Fourier Representation of Quadratic Friction

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ABSTRACT

If a current is composed of a number of constituents with different frequencies, then quadratic friction may be analyzed at the same frequencies. The ratios of the constituents of the friction differ from the ratios for the current itself, with a classic result being that for unidirectional flow a very weak current constituent experiences proportionately 50% more friction than a strong constituent. Here, exact results for the magnitude of the friction constituents are derived and confirmed numerically. The results are applied to the tidal currents in Juan de Fuca Strait and the Strait of Georgia, showing that minor constituents experience proportionately more friction than the main constituent by an amount that varies spatially but is typically less than the classic result of 50%. For two-dimensional currents it is shown that, if there are two current constituents with the same ellipticity and major axis direction, the friction coefficients are separable functions of the current constituent ratio and the ellipticity. Some results are derived for two constituents with different ellipticity and major axis direction. For the case of two constituents with rectilinear but misaligned currents, each constituent experiences friction inclined at an angle to its current. Last, the effect of a tidal current on the bottom friction experienced by a steady flow is investigated for arbitrary relative magnitudes and directions of the tide and steady flow. In particular, the inclination of the mean friction to the mean flow is quantified.

1. Introduction

Bottom friction is an important process in the ocean and is usually expressed as

$$\boldsymbol{\tau} = \rho C_D \mathbf{u}|\mathbf{u}|. \tag{1}$$

Here, ρ is density, C_D is the drag coefficient, and $\mathbf{u}$ is the velocity vector. The value of C_D depends on the bottom roughness and roughness height. In barotropic tidal models, for example, a smaller value of C_D is used with the depth-averaged current than with the current at some prescribed height above the bottom (typically 1 m). The drag coefficient may also be weakly dependent on the magnitude of $\mathbf{u}$. Ignoring this, the drag has the interesting property that it is mathematically an odd, rather than even, function of the current even though it is quadratic in magnitude.

This structure leads to interesting interactions between different components of the flow that one might wish to treat separately. For example, if there are many tidal constituents the friction experienced by one is affected by the presence of the others. Similarly, akin to a special case of the tidal situation, the friction experienced by a mean flow is affected by the presence of tidal currents. These issues have been much analyzed in the past, but the parameter space is rich.

One simple but important example occurs if the current is rectilinear and composed of one dominant constituent and a much smaller constituent. We assume noncommensurate frequencies ω_0, ω_1, such that $p\omega_0 \neq q\omega_1$ for small integers p and q. In that case, the ratio for the constituents of the bottom friction differs from the ratio for the currents, and the weaker constituent feels proportionately 50% more friction than the dominant constituent (Jeffreys 1970). This makes physical sense; the quadratic nature of the friction means that it is disproportionately stronger at the maximum of the modulated tides (spring tide if the constituents are M_2 and S_2) than at the minimum (neap tides). Thus, friction acts to reduce the modulation, which is equivalent to there being more damping of the weaker constituent. This effect is a key factor in the analysis of resonance in bays (e.g., Garrett 1972), and in frequency-space nu-

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merical simulations of regional tidal currents (e.g., Snyder et al. 1979; Pingree and Griffiths 1981; Le Provost and Fornerino 1985).

The 50% factor here relies on the assumption that $\varepsilon \ll 1$, where ε is the ratio of the magnitudes of the weaker and dominant constituents. Corrections of $O(\varepsilon^2)$, for both the ratio and the actual drag forces, were derived by Jeffreys (1970). Other authors have extended the analysis to allow for more than two constituents and two-dimensional currents. The most comprehensive study has been by Pingree (1983) who considered a large number of situations and conducted numerical Fourier analysis to allow for arbitrary values of ε in the case of two constituents.

In this paper, we show how a new approach to the problem leads to simple exact results in a number of situations. The results are readily confirmed and extended using the FFT routine in MATLAB. There is a wide range of situations to be considered, given that there can be many tidal constituents with different amplitudes, different current ellipticities, and different directions of their major axes. We start in section 2 with the comparatively simple case of rectilinear currents (as might be found in channeled tidal flows in coastal waters). Results for many constituents are then applied in section 3 to the recent analysis (Sutherland et al. 2005) of tides in Juan de Fuca Strait and the Strait of Georgia. In section 4 we allow for two-dimensional currents as occur in the open ocean, at first assuming that tidal constituents, while having different amplitudes, have the same ellipticity and major axis orientation. This may be a reasonable assumption for tidal constituents that are close in frequency, such as M_2 and S_2, but not for constituents from, for example, semidiurnal and diurnal bands. The full range of possibilities is very large, but we examine some special cases to give a guide to possible effects, such as a misalignment of current and stress for each constituent. A special case of great importance occurs when one of the tidal currents has zero frequency, and so represents a rectilinear steady current that experiences a drag that is greatly influenced by the presence of a tidal current with arbitrary magnitude, ellipticity, and major axis direction. We consider this problem in section 5, reviewing and extending the work of Saunders (1977).

2. Unidirectional currents

a. Exact solutions

With a single constituent $u = u_0 \cos\omega_0 t$, the Fourier expansion of $u|u|$ is

$$u|u| = \frac{8u_0^2}{3\pi}\cos\omega_0 t + \cdots, \tag{2}$$

where the ellipsis implies terms with frequencies other than ω_0 (in this case, the frequencies are $3\omega_0$, $5\omega_0$, etc.). Adding a second constituent, we write

$$u = u_0 \cos\omega_0 t + u_1 \cos\omega_1 t = u_0(\cos\omega_0 t + \varepsilon_1 \cos\omega_1 t), \tag{3}$$

where $\varepsilon_1 = u_1/u_0$ is taken between 0 and 1 (assuming without loss of generality that the ω_0 constituent remains dominant). We are assuming that the two constituents are in phase at $t = 0$. This is justified in appendix A. We may write the Fourier expansion of $u|u|$ as

$$u|u| = \frac{8u_0^2}{3\pi}(F_0 \cos\omega_0 t + \varepsilon_1 F_1 \cos\omega_1 t + \cdots), \tag{4}$$

so that $F_0 = 1$ when $\varepsilon_1 = 0$. By symmetry there are no terms in (4) involving $\sin\omega_0 t$ or $\sin\omega_1 t$. Now the ellipsis implies terms with frequencies other than ω_0 and ω_1.

Our new approach to the determination of F_0 and F_1 starts with defining

$$\alpha_1 = \omega_0 t - \omega_1 t \tag{5}$$

and writing

$$\begin{aligned} u &= u_0 \cos\omega_0 t + u_1 \cos\omega_1 t = u_0[\cos\omega_0 t + \varepsilon_1 \cos(\omega_0 t - \alpha_1)] \\ &= u_0[(1 + \varepsilon_1 \cos\alpha_1)\cos\omega_0 t + \varepsilon_1 \sin\alpha_1 \sin\omega_0 t] = u_0(1 + 2\varepsilon_1 \cos\alpha_1 + \varepsilon_1^2)^{1/2}\cos(\omega_0 t - \zeta_1), \end{aligned} \tag{6}$$

with

$$\zeta_1 = \mathrm{atan}(\varepsilon_1 \sin\alpha_1, 1 + \varepsilon_1 \cos\alpha_1), \tag{7}$$

where atan(y, x) is just the arctangent of y/x but with the specific quadrant specified by the signs of x and y.

If α_1, and hence ζ_1, were constant, then the Fourier expansion of $u|u|$ would have an amplitude corresponding to a current of strength $u_0(1 + 2\varepsilon_1 \cos\alpha_1 + \varepsilon_1^2)^{1/2}$ with frequency ω_0 and may be written

$$u|u| = \frac{8u_0^2}{3\pi}[(1 + 2\varepsilon_1 \cos\alpha_1 + \varepsilon_1^2)(\cos\zeta_1 \cos\omega_0 t + \sin\zeta_1 \sin\omega_0 t) + \cdots]. \tag{8}$$

The phase α_1 varies, of course, but it is uniformly distributed. This is the key feature of the new approach. Hence, we average (8) using (7) to obtain

$$F_0 = \frac{1}{\pi}\int_0^{\pi}(1+\varepsilon_1\cos\alpha_1)(1+2\varepsilon_1\cos\alpha_1+\varepsilon_1^2)^{1/2}\,d\alpha_1 \quad \text{and}$$
$$F_1 = \frac{1}{\varepsilon_1\pi}\int_0^{\pi}(\varepsilon_1+\cos\alpha_1)(1+2\varepsilon_1\cos\alpha_1+\varepsilon_1^2)^{1/2}\,d\alpha_1, \qquad (9)$$

where we recognize that the coefficient of sin $\omega_0 t$ averages to zero and where we have included the value of F_1 derived in a similar manner. The average in (9) should be over 0 to 2π, but symmetry allows us to restrict the range to 0 to π. We note that, as required by symmetry, $F_1(\varepsilon_1) = \varepsilon_1 F_0(1/\varepsilon_1)$.

