Unexpected Waves

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ABSTRACT

Extreme, or "rogue," waves are those in the tail of the probability distribution and are a matter of great concern and considerable research. They may be partly associated with non-Gaussian behavior caused by resonant nonlinear interactions. Here it is shown that even in a Gaussian sea, "unexpected" waves, in the sense of, for example, waves twice as large as any in the preceding 30 periods, occur with sufficient frequency to be of interest and importance. The return period of unexpected waves is quantified as a function of the height multiplier and prior quiescent interval for various spectral shapes, and it is shown how the return period is modified if allowance is made for nonlinear changes in wave shape and/or a buildup of one or more waves prior to the unexpected wave. The return period of "two-sided" unexpected waves, with subsequent as well as prior quiescence, is also evaluated.

1. Introduction

Large ocean surface waves are a matter of great concern for mariners, designers of ships and offshore structures such as oil platforms, and many users of the seashore. In many situations, the overall roughness of the sea is of primary interest, but particular attention has been paid to the largest waves within a given sea state. A common criterion for designating a given wave as a "rogue" is that its trough to crest height H is at least 2.2 times the significant wave height H_s, defined as four times the standard deviation of the surface elevation. [See Dysthe et al. (2008) for a recent review.]

For the simple case of a narrowband spectrum of linear and independent waves, the surface elevation distribution is Gaussian and the height H has a probability density distribution (Longuet-Higgins 1952):

$$P(H) = \frac{4H}{H_s^2}\exp\left(-\frac{2H^2}{H_s^2}\right). \tag{1}$$

The probability that a wave exceeds νH_s is $\exp(-2\nu^2)$, or $(3.4 \times 10^{-4}, 6.3 \times 10^{-5}, 3.7 \times 10^{-6})$ for $\nu = (2, 2.2, 2.5)$. The corresponding return periods for waves of period 10 s are 8.3 h, 2 days, 1 month. For narrowbanded Gaussian seas, similar results apply for the wave crest $\eta_c = H/2$.

These results need modification if the narrowbanded assumption is relaxed (Naess 1985) or if the probability of rogue waves is determined from data (Forristall 2005). The more rapid change in wave heights means that a large trough is less likely to be followed by a large crest so that extreme wave heights are now considerably less common (typically by a factor of 5 or more) than in the limit of a very narrow band. Allowance may also be made for nonresonant nonlinearity in the waves. In particular, allowing for phase-locked second and higher harmonics, crests become sharper and troughs flatter. This does not affect the distribution of wave heights, at least for a narrowband spectrum, but does mean that a crest height that occurs as frequently as a wave height of $2.2H_s$ is no longer just half this but, rather, $1.34H_s$ for typical situations (Dysthe et al. 2008).

What has been a matter of great concern is the possibility that extreme waves, whether recorded as wave height or crest height, occur more frequently than expected based on Gaussian theory, or on the modification of this theory to allow for shape changes associated with nonresonant nonlinear interactions. In particular, numerical simulations of a reduced equation set have

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suggested that, in seas with a narrow directional spread as well as a narrow peak in frequency, resonant nonlinear interactions (as in the Benjamin–Feir instability) will tend to transfer energy within a wave group, leading to amplification of one or more waves within that group and, hence, the generation of particularly large waves (Socquet-Juglard et al. 2005; Gramstad and Trulsen 2007). On the other hand, the observational evidence for this is limited, partly because of difficulties in measurement and partly because the statistical basis is not strong. [For reviews and discussion see Dysthe et al. (2008), the collection of papers in Müller and Henderson (2005), or Müller and Garrett (2007)].

What is clear from many cited examples of what observers describe as "freak" waves is that they tend to be much larger than the waves in the surrounding sea state, often appearing either singly or in small groups, without warning. In many situations, this unexpectedness is more dangerous than the wave height itself, for example, if mariners interpret an interval of several minutes of relatively small waves as an indication of a decreasing sea state. The question to be addressed in this note is whether unexpectedness is necessarily evidence for strongly nonlinear behavior or whether it could arise frequently from simple linear superposition (perhaps with some shape modification, as described above, to allow for nonresonant nonlinear interactions).

