended to allow for simple cross-section shapes. In the linear case the integral formulation proposed by Garrett and Cummins (2005) (see section 2) allows for a variable depth along the strait. Rocha and Clarke (1987) find that it is the depth of the two basins on either side that matter and not that of the strait itself. Extending the nonlinear results to allow for variable depth along the strait would be straightforward since the physical principle of hydraulic limitation would still apply.

Using a nonlocal integrated momentum-balance approach Nof (1993, 1995a,b) calculates the highly nonlinear flow through a gap separating two ocean basins on a β plane. In all of these studies the gap is assumed to be wider than R, yet less than 100% of the steady-state boundary current passes through. This difference may be due in part to the different behavior of a gap compared with a slowly varying channel, for example the different forces acting on the sidewalls. However, the effect of β for different orientations of strait must also be established, and the results of Nof (1993, 1995a,b) must be fully reconciled with those presented here.

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