When $\varepsilon_1 = 1$,

$$F_0 = F_1 = \frac{2^{1/2}}{\pi}\int_0^{\pi}(1+\cos\alpha_1)^{3/2}\,d\alpha_1$$
$$= \frac{8}{\pi}\int_0^{\pi/2}\cos^3\xi\,d\xi = \frac{16}{3\pi} = 1.698, \qquad (10)$$

using $\xi = \alpha_1/2$. We note that $F_1/F_0 = 1.0$ (when the constituents are equal in magnitude, there is no difference in the friction they experience).

For $\varepsilon_1 \ll 1$, we expand the square root term in the integrals in (9) and obtain approximate solutions,

$$F_0 = 1 + \frac{3}{4}\varepsilon_1^2 - \frac{3}{64}\varepsilon_1^4 \quad \text{and} \quad F_1 = 1.5\left(1+\frac{1}{8}\varepsilon_1^2+\frac{1}{192}\varepsilon_1^4\right), \qquad (11)$$

to fourth order in ε_1. These expressions contain the classic result that, for $\varepsilon_1 \ll 1$, the weaker constituent experiences proportionately 50% more friction than the dominant constituent. The second-order terms were obtained by Jeffreys (1970) and the fourth-order term for F_0, but not for F_1, by Pingree (1983). The fourth-order expansion is surprisingly accurate; when $\varepsilon_1 = 1$, (11) gives 1.703, 1.695, and 0.995 for F_0, F_1, and F_1/F_0, respectively, instead of the exact values of 1.698, 1.698, and 1. The exact solution from (9) is shown in Fig. 1.[1]

We note that the magnitudes of other Fourier components are also readily obtained using this new approach (see appendix B).

[1] The above result no longer holds if ω_0 and ω_1 are commensurate. A simple example is with $\omega_1 = 2\omega_0$, in which case $F_1 = 1.4$ to the lowest order in ε_1. We will not examine this, or similar, situations further.

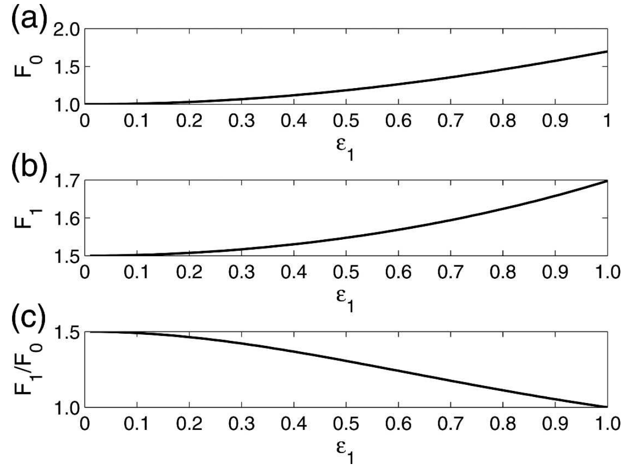

FIG. 1. The Fourier coefficients of $u|u|$ for two constituents: (a) F_0, (b) F_1, and (c) F_1/F_0. The solid line is the exact solution.

b. Numerical integration

The exact results of (9) are easily evaluated, but we could also obtain the coefficients F_0 and F_1 by using the FFT package in MATLAB for any value of ε_1. In doing this we have arbitrarily chosen the frequencies ω_0 and ω_1 as 31.23 and 28.27 Hz, respectively. The results are insensitive to the values of the frequencies as long as they are noncommensurate for small integral multiples. The sampling frequency must be high enough to avoid any aliasing from high-frequency spectral peaks in $u|u|$ and is taken to be 1024 Hz. The record length must be long enough to sample all the different phase relationships between the two constituents and is taken to be 102 400. Using these values leads to results in agreement with Fig. 1. Finer sampling frequencies and longer data lengths do not lead to any changes.

c. Many tidal constituents

We can extend our new approach to include many constituents. We first consider three, with

$$u = u_0(\cos\omega_0 t + \varepsilon_1\cos\omega_1 t + \varepsilon_2\cos\omega_2 t), \qquad (12)$$

where $\varepsilon_2 = u_2/u_0$. Once again we assume that we may take the three constituents to be in phase at $t = 0$ (see appendix A). We write

$$u|u| = \frac{8u_0^2}{3\pi}(F_0\cos\omega_0 t + \varepsilon_1 F_1\cos\omega_1 t + \varepsilon_2 F_2\cos\omega_2 t + \cdots). \qquad (13)$$

Proceeding as before by defining $\alpha_1 = \omega_0 t - \omega_1 t$ and $\alpha_2 = \omega_1 t - \omega_2 t$, we obtain

$$F_0 = \frac{1}{4\pi^2}\int_0^{2\pi}\int_0^{2\pi}(1 + \varepsilon_1\cos\alpha_1 + \varepsilon_2\cos\alpha_2)[1 + 2\varepsilon_1\cos\alpha_1 + 2\varepsilon_2\cos\alpha_2 + \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1\varepsilon_2\cos(\alpha_1 - \alpha_2)]^{1/2}\,d\alpha_1\,d\alpha_2,$$

$$F_1 = \frac{1}{4\varepsilon_1\pi^2}\int_0^{2\pi}\int_0^{2\pi}[\cos\alpha_1 + \varepsilon_1 + \varepsilon_2\cos(\alpha_1 - \alpha_2)][1 + 2\varepsilon_1\cos\alpha_1 + 2\varepsilon_2\cos\alpha_2 + \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1\varepsilon_2\cos(\alpha_1 - \alpha_2)]^{1/2}\,d\alpha_1\,d\alpha_2,$$

and

$$F_2 = \frac{1}{4\varepsilon_2\pi^2}\int_0^{2\pi}\int_0^{2\pi}[\cos\alpha_2 + \varepsilon_1\cos(\alpha_1 - \alpha_2) + \varepsilon_2][1 + 2\varepsilon_1\cos\alpha_1 + 2\varepsilon_2\cos\alpha_2 + \varepsilon_1^2 + \varepsilon_2^2 + 2\varepsilon_1\varepsilon_2\cos(\alpha_1 - \alpha_2)]^{1/2}\,d\alpha_1\,d\alpha_2, \tag{14}$$

where the integrals over the phases α_1 or α_2 must now be from 0 to 2π to include all possibilities.

For small ε_1 and ε_2, the fourth-order expansions are

$$F_0 = 1 + \frac{3}{4}(\varepsilon_1^2 + \varepsilon_2^2) - \frac{3}{64}(\varepsilon_1^4 + \varepsilon_2^4) - \frac{3}{16}\varepsilon_1^2\varepsilon_2^2, \quad F_1 = 1.5\left(1 + \frac{1}{8}\varepsilon_1^2 + \frac{1}{4}\varepsilon_2^2 + \frac{1}{192}\varepsilon_1^4 + \frac{1}{64}\varepsilon_2^4 + \frac{3}{32}\varepsilon_1^2\varepsilon_2^2\right), \quad \text{and}$$

$$F_2 = 1.5\left(1 + \frac{1}{8}\varepsilon_2^2 + \frac{1}{4}\varepsilon_1^2 + \frac{1}{192}\varepsilon_2^4 + \frac{1}{64}\varepsilon_1^4 + \frac{3}{32}\varepsilon_1^2\varepsilon_2^2\right). \tag{15}$$

These expressions show expected symmetries in ε_1 and ε_2 and reduce to (11) when $\varepsilon_2 = 0$. While F_0 depends equally on both ε_1 and ε_2, F_2 has a weaker dependence on ε_2 than on ε_1. Pingree (1983) obtained the fourth-order terms for F_0 but not for F_1 or F_2.

Figure 2 shows the coefficients for ω_0 and ω_2 from the integral expression and numerical solutions. When $\varepsilon_1 = \varepsilon_2 = 1$, the exact solutions for F_0, F_2, and F_2/F_0 are 2.151, 2.151 and 1, respectively, whereas (15) gives 2.219, 2.234, and 1.007, respectively.