Here we report on Monte Carlo simulations of ocean surface waves, based on simple superposition of random phase sinusoids making up typical wave spectra, and pick out waves that are much larger than a significant number of previous waves, perhaps ignoring one or more waves immediately prior to the "unexpected" one. Our study complements the investigation of the average number of waves, in a group, with height above a prescribed threshold (Tucker and Pitt 2001, and references therein). In that situation, the prior waves need not be much smaller.

2. Simulations

We first choose the wave spectrum $S(f)$ as a function of frequency f (in hertz) [with $\int_0^\infty S(f)df$ giving the variance of the surface elevation] to be one of the Joint North Sea Wave Project (JONSWAP) family (e.g., Holthuijsen 2007), with

$$S(f) = Ag^2 f^{-5} \exp\left[-\frac{5}{4}\left(\frac{f}{f_p}\right)^{-4}\right]\gamma^r;$$

$$r = \exp\left[\frac{1}{2}\left(\frac{f}{f_p} - 1\right)^2 \sigma^{-2}\right], \qquad (2)$$

where $A = 0.008$ and $\sigma = (0.07, 0.09)$ for $f(\leq, >)\, f_p$. Here γ is the peak enhancement factor, taken to be 1.0, 2.0, 3.3 for fully developed, developing, and young seas, respectively. We also take $f_p = 0.1$ Hz, though clearly all our results scale in time with this.

A random sea with a given spectrum may be simulated by adding contributions from a number of independent frequencies separated by a frequency interval Δf, which is then also the lowest frequency present. To avoid repetition, Δf must be taken to be no larger than the reciprocal of the record length (e.g., Tucker et al. 1984). We have chosen to generate time series that are each 10 h long so that $\Delta f = 2.8 \times 10^{-5}$ Hz. We then have 216 000 constituents between 0 Hz and our chosen cutoff frequency of 6 Hz.

We could take the phase of each frequency to be random and assign an amplitude of the square root of twice the required spectral amplitude. However, as discussed by Tucker et al. (1984), the spectral amplitudes should be random and come from a Rayleigh distribution, corresponding to separate sine and cosine terms having amplitudes that are independent and Gaussian, each with a mean square equal to the spectral amplitude. This scheme is implemented in the Matlab toolbox Wave Analysis for Fatigue and Oceanography (WAFO) (WAFO Group 2000), which employs inverse fast Fourier transform techniques to speed up the simulations. We use WAFO and, in order to generate reliable statistics, generate 60 000 independent time series for each situation.

A wave crest is defined as the maximum elevation η_c between a zero upcrossing and a zero downcrossing, thus ignoring possible intermediate crests. The wave height is taken as the elevation change from a similarly defined trough to the following crest. (We return later to consideration of including the change from crest to trough.)

The wave period is the time between a zero crossing and the second successor. The average wave period T_a for a JONSWAP spectrum is given approximately in terms of the peak period $T_p = f_p^{-1}$ by

$$T_a/T_p = (0.71, 0.75, 0.78) \quad \text{for} \quad \gamma = (1.0, 2.0, 3.3). \qquad (3)$$

We define an unexpected wave as one having its height H or crest height η_c greater by a factor α than the wave or crest height of any of the preceding η_a waves (based on the average period T_a), excluding the m waves immediately prior to the unexpected one. In other words, the unexpected wave is preceded by $na - m$ waves, none more than $1/\alpha$ as big as the unexpected wave, and then a buildup of m intermediate waves.

Nonresonant nonlinearity

As remarked above, large waves tend to have sharper crests and flatter troughs than do sinusoids. This is a consequence of nonresonant nonlinearities; to second order in the wave steepness a periodic wave has an elevation (e.g., Holthuijsen 2007)

$$\eta = a\cos\theta + \frac{1}{2}ka^2\cos(2\theta), \tag{4}$$

where θ is the wave phase, $kx - 2\pi ft$, for a wave with wavenumber k propagating in the x direction, and a is the amplitude of the first harmonic. The second-order correction is a forced wave, nonresonant since $(2f, 2k)$ do not satisfy the dispersion relation. The correction flattens the troughs and steepens the crests, but without any change in the trough to crest height for a periodic wave. Allowing for the correction has been found to better account for the shape of large waves (e.g., Walker et al. 2004). Some further improvement can be obtained by allowing for even higher harmonics as well, but we restrict our analysis to allowing for the second harmonic only so as to illustrate as simply as possible the consequence of nonresonant nonlinearity.

For the slowly varying wave trains considered here we implement the correction for crest heights by deriving the local frequency f as the reciprocal of twice the time from upcrossing to downcrossing, determining $k = (2\pi f)^2/g$ from the dispersion relation and adding $(1/2)k\eta_c^2$ to the linear crest height η_c. This may slightly exaggerate the correction if the crest is not midway between the up- and downcrossings, but we ignore this. We could adopt a similar procedure for trough correction and would now get a slight change in the wave height as the linear crests and troughs will be of unequal magnitudes in a slowly varying wave train, but we ignore this detail and only concern ourselves with the correction to the crest height.

Two minor points need to be discussed. The first is that the mean square steepness per octave for the JONSWAP spectra is independent of f_p, since the steepness spectral density is proportional to f^{-1} times the peak enhancement factor. Thus, even with the nonlinear crest height increase, our results are independent of our arbitrary choice of $f_p = 0.1$ Hz and continue to scale in time with f_p^{-1}.

Second, we note that by adding the second harmonic we are altering the spectrum of the waves. We have ignored this as the effect is small; the spectrum is increased by less than 1% up to $1.5f_p$, approximately 2% at $2f_p$, increasing to approximately 10% between $3f_p$ and $4f_p$ by which point the spectral contribution to the mean square wave height is very small.

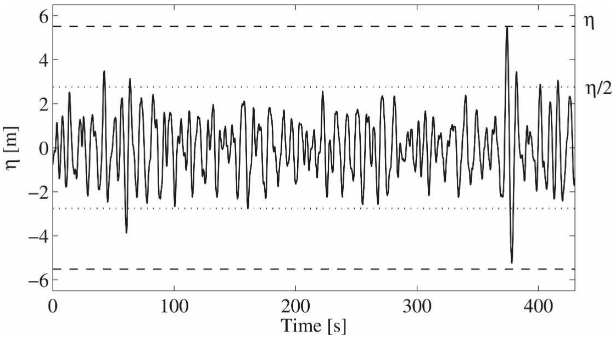

FIG. 1. An example of an unexpected wave, at 374 s, from a Monte Carlo simulation of the sea surface with linear waves of random phase and spectral density corresponding to a JONSWAP spectrum with $\gamma = 1.0$ and peak period of 10 s.

3. Results

Figure 1 shows an example of an unexpected wave. In this case, the crest height at 374 s is 2.1 times as great as any in the preceding 310 s, which is 31 times the peak period of 10 s, or 43 times the average wave period of 7.1 s. In this example, there is no buildup, so the wave would qualify as unexpected, with $n_a = 43$ and $\alpha = 2.1$ for any value of m including zero. The crest height of this particular unexpected wave is $1.4H_s$—greater than the $1.1H_s$ that would qualify it as a rogue wave using linear theory. The trough to crest height, however, is only $1.9H_s$, less than the $2.2H_s$ required for rogue designation (though the height change from the crest to the subsequent trough is $2.7H_s$). We return later to a more general discussion of the extent to which unexpected waves meet a criterion for designation as rogue waves.