We can generalize the problem to many tidal constituents, as is likely to be necessary for the real ocean. We take

$$u = u_0(\cos\omega_0 t + \varepsilon_1\cos\omega_1 t + \varepsilon_2\cos\omega_2 t + \cdots + \varepsilon_n\cos\omega_n t + \cdots), \tag{16}$$

and express the Fourier expansion of $u|u|$ as

$$u|u| = \frac{8u_0^2}{3\pi}(F_0\cos\omega_0 t + \varepsilon_1 F_1\cos\omega_1 t + \varepsilon_2 F_2\cos\omega_2 t + \cdots + \varepsilon_n F_n\cos\omega_n t + \cdots). \tag{17}$$

By assuming $\varepsilon_1, \varepsilon_2, \varepsilon_3, \ldots, \varepsilon_n, \ldots$ small, the coefficients for the main and n constituents in (17) are derived by the same procedure used in (15), giving

$$F_0 = 1 + \frac{3}{4}(\varepsilon_1^2 + \varepsilon_2^2 + \cdots + \varepsilon_n^2 + \cdots) - \frac{3}{64}(\varepsilon_1^4 + \varepsilon_2^4 + \cdots + \varepsilon_n^4 + \cdots) - \frac{3}{16}(\varepsilon_1^2\varepsilon_2^2 + \varepsilon_1^2\varepsilon_n^2 + \varepsilon_2^2\varepsilon_n^2 + \cdots),$$

$$F_n = 1.5\left[1 + \frac{1}{8}\varepsilon_n^2 + \frac{1}{4}(\varepsilon_1^2 + \varepsilon_2^2 + \cdots)\right] + \text{fourth-order terms}. \tag{18}$$

3. An application

Before moving onto two-dimensional problems, we apply results in the one-dimensional problem to Juan de Fuca Strait and the Strait of Georgia (Sutherland et al. 2005, hereinafter SGF), since the Rossby radius of deformation is much greater than the channel width (SGF) and the different tidal constituents have currents that are reasonably rectilinear and parallel (Foreman et al. 2004). The ratios of the weaker constituent current amplitudes to the dominant one ($\varepsilon_n = u_n/u_{M_2}$) are estimated for each constituent (Table 1).

The coefficients F_n are then estimated from (18) and shown in Fig. 3. We note that, because of the presence of harmonics (K_1 and K_2), the approximate expression (18) is not mathematically exact. However, as long as M_2 dominates and the current amplitude of K_2 is small, this effect can be neglected. In Seymour Narrows, M_2 dominates over other constituents, so that the other constituents feel more friction; however, the values of F_n/F_0 are smaller than 1.4. In the eastern and western sections of Juan de Fuca Strait, M_2 still dominates over other constituents, but ε_{K_1} goes up to 0.6 in these locations so that the values of F_n/F_0 are smaller

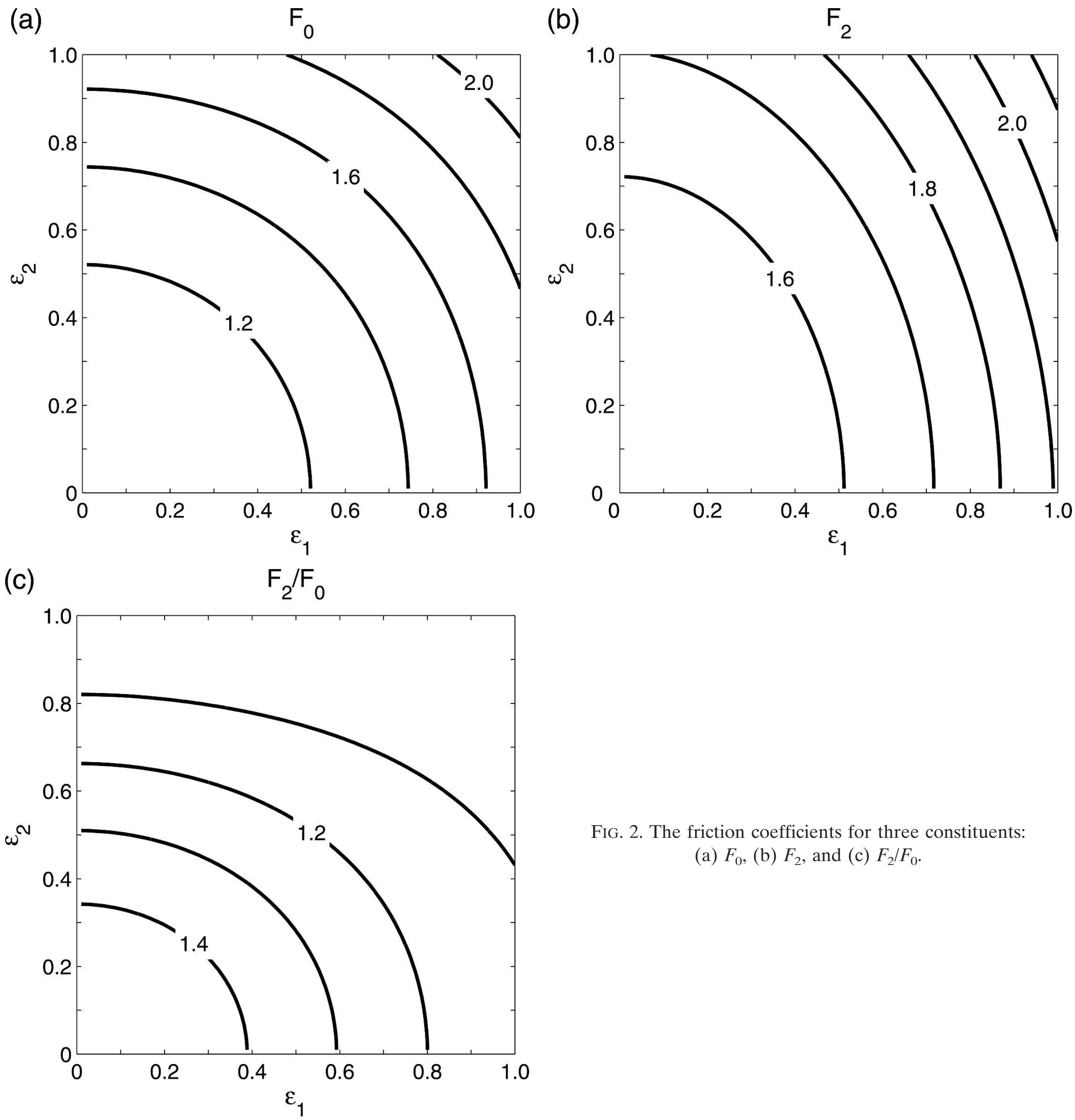

FIG. 2. The friction coefficients for three constituents: (a) F_0, (b) F_2, and (c) F_2/F_0.

than in Seymour Narrows and F_{K_1}/F_0 is smaller than the corresponding ratio for other constituents. However, in Haro Strait, the K_1 current becomes comparable to that of M_2 (ε_{K_1} goes up to 0.84), and values of F_n/F_0 are closer to 1. These results suggest that the assumption sometimes used—that the minor constituent feels friction 50% more strongly than the main constituent, $F_n/F_0 = 1.5$—should be applied with caution. In the present situation it confirms the suggestion by SGF that the discrepancy between the observed and predicted responses for the M_2 tide in their Helmholtz model is at least partly a consequence of their initial use of 50% greater friction for the weaker constituents.

4. Two-dimensional currents

Primarily because of the effect of the Coriolis force, tidal currents are not necessarily rectilinear. For two-dimensional currents, the linearization of quadratic friction has been performed in past studies and applied to spectral models (e.g., Snyder et al. 1979; Le Provost

TABLE 1. Each site shows, in order for each constituent, the observed along-channel tidal current amplitude (m s^{-1}), the current ratio to the M_2 tide ($\varepsilon_n = u_n/u_{M_2}$), and the friction coefficient from (18). The current observation sites are shown in Fig. 2 in Foreman et al. (2004).