The occurrence of the large wave in Fig. 1 following a period of much smaller waves, purely as a consequence of random superposition, is not surprising qualitatively. What is not obvious is the frequency with which such events occur. Figure 2 shows the return period of an unexpected wave as a function of the amplitude multiplier α and number of prior waves no more than $1/\alpha$ times as big, for $m = 0$, that is, allowing for no buildup. The rows are for different sea states and the columns for wave height based on linear waves, crest height η_{lin} for linear waves and crest height η_{nl} using the second-order correction described above.

For $\gamma = 1$, a wave with height H at least twice that of any of the preceding 30 waves (corresponding to 21 peak periods) occurs once every 7×10^4 peak periods on average, giving a return period of 8 days if the peak period of the waves is 10 s. Also for $\gamma = 1$, unexpected crests are more probable than unexpected waves that are high from trough to crest; with $\alpha = 2$ and $n_a = 30$ the return period is 4 days for the linear simulation and 2 days if the nonlinear enhancement is included.

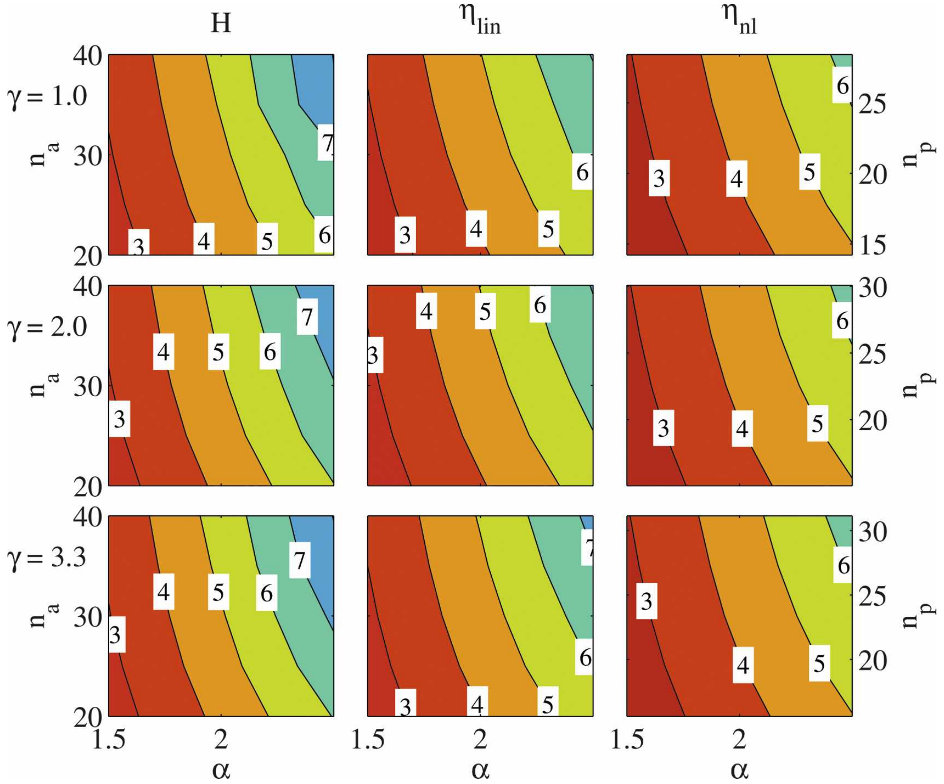

FIG. 2. Contours of $\log_{10}$ of the return period in peak periods for the occurrence of an unexpected wave α times larger than any of the preceding n_a waves ($m = 0$). The rows are for different values of the JONSWAP parameter γ and the columns for, respectively, the wave height, the crest height using linear waves, and the crest height with the second-order correction. The abscissa is labeled on the right with the number n_p of peak periods.

Figure 2 shows that there is no major dependence of the probability on γ for a given α and n_a. For larger γ a given n_a corresponds to a larger n_p, but typically less real time as the peak period will be less for young and developing seas than for a fully developed sea.

Figure 3 is similar to Fig. 2, but now with $m = 2$, allowing for some buildup to the unexpected wave. This leads to an increase in the probability of an unexpected wave, more so for the narrowband spectrum characterized by $\gamma = 3.3$ than for a spectrum with $\gamma = 1$.

To illustrate the dependence of the probability on the group parameter m and the JONSWAP parameter γ, we focus on the particular choices $\alpha = 2$ and $n_a = 30$, corresponding to the point at the center of the plots in Figs. 2 and 3. The figure confirms that, for all three values of m, unexpected crests are more probable than unexpected wave heights from trough to crest, particularly if the nonlinear enhancement is included. The circles in Fig. 4 show that for $m = 0$ the return period R in peak wave periods increases with γ, perhaps because increasing γ corresponds to a peakier spectrum, longer groups, and hence less likelihood of an isolated unexpected wave. Interestingly, the triangles and diamonds ($m = 1$ and 2) show the opposite trend to that for $m = 0$.

4. Discussion

The frequency with which unexpected waves occur even without the extra possible physics of resonant nonlinear interactions is remarkable and of scientific interest. It suggests that many reported freak waves may not be so freakish after all, but merely the simple consequence of linear superposition. Our predictions could be incorporated into maritime safety manuals or coastal warnings. To be sure, we have based our simulations on deep water scenarios and further analysis is required for nearshore situations, but it seems likely that similar results will emerge. Thus, mariners and tourists can be warned that even a few minutes of waves can be followed by a wave at least twice as high, with such an event likely every few days.

There are other possible interpretations and extensions of our analysis. We discuss these next.

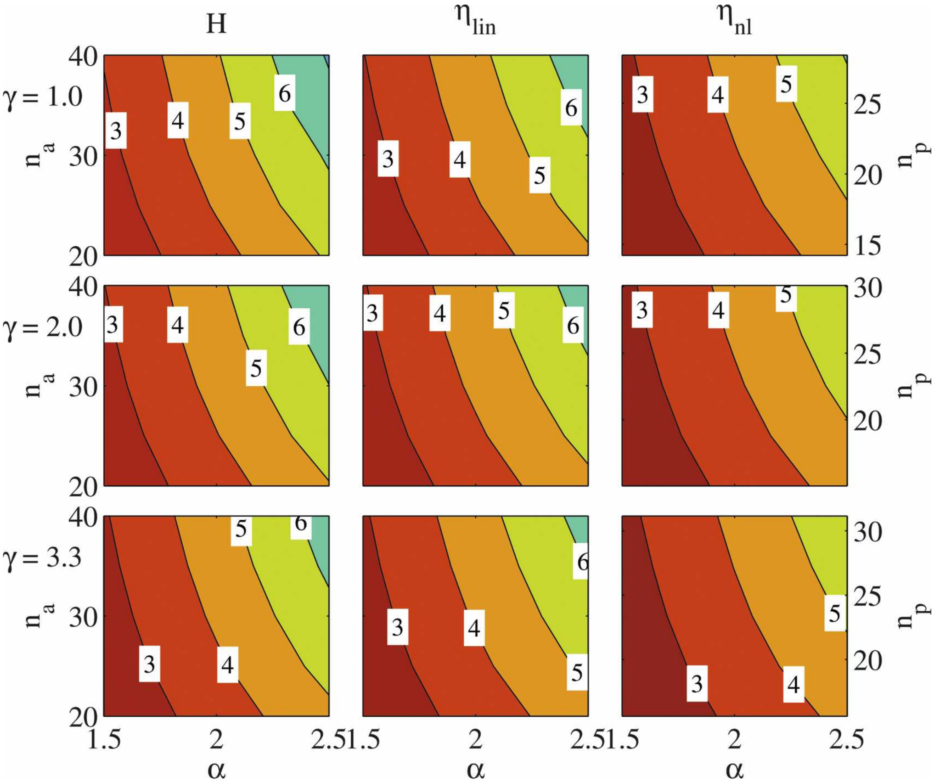

FIG. 3. Contours of $\log_{10}$ of the return period in peak periods for the occurrence of an unexpected wave α times larger than any of the preceding n waves except the immediately prior 2 ($m = 2$). Otherwise, as in Fig. 2.

a. Are unexpected waves rogues?