Tidal constituent Frequency (cpd)	Q_1 0.893	O_1 0.930	P_1 0.995	K_1 1.003	N_2 1.0896	M_2 1.932	S_2 2.000	K_2 2.005
Juan de Fuca West	0.014	0.095	0.070	0.210	0.074	0.382	0.090	0.017
	0.037	0.249	0.183	0.550	0.194	1.000	0.236	0.045
	1.685	1.674	1.679	1.629	1.678	1.352	1.675	1.685
Juan de Fuca East	0.038	0.239	0.146	0.449	0.159	0.735	0.198	0.043
	0.052	0.325	0.199	0.611	0.216	1.000	0.269	0.059
	1.741	1.722	1.734	1.671	1.733	1.451	1.728	1.741
Haro Strait	0.050	0.335	0.165	0.497	0.097	0.589	0.146	0.040
	0.085	0.569	0.280	0.844	0.165	1.000	0.248	0.068
	1.954	1.895	1.941	1.822	1.950	1.802	1.944	1.955
Seymour Narrows	0.087	0.540	0.334	1.008	0.772	4.661	1.163	0.334
	0.019	0.116	0.072	0.216	0.166	1.000	0.250	0.072
	1.560	1.558	1.559	1.551	1.555	1.118	1.549	1.559

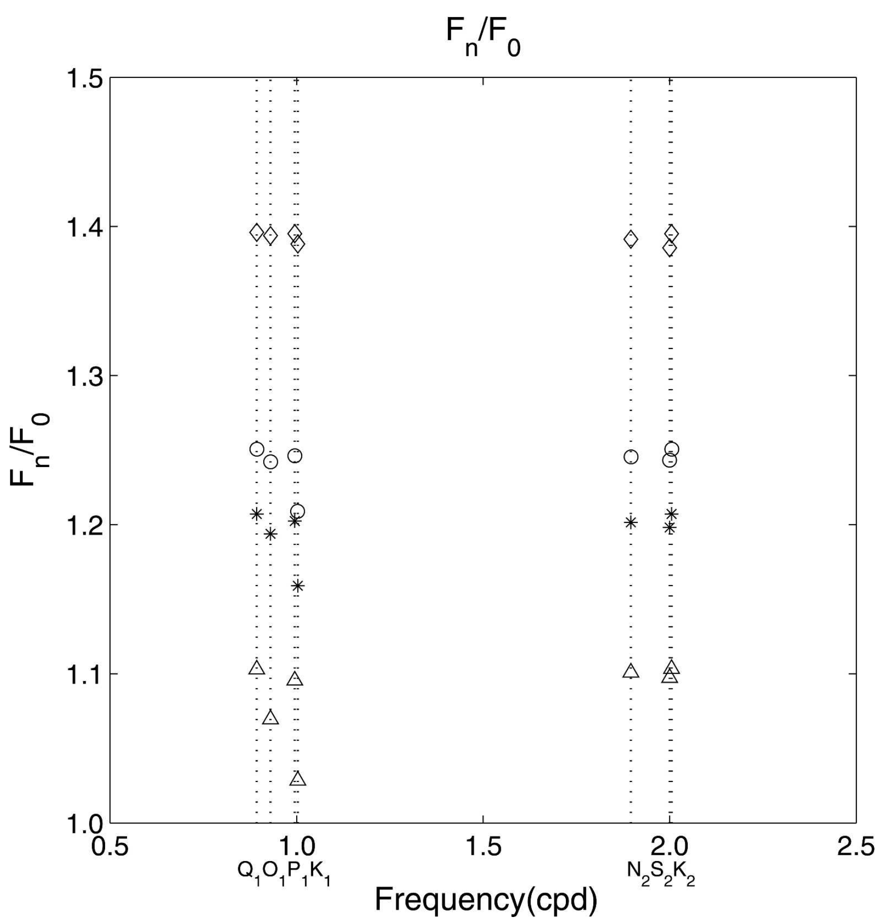

FIG. 3. The ratio of the friction coefficients for the minor constituents to that for the main constituent M_2 from (18): the triangles are estimated from current-meter data in Haro Strait, the circles in the western end of Juan de Fuca Strait; the asterisks in the eastern end of Juan de Fuca Strait; and the diamonds in Seymour Narrows. The dotted lines show the frequencies for each tidal constituent.

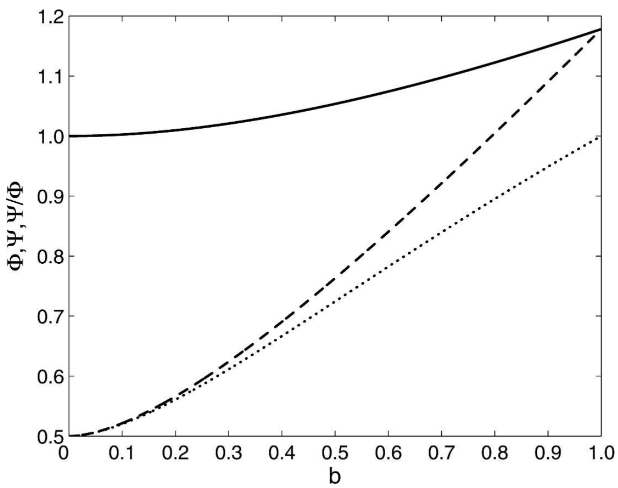

FIG. 4. The dependence on the ellipticity b of the friction coefficients for a single constituent (Pingree 1983): Φ (solid line), Ψ (dashed line), and Ψ/Φ (dotted line).

and Fornerino 1985) and reviews of the friction term in models are given by Le Provost (1991) and Walters and Werner (1991). Analytical expressions for the bottom friction for two-dimensional currents comprising a tide and mean flow were obtained by Hunter (1975) and Heaps (1978) for special cases. Following these studies, Pingree (1983) examined the Fourier components of quadratic friction for a mix of tidal constituents in a variety of two-dimensional situations. He derived analytical results for situations with $\varepsilon_1 \ll 1$, and presented some numerical results for $\varepsilon_1 = 0.33$. In this section, we apply our new approach and FFT methods.

a. The same ellipticity and major axis orientation

We first consider a current made up of a single constituent with $\mathbf{u} = u_0(\cos\omega_0 t,\ b\sin\omega_0 t)$, where b is the ellipticity. We write the Fourier expansion of $\mathbf{u}|\mathbf{u}|$ as

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(x)}\cos\omega_0 t + \cdots] \quad \text{and}$$

$$v|\mathbf{u}| = \frac{8bu_0^2}{3\pi}[F_0^{(y)}\sin\omega_0 t + \cdots]. \tag{19}$$

The scaling factor for the friction in the x direction is chosen to allow for easy comparison with the one-dimensional case. In the y direction we add a factor b as we expect the friction to go to zero if b does. Following Pingree (1983), and writing $\theta = \omega_0 t$ in the Fourier transform, we obtain

$$F_0^{(x)} \equiv \Phi = \frac{3}{2}\int_0^{\pi/2}(1 - k^2\sin^2\theta)^{1/2}\cos^2\theta\, d\theta = \frac{1}{2}\left\{E(k) + K(k) + \frac{1}{k^2}[E(k) - K(k)]\right\} \quad \text{and}$$

$$F_0^{(y)} \equiv \Psi = \frac{3}{2}\int_0^{\pi/2}(1 - k^2\sin^2\theta)^{1/2}\sin^2\theta\, d\theta = \frac{1}{2}\left\{2E(k) - K(k) - \frac{1}{k^2}[E(k) - K(k)]\right\}, \tag{20}$$

where k is eccentricity $[k = (1 - b^2)^{1/2}]$,

$$E(k) = \int_0^{\pi/2}(1 - k^2\sin^2\theta)^{1/2}\, d\theta, \quad \text{and}$$

$$K(k) = \int_0^{\pi/2}(1 - k^2\sin^2\theta)^{-1/2}\, d\theta\,. \tag{21}$$

Figure 4 shows Φ, Ψ, and Ψ/Φ as functions of b. Focusing on Φ for the moment, we note that $b = 0$ corresponds to a rectilinear current and has $\Phi = 1$, so that the x component of friction is as before. As the ellipticity b increases, Φ only increases to $3\pi/8 = 1.18$ at $b = 1$. Thus, adding a y component to the current, but still scaling the friction on the x component, adds less than 20% to the x component of friction. Turning to the y component, we note that Ψ increases from ½ to $3\pi/8 = 1.18$ as b increases from 0 to 1. Thus, the y component of friction is more sensitive than the x component to the presence of a y as well as an x component in the current. Interestingly, Ψ/Φ is always less than 1, so that the ellipticity $b\Psi/\Phi$ of the stress ellipses is always less than the ellipticity b of the current (Pingree 1983). We note that the above multiplier $3\pi/8$ at $b = 1$ simply converts the factor outside the parentheses in (19) to u_0^2, as expected for a circular current with constant speed u_0.

For a current made up of two tidal constituents with the same ellipticity and major axis orientation, taken to be along the x axis, the vector current becomes

$$\mathbf{u} = u_0(\cos\omega_0 t + \varepsilon_1\cos\omega_1 t,\ b\sin\omega_0 t + \varepsilon_1 b\sin\omega_1 t). \tag{22}$$

As before, ε_1 is the ratio of the amplitudes of the two constituents and b is their ellipticity. We express the Fourier expansion of $\mathbf{u}|\mathbf{u}|$ for the x and y directions as

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(x)}\cos\omega_0 t + \varepsilon_1 F_1^{(x)}\cos\omega_1 t + \cdots] \quad \text{and}$$

$$v|\mathbf{u}| = \frac{8bu_0^2}{3\pi}[F_0^{(y)}\sin\omega_0 t + \varepsilon_1 F_1^{(y)}\sin\omega_1 t + \cdots]. \tag{23}$$

The four factors $F_0^{(x)}$, $F_1^{(x)}$, $F_0^{(y)}$ and, $F_1^{(y)}$ are now functions of b as well as of ε_1; we are interested in the range from 0 to 1 for both b and ε_1.

With α_1 from (5), ζ_1 from (7), and defining

$$\eta_1 = \omega_0 t - \zeta_1, \tag{24}$$

we can write

$$\mathbf{u} = u_0(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}(\cos\eta_1, b\sin\eta_1). \tag{25}$$

Then $u|\mathbf{u}|$ becomes

$$u|\mathbf{u}| = u_0^2(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)(1 - k^2\sin^2\eta_1)^{1/2}\cos\eta_1, \tag{26}$$

with a similar expression for $v|\mathbf{u}|$. To obtain $F_0^{(x)}$ we need to Fourier transform this at frequency ω_0 and also average over α_1. Hence, denoting $\omega_0 t$ by θ as before so that $\eta_1 = \theta - \zeta_1$, we have

$$\begin{aligned}
F_0^{(x)} &= \frac{3}{16\pi}\int_0^{2\pi}\left[\int_0^{2\pi}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)(1 - k^2\sin^2\eta_1)^{1/2}\cos\eta_1\cos\theta\, d\theta\right] d\alpha_1 \\
&= \frac{3}{16\pi}\int_0^{2\pi}\left[\int_{\zeta_1}^{2\pi+\zeta_1}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)(1 - k^2\sin^2\eta_1)^{1/2}\cos\eta_1\cos(\eta_1 + \zeta_1)\, d\eta_1\right] d\alpha_1 \\
&= \frac{3}{16\pi}\int_0^{2\pi}\left[\int_{\zeta_1}^{2\pi+\zeta_1}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)(1 - k^2\sin^2\eta_1)^{1/2}\cos\zeta_1\cos^2\eta_1\, d\eta_1\right] d\alpha_1 \\
&\quad - \frac{3}{16\pi}\int_0^{2\pi}\left[\int_{\zeta_1}^{2\pi+\zeta_1}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)(1 - k^2\sin^2\eta_1)^{1/2}\sin\zeta_1\sin\eta_1\cos\eta_1\, d\eta_1\right] d\alpha_1 \\
&= \frac{3}{16\pi}\int_0^{2\pi}(1 + \varepsilon_1\cos\alpha_1)(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}\, d\alpha_1\int_{\zeta_1}^{2\pi+\zeta_1}(1 - k^2\sin^2\eta_1)^{1/2}\cos^2\eta_1\, d\eta_1 \\
&= \frac{1}{\pi}\int_0^{\pi}(1 + \varepsilon_1\cos\alpha_1)(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}\, d\alpha_1 \times \Phi,
\end{aligned} \tag{27}$$

where symmetry has allowed us to reduce the range of the integral in the final expression and Φ is from (20). Similarly, the other coefficients are

$$F_0^{(y)} = \frac{1}{\pi}\int_0^{\pi}(1 + \varepsilon_1\cos\alpha_1)(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}\, d\alpha_1 \times \Psi,$$

$$F_1^{(x)} = \frac{1}{\varepsilon_1\pi}\int_0^{\pi}(\varepsilon_1 + \cos\alpha_1)(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}\, d\alpha_1 \times \Phi,$$

and

$$F_1^{(y)} = \frac{1}{\varepsilon_1\pi}\int_0^{\pi}(\varepsilon_1 + \cos\alpha_1)(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2}\, d\alpha_1 \times \Psi, \tag{28}$$

with Ψ also from (20).

Thus, the four coefficients are separable functions of ε_1 and b. The multipliers of Φ and Ψ are just the expressions in (9) for F_0 and F_1 in the case of unidirectional currents. Separability also occurs if there are many constituents with the same major axis orientation and ellipticity. Hence, the factors involving ε_1 may be replaced by the same factors as in the unidirectional case.

b. Different ellipticities and major axis orientations

In the open ocean it is likely that constituents close in frequency, within either the diurnal or semidiurnal frequency bands, have similar tidal ellipses. It is also likely, however, that constituents from different bands (e.g., K_1 and M_2) have tidal ellipses that differ in both major axis orientation and ellipticity. This provides for a rich parameter space.

Pingree (1983) allowed for two tidal constituents with the same major axis orientation but different ellipticities (his Fig. 3 and Table 2), presenting results for $\varepsilon_1 \ll 1$ and $\varepsilon_1 = 0.33$. Our new approach can readily provide estimates of the friction coefficients for tidal currents having both different ellipticities and major axis orientations. We explore some of these situations next, presenting closed-form expressions and also confirming our results with FFT analysis of synthetic time series. In this analysis, the frequencies ω_0 and ω_1, the sampling frequency, and the data length are all the same as in section 2.

Some insight may be obtained by considering the case of orthogonal major axes with different ellipticities. We take the major axis of the smaller constituent to be along the y axis and write

$$\mathbf{u} = u_0(\cos\omega_0 t + \varepsilon_1 b_1 \sin\omega_1 t, b_0 \sin\omega_0 t + \varepsilon_1 \cos\omega_1 t). \tag{29}$$

Here, b_0 and b_1 are ellipticities of the dominant and weaker constituents, respectively. Although the major axes of the two constituents are orthogonal if $b_1 < 1$, the major axes coincide if we allow $b_1 > 1$. We could take a minus sign between the two contributions to the x component of $\mathbf{u}$ in order to have the same sense of rotation for both constituents, but the results for friction are independent of this so we take a plus sign for convenience. We express the Fourier expansion of $\mathbf{u}|\mathbf{u}|$ for the x and y directions as

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(x)} \cos\omega_0 t + \varepsilon_1 b_1 F_1^{(x)} \sin\omega_1 t + \cdots] \quad \text{and}$$

$$\upsilon|\mathbf{u}| = \frac{8u_0^2}{3\pi}[b_0 F_0^{(y)} \sin\omega_0 t + \varepsilon_1 F_1^{(y)} \cos\omega_1 t + \cdots]. \tag{30}$$

Writing $\theta_0 = \omega_0 t$ and $\theta_1 = \omega_1 t$, we proceed in a similar manner as before and obtain

$$F_0^{(x)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)\cos^2(\theta_0 - \zeta_2) + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_3)]^{1/2} \times (1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)^{1/2}\cos(\theta_0 - \zeta_2)\cos\theta_0\, d\theta_0\right\} d\alpha_1,$$

$$F_0^{(y)} = \frac{3}{16 b_0 \pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)\cos^2(\theta_0 - \zeta_2) + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_3)]^{1/2} \times (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)^{1/2}\cos(\theta_0 - \zeta_3)\sin\theta_0\, d\theta_0\right\} d\alpha_1,$$

$$F_1^{(x)} = \frac{3}{16\varepsilon_1 b_1 \pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)\cos^2(\theta_1 - \zeta_4) + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_1 - \zeta_5)]^{1/2} \times (1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)^{1/2}\cos(\theta_1 - \zeta_4)\sin\theta_1\, d\theta_1\right\} d\alpha_1, \quad \text{and}$$

$$F_1^{(y)} = \frac{3}{16\varepsilon_1 \pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1 - 2\varepsilon_1 b_1 \sin\alpha_1 + \varepsilon_1^2 b_1^2)\cos^2(\theta_1 - \zeta_4) + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_1 - \zeta_5)]^{1/2} \times (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)^{1/2}\cos(\theta_1 - \zeta_5)\cos\theta_1\, d\theta_1\right\} d\alpha_1, \tag{31}$$

where

$$\zeta_2 = \mathrm{atan}(\varepsilon_1 b_1 \cos\alpha_1, 1 - \varepsilon_1 b_1 \sin\alpha_1), \quad \zeta_3 = \mathrm{atan}(b_0 + \varepsilon_1 \sin\alpha_1, \varepsilon_1 \cos\alpha_1),$$
$$\zeta_4 = \mathrm{atan}(-\sin\alpha_1 + \varepsilon_1 b_1, \cos\alpha_1), \quad \text{and} \quad \zeta_5 = \mathrm{atan}(b_0 \cos\alpha_1, b_0 \sin\alpha_1 + \varepsilon_1). \tag{32}$$

The coefficients are no longer separable functions of ε_1 and b_0 or b_1, but the integration is easily performed. Symmetries allow the ranges of integration to be reduced in some situations, but this is not necessary.