For each unexpected wave in our simulations we have evaluated its height as a multiple of H_s and calculated the fraction of the waves with a multiplier of more than 2.2. The contours for fixed fractions are broadly similar in shape to those of Figs. 2 and 3. For our canonical situation with $\alpha = 2$ and $n_a = 30$, only about 10%–20% of the unexpected waves are rogues. The percentage increases as α and n_a increase, reaching 70% or so for the very rare unexpected waves with $\alpha = 3$ and $n_a = 40$ in the top right-hand corners of our plots.

b. Wave steepness

The steepness of a large wave is often as important as its height. We have therefore examined the steepness of the unexpected waves in our simulations, first deriving the wavenumber k from the linear dispersion relation. In this, and for evaluation of the steepness using the wave height, the period is defined as the time between the zero downcrossings containing the wave. Using just the crest height to evaluate the steepness, the period is defined, as earlier, as twice the time between the up- and downcrossings containing the crest. We find a tendency for the period to be longer than the average given by (3), by up to 30% for H, though only 5%–10% for η_{nl}. We define the wave steepness as ka, with a given by half the wave height or by the crest height.

The contours of wave steepness are again similar in shape to those of Figs. 2 and 3, with a steepness, for our canonical point in the center of the plots, of approximately 0.17 using H, but 0.26 or so using η_{lin} and 0.35 or so using η_{nl}. If we allow for the change in the dispersion relation for finite amplitude waves, the steepness would be reduced by a factor of approximately $1 - s^2$, where s is the initial estimate (e.g., Grue et al. 2003). This would reduce the estimate of 0.35 to 0.31. Either value is approaching the theoretical limit of 0.44 (e.g., Holthuijsen 2007). The expected steepness increases further toward the top right-hand corner of our graphs so that unexpected waves there are not only very rare but also likely to be constrained by breaking. We remark that wave breaking is commonly observed at a wave steepness (based on H) as small as 0.15 to 0.2 (Banner et al. 2002), though these authors find that the steepness alone is not an adequate guide, and it seems possible that the sudden superposition leading to the

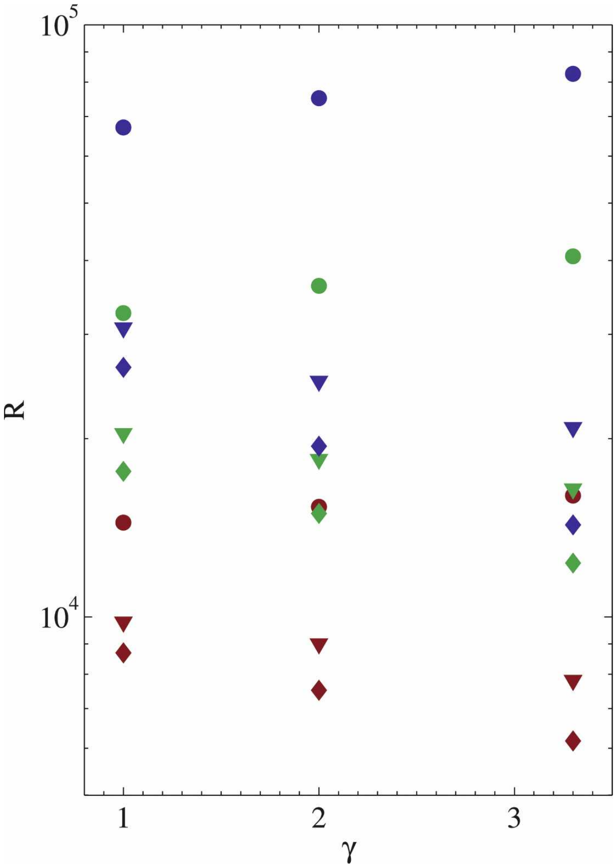

FIG. 4. The return period R, in peak periods, of an unexpected wave twice as large as any of the preceding 30 waves as a function of γ, for various values of m. Values for H are in blue, for η_{lin} in green, and for η_{nl} in red. The circles, triangles, and diamonds are for $m = 0, 1, 2$, respectively.