We examine the case in which both constituents have rectilinear currents ($b_0 = b_1 = 0$) so that

$$\mathbf{u} = u_0(\cos\omega_0 t, \varepsilon_1 \cos\omega_1 t). \tag{33}$$

Then the formulas for nonzero friction coefficients are

$$F_0^{(x)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\cos^2\theta_0 + \varepsilon_1^2\cos^2(\theta_0 - \alpha_1)]^{1/2}\cos^2\theta_0\,d\theta_0\right\}d\alpha_1 \quad \text{and}$$

$$F_1^{(y)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\cos^2(\theta_1 + \alpha_1) + \varepsilon_1^2\cos^2\theta_1]^{1/2}\cos^2\theta_1\,d\theta_1\right\}d\alpha_1. \tag{34}$$

We have confirmed that the FFT method gives the same results as these equations for all values of ε_1.

As first derived by Hunter (1975) for the case of a tide and mean flow, $F_0^{(x)} = 1$ and $F_1^{(y)} = 0.75$ for $\varepsilon_1 \ll 1$. This shows that the minor constituent now feels proportionately less friction than the major constituent, in contrast to the results in previous sections. Numerically we find that $F_0^{(x)} = F_1^{(y)} = 1.29$ at $\varepsilon_1 = 1$; the bottom friction has the same magnitude for both constituents, but with a smaller coefficient than the 1.70 at $\varepsilon_1 = 1$ for parallel constituents.

The change from 1.5 to 0.75 in F_1 for small ε_1 as the major axes change from parallel to orthogonal is interesting. To examine the transition further, we take the two constituents to be inclined at an angle ϕ to each other, but with the smaller constituent still along the y axis, then

$$\mathbf{u} = u_0(\sin\phi\cos\omega_0 t, \cos\phi\cos\omega_0 t + \varepsilon_1\cos\omega_1 t). \tag{35}$$

Without loss of generality, we take $0° \le \phi \le 90°$. The Fourier expansion of $\mathbf{u}|\mathbf{u}|$ is of the following form:

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(x)}\cos\omega_0 t + \varepsilon_1 F_1^{(x)}\cos\omega_1 t + \cdots] \quad \text{and}$$

$$v|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(y)}\cos\omega_0 t + \varepsilon_1 F_1^{(y)}\cos\omega_1 t + \cdots]. \tag{36}$$

The coefficients are

$$F_0^{(x)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\sin^2\phi\cos^2\theta_0 + (\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_6)]^{1/2}\sin\phi\cos^2\theta_0\,d\theta_0\right\}d\alpha_1,$$

$$F_0^{(y)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\sin^2\phi\cos^2\theta_0 + (\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_6)]^{1/2}(\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)^{1/2}\cos(\theta_0 - \zeta_6)\cos\theta_0\,d\theta_0\right\}d\alpha_1,$$

$$F_1^{(x)} = \frac{3}{16\varepsilon_1\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\sin^2\phi\cos^2(\theta_1 + \alpha_1) + (\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)\cos^2(\theta_1 - \zeta_7)]^{1/2}\sin\phi\cos(\theta_1 + \alpha_1)\cos\theta_1\,d\theta_1\right\}d\alpha_1, \quad \text{and}$$

$$F_1^{(y)} = \frac{3}{16\varepsilon_1\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\sin^2\phi\cos^2(\theta_1 + \alpha_1) + (\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)\cos^2(\theta_1 - \zeta_7)]^{1/2}(\cos^2\phi + 2\varepsilon_1\cos\phi\cos\alpha_1 + \varepsilon_1^2)^{1/2}\cos(\theta_1 - \zeta_7)\cos\theta_1\,d\theta_1\right\}d\alpha_1, \tag{37}$$

where

$$\zeta_6 = \mathrm{atan}(\varepsilon_1\sin\alpha_1, \cos\phi + \varepsilon_1\cos\alpha_1) \quad \text{and}$$
$$\zeta_7 = \mathrm{atan}(-\cos\phi\sin\alpha_1, \cos\phi\cos\alpha_1 + \varepsilon_1). \tag{38}$$

We have to allow for rotation of the frictional force away from the current directions, so the ratio of $F_0^{(x)}$ to $F_0^{(y)}$ is not simply $\tan\phi$. There is also an x component of friction on the smaller constituent. We describe the magnitude of bottom friction by

$$F_0 = [F_0^{(x)2} + F_0^{(y)2}]^{1/2} \quad \text{and} \quad F_1 = [F_1^{(x)2} + F_1^{(y)2}]^{1/2}, \tag{39}$$

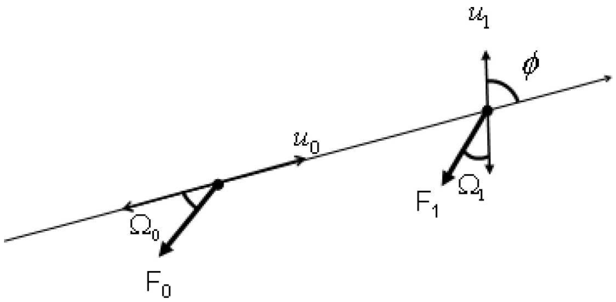

FIG. 5. A schematic of the relation between two tidal constituents, with u_0 and u_1 denoting the major axis currents for the dominant and smaller constituents, respectively. Here F_0 and F_1 are the friction coefficients, ϕ is the angle between the directions of the dominant constituent and the weaker constituent, and Ω_0 and Ω_1 are the angles between the directions of a constituent's current and its friction.

and the changes in direction from the current axes by

$$\Omega_0 = \phi - \tan^{-1}[F_0^{(x)}/F_0^{(y)}] \quad \text{and} \quad \Omega_1 = \tan^{-1}[F_1^{(x)}/F_1^{(y)}] \tag{40}$$

as shown schematically in Fig. 5. For $\varepsilon_1 \ll 1$,

$$F_0 = 1, F_1 = \frac{1}{2} \times 1.5(1 + 3\cos^2\phi)^{1/2}, \quad \Omega_0 = 0, \quad \text{and}$$

$$\Omega_1 = \tan^{-1}\left(\frac{\cos\phi \sin\phi}{1 + \cos^2\phi}\right). \tag{41}$$

The expressions here for F_1 and Ω_1 were derived by Saunders (1977) for the situation in which $\omega_1 = 0$, representing a steady flow. In particular, for $\varepsilon_1 \ll 1$, F_1 changes smoothly from 1.5 when $\phi = 0$ to 0.75 when $\phi = 90°$, with a value of 1.19 when $\phi = 45°$. The frictional force on the weaker constituent is rotated toward the axis of the stronger constituent by the angle Ω_1. This is zero, as expected, for ϕ equal to zero or 90°, with a maximum of 19.5° when $\phi = 55°$.

These results are for $\varepsilon_1 \ll 1$. As ε_1 increases for a given ϕ, F_1 increases and Ω_1 decreases (Fig. 6). In particular, for $\varepsilon_1 = 1$, the maximum value is reduced to 10.1° and occurs when $\phi = 46°$. Finite values of ε_1 also lead to increases in F_0 and Ω_0, with the friction felt by the main constituent now rotated slightly toward the axis of the smaller constituent (Fig. 6). As expected by symmetry, when $\varepsilon_1 = 1$ we have $F_0 = F_1$ and $\Omega_0 = \Omega_1$ for all ϕ. These results for finite ε_1 only apply for nonzero values of the frequency ω_1. The case examined by Saunders (1977), with $\omega_1 = 0$, will be discussed further in the next section.

We now consider the effect of finite ellipticity of the tidal currents, restricting our attention to the situation where the major axes of the two constituents are orthogonal. We start with a rectilinear minor constituent ($b_1 = 0$) so that

$$\mathbf{u} = u_0(\cos\omega_0 t, b_0 \sin\omega_0 t + \varepsilon_1 \cos\omega_1 t). \tag{42}$$

The Fourier expansion of $\mathbf{u}|\mathbf{u}|$ is given by (30) with $b_1 = 0$ so that

$$F_0^{(x)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\cos^2\theta_0 + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_3)]^{1/2}\cos^2\theta_0\, d\theta_0\right\} d\alpha_1,$$

$$F_0^{(y)} = \frac{3}{16 b_0 \pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\cos^2\theta_0 + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_0 - \zeta_3)]^{1/2}[\varepsilon_1\cos\alpha_1\cos\theta_0 + (b_0 + \varepsilon_1\sin\alpha_1)\sin\theta_0] \times \sin\theta_0\, d\theta_0\right\} d\alpha_1, \quad \text{and}$$

$$F_1^{(y)} = \frac{3}{16\varepsilon_1\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[\cos^2(\theta_1 + \alpha_1) + (b_0^2 + 2\varepsilon_1 b_0 \sin\alpha_1 + \varepsilon_1^2)\cos^2(\theta_1 - \zeta_5)]^{1/2}[(b_0\sin\alpha_1 + \varepsilon_1)\cos\theta_1 + b_0\cos\alpha_1\sin\theta_1]\cos\theta_1\, d\theta_1\right\} d\alpha_1, \tag{43}$$

using $\zeta_2 = 0$, $\zeta_4 = -\alpha_1$, and ζ_3 and ζ_5 from (32). With $b_1 = 0$ there is no friction in the x direction with frequency ω_1. For $0 < b_0 < 1$ and $\varepsilon_1 \ll 1$, $F_1^{(y)}$ is approximately 1.5Ψ. For $b_0 = 1$, $F_0^{(x)} = F_0^{(y)} = 3\pi/8$ and $F_1^{(y)} = 1.5 \times 3\pi/8$ if $\varepsilon_1 \ll 1$. The coefficients increase slightly as ε_1 increases; at $\varepsilon_1 = 1$: $F_0^{(x)} = 1.39$, $F_0^{(y)} = 1.80$, and $F_1^{(y)} = 1.94$. Thus, the main constituent feels more friction in the direction of the minor constituent than in the orthogonal direction, but both constituents feel less friction than for the situation discussed earlier where

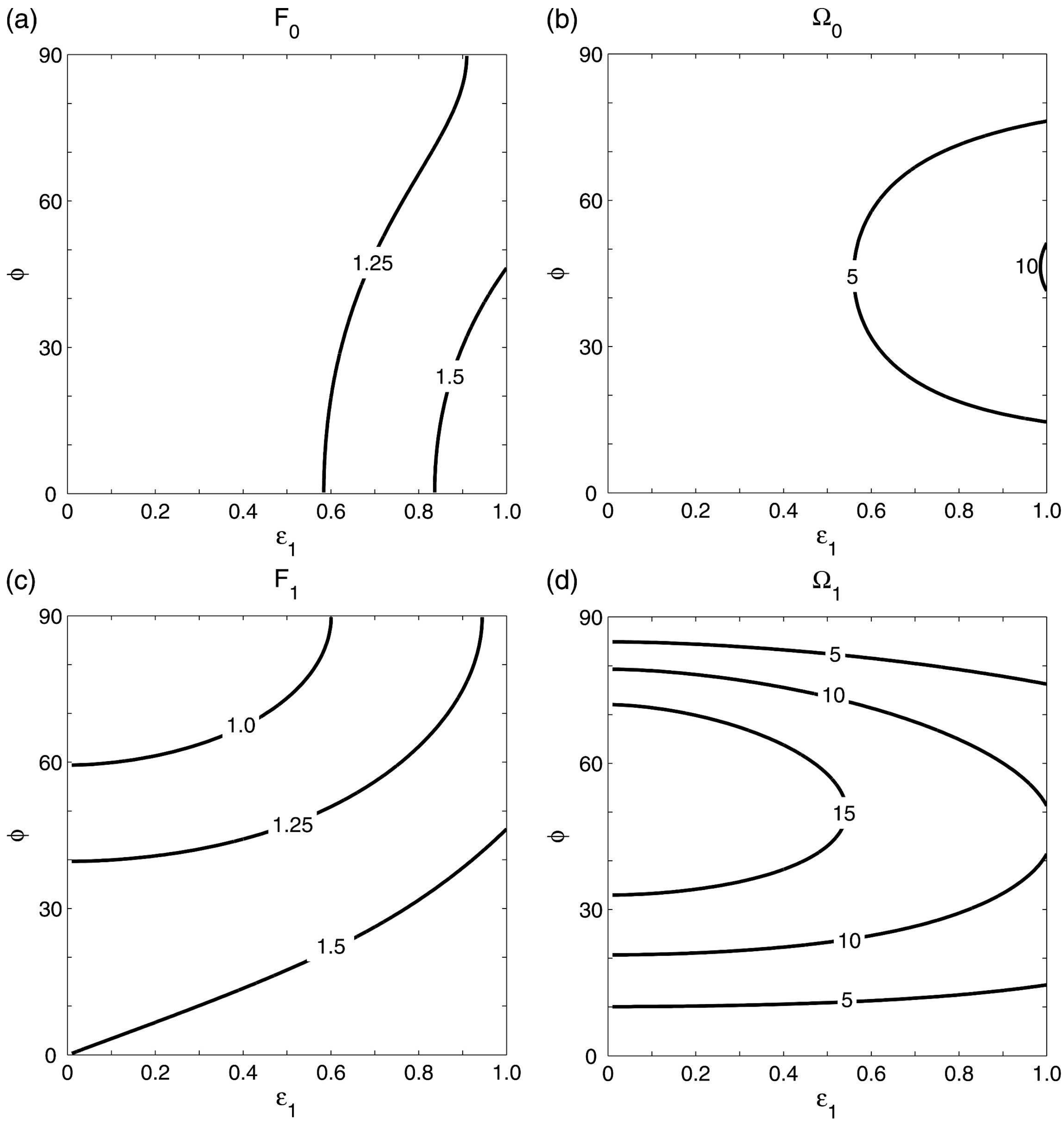

FIG. 6. The friction coefficients and rotation angles for two rectilinear constituents with amplitude ratio ε_1 and relative inclination ϕ: (a) F_0, (b) Ω_0, (c) F_1, and (d) Ω_1.

both constituents have circular currents. If the constituent major axes are parallel instead of orthogonal, but both in the y direction, then for $0 < b_0 < 1$ and $\varepsilon_1 \ll 1$ the coefficient $F_1^{(y)}$ is approximately 1.5Φ. For $b_0 = 1$, of course, we obtain the same results as for orthogonal major axes.

Taking the opposite situation, we now take the major constituent to be rectilinear ($b_0 = 0$) but allow the minor constituent to be elliptical. Considering only the limiting case of a circular minor constituent ($b_1 = 1$), we have

$$\mathbf{u} = u_0(\cos\omega_0 t + \varepsilon_1 \sin\omega_1 t, \varepsilon_1 \cos\omega_1 t). \qquad (44)$$

The relevant friction coefficients from (30) are then

$$F_0^{(x)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1-2\varepsilon_1\sin\alpha_1+\varepsilon_1^2)\cos^2(\theta_0-\zeta_2)+\varepsilon_1^2\cos^2(\theta_0-\alpha_1)]^{1/2}[(1-\varepsilon_1\sin\alpha_1)\cos\theta_0 + \varepsilon_1\cos\alpha_1\sin\theta_0]\cos\theta_0\,d\theta_0\right\}d\alpha_1,$$

$$F_1^{(x)} = \frac{3}{16\varepsilon_1\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1-2\varepsilon_1\sin\alpha_1+\varepsilon_1^2)\cos^2(\theta_1-\zeta_4)+\varepsilon_1^2\cos^2\theta_1]^{1/2}[\cos\alpha_1\cos\theta_1+(-\sin\alpha_1+\varepsilon_1)\times\sin\theta_1]\sin\theta_1\,d\theta_1\right\}d\alpha_1,\quad\text{and}$$

$$F_1^{(y)} = \frac{3}{16\pi}\int_0^{2\pi}\left\{\int_0^{2\pi}[(1-2\varepsilon_1\sin\alpha_1+\varepsilon_1^2)\cos^2(\theta_1-\zeta_4)+\varepsilon_1^2\cos^2\theta_1]^{1/2}\cos^2\theta_1\,d\theta_1\right\}d\alpha_1, \tag{45}$$

using $\zeta_3 = \alpha_1$, $\zeta_5 = 0$, and with ζ_2 and ζ_4 from (32) given by

$$\zeta_2 = \operatorname{atan}(\varepsilon_1\cos\alpha_1, 1-\varepsilon_1\sin\alpha_1)\quad\text{and}$$
$$\zeta_4 = \operatorname{atan}(-\sin\alpha_1+\varepsilon_1, \cos\alpha_1). \tag{46}$$

For $\varepsilon_1 \ll 1$, we obtain $F_0^{(x)} = 1$, $F_1^{(x)} = 1.5$, and $F_1^{(y)} = 0.75$. The minor constituent feels more friction in the direction of the main constituent than in the orthogonal direction. Moreover, in this case, F_0 and F_1 do not have the factor $3\pi/8$ that arises if the main constituent has circular currents. This is because the y component of the current now has only a contribution from the minor constituent. The coefficients increase as ε_1 increases, with $F_0^{(x)} = 1.94$, $F_1^{(x)} = 1.80$, and $F_1^{(y)} = 1.39$ at $\varepsilon_1 = 1$, simply representing an interchange with the coefficients of the previous case ($b_0 = 1$, $b_1 = 0$, and $\varepsilon_1 = 1$).