isolated unexpected waves discussed here might not leave time for the onset of breaking.

c. Different definitions

We have defined the wave height to be the elevation change from trough to subsequent crest. A more general alternative might be to define the wave as the maximum of the elevation changes from a crest to the troughs immediately after as well as before. With this definition we find a slightly reduced probability of an unexpected wave for a given α and n_a. The statistics for the crests are unaffected.

d. Subsequent quiescence

We have focused on periods of quiescence prior to unexpected waves, with no restrictions on the size of subsequent waves. There are, however, some remarkable observations of extreme waves that are much larger than subsequent as well as previous waves for many wave periods (Dysthe et al. 2008). Kimura and Ohta (1994) evaluated the frequency of occurrence, for narrowband spectra, of freak waves twice as high as the immediately prior and subsequent waves, but placed no conditions on the wave height outside this group of three waves. Our simulations can easily be used to examine the probability of large waves with any selection for the duration of prior and subsequent quiescence. We examine the situation assuming n_a subsequent as well as n_a prior smaller waves, excluding m waves on each side of the unexpected one.

Not surprisingly, the return period of unexpected waves is now very much longer than for waves conditioned only on prior quiescence. For our canonical point at the center of our plots the return period increases to the order of 10^8 peak periods, or so, if $m = 0$. This return period is 30 yr for waves with a peak period of 10 s.

The return period decreases by more than an order of magnitude if we take η_{nl}, as shown in Fig. 5. This shows that, for any value of γ, a wave with a crest twice as high as that of any of the preceding and subsequent 30 waves occurs approximately every 4×10^6 peak periods, or about once every 15 months if the peak period is 10 s. The expected return period is even less if we

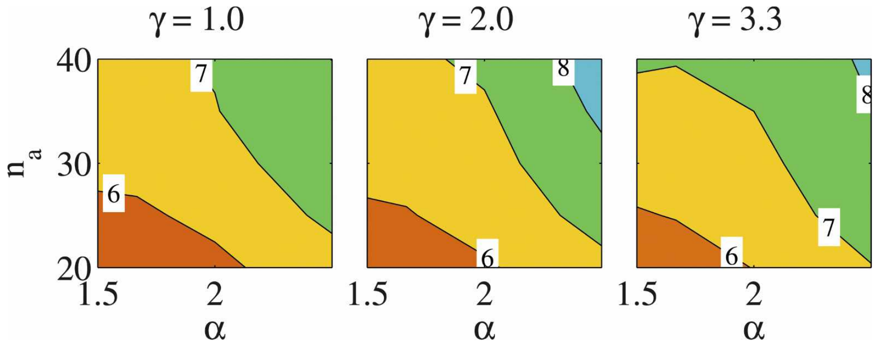

FIG. 5. Contours of $\log_{10}$ of the return period in peak periods for the occurrence of an unexpected wave with a crest height η_{nl} that is α times larger than any of the preceding and subsequent n_a waves, for different values of γ.

take $m = 1$ or larger, allowing for waves just before and just after the largest wave to be intermediate in height, particularly for $\gamma = 3.3$. It does seem possible that the very few isolated extreme waves that have been reported could have arisen from simple superposition in quasi-Gaussian seas, with an enhancement of crest height only from the nonresonant harmonics.

e. Observations

Our future plans are to examine actual wave records for unexpected waves and to compare their frequency of occurrence with that in synthetic time series made up of waves with a power spectrum equal to that of the observations but with random phase. Preliminary analysis of datasets from one location off the coast of British Columbia indicates a frequency of occurrence of unexpected crest heights close to that in the simulated records of nonlinear crest heights if the amplification factor over prior waves is 2, but about double the predicted frequency if the amplification is only 1.67 or 1.5. Further analysis is required and will be reported in due course.

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