Last, when both constituents become circular ($b_0 = 1$ and $b_1 = 1$), we are back to the case considered in section 4a with parallel major axes and ellipticity $b = 1$ (so that the eccentricity $k = 0$). For $\varepsilon_1 \ll 1$ we have $F_0^{(x)} = F_0^{(y)} = 3\pi/8$ and $F_1^{(x)} = F_1^{(y)} = 1.5 \times 3\pi/8$. For $\varepsilon_1 = 1$ all the coefficients have a value of 2.0.

Our approach can easily be used for any ellipticities, any value of ε_1, and any inclination of the major axes. We will not explore this large parameter space in detail here, but the results presented so far give a good idea of what to expect. We have also presented some results for intermediate relative inclinations of rectilinear currents. In the next section, however, we will pursue further the special case in which the main constituent has arbitrary ellipticity and major axis orientation relative to the smaller constituent, taking this to have zero frequency. The problem represents the important practical case of a steady flow in the presence of a stronger tidal current.

5. Tidal current and steady flow

As mentioned above, Saunders (1977) evaluated the mean friction for a situation with a rectilinear oscillatory current and a weaker steady flow with different orientation. The current is described by

$$\mathbf{u} = u_0(\sin\phi\cos\omega_0 t, \cos\phi\cos\omega_0 t + \varepsilon_1), \tag{47}$$

where $\varepsilon_1 = \bar{v}/u_0$ is the ratio of the magnitude of the mean current $\bar{\mathbf{u}} = (0, \bar{v})$ to the tidal current amplitude u_0. We take $u_0 > \bar{v}$ so that $\varepsilon_1 = \bar{v}/u_0 < 1$, but the analysis could easily be extended to $\varepsilon_1 > 1$; the situation is no longer one of just switching which constituent is taken to be dominant, as it is in the tidal case.

The Fourier expansion of $\mathbf{u}|\mathbf{u}|$ is of the following form:

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(x)}\cos\omega_0 t + \varepsilon_1\overline{F^{(x)}} + \cdots]\quad\text{and}$$
$$v|\mathbf{u}| = \frac{8u_0^2}{3\pi}[F_0^{(y)}\cos\omega_0 t + \varepsilon_1\overline{F^{(y)}} + \cdots]. \tag{48}$$

Denoting $\theta = \omega_0 t$ as before, we obtain

$$F_0^{(x)} = \frac{3}{8}\int_0^{2\pi}(\cos^2\theta + 2\varepsilon_1\cos\phi\cos\theta + \varepsilon_1^2)^{1/2}\sin\phi\cos^2\theta\,d\theta,$$

$$F_0^{(y)} = \frac{3}{8}\int_0^{2\pi}(\cos^2\theta + 2\varepsilon_1\cos\phi\cos\theta + \varepsilon_1^2)^{1/2}(\cos\phi\cos\theta + \varepsilon_1)\cos\theta\,d\theta,$$

$$\overline{F^{(x)}} = \frac{3}{16\varepsilon_1}\int_0^{2\pi}(\cos^2\theta + 2\varepsilon_1\cos\phi\cos\theta + \varepsilon_1^2)^{1/2}\sin\phi\cos\theta\,d\theta,\quad\text{and}$$

$$\overline{F^{(y)}} = \frac{3}{16\varepsilon_1}\int_0^{2\pi}(\cos^2\theta + 2\varepsilon_1\cos\phi\cos\theta + \varepsilon_1^2)^{1/2}(\cos\phi\cos\theta + \varepsilon_1)\,d\theta. \tag{49}$$

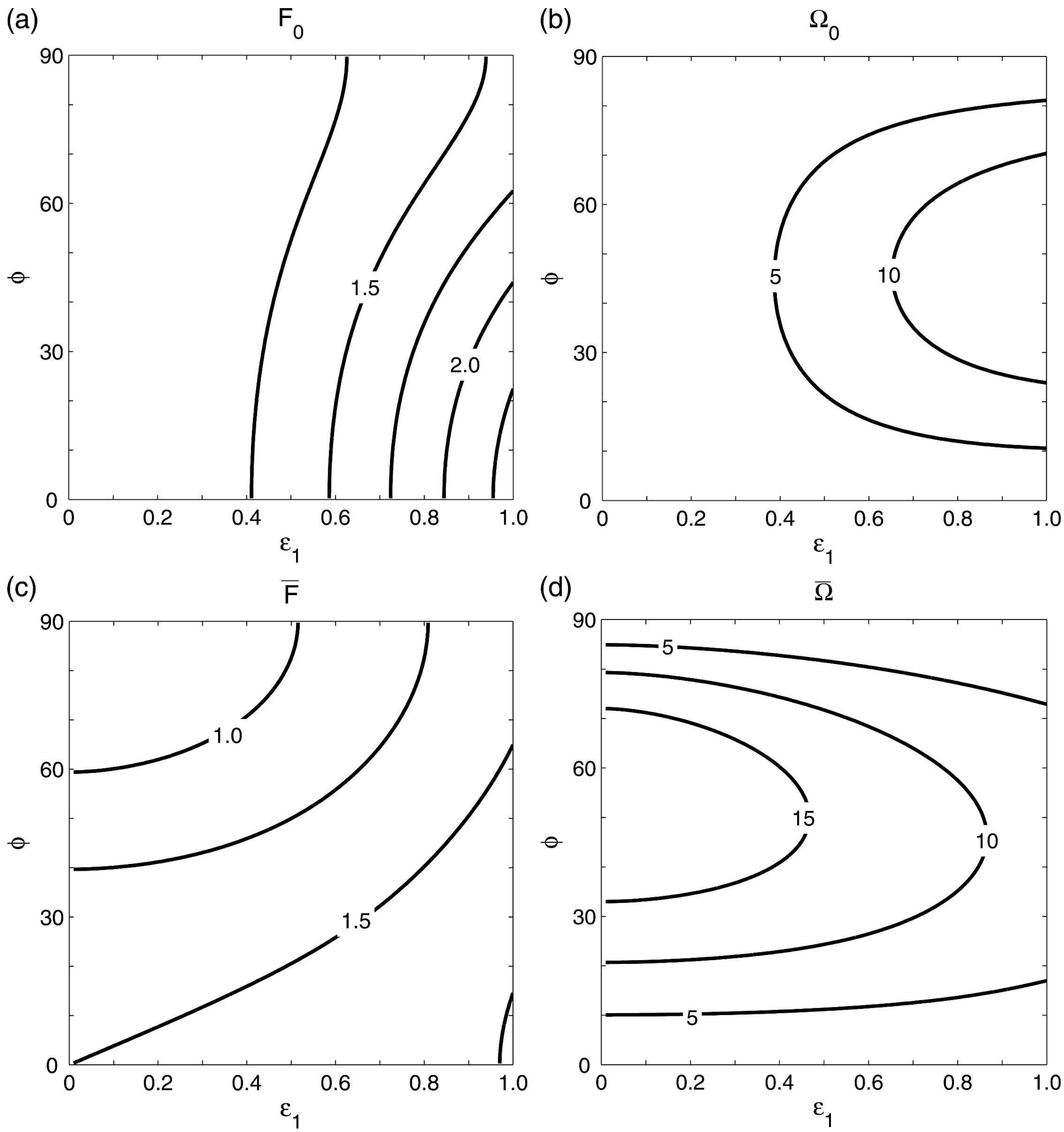

FIG. 7. The friction coefficients and rotation angles for a rectilinear tide and a steady flow with amplitude ratio ε_1 and relative inclination ϕ: (a) F_0, (b) Ω_0, (c) $\overline{F}$, and (d) $\overline{\Omega}$.

The friction coefficients and rotation angles are given by (39) and (40), where we use $\overline{F}$ and $\overline{\Omega}$ instead of F_1 and Ω_1 (see also Fig. 5). For $\varepsilon_1 \ll 1$, (41) applies. As ε_1 increases, however, the results are different from the purely tidal case in which the strength of the smaller constituent varies. The difference is clearest if we consider unidirectional currents, with $\phi = 0$. Writing $\overline{F}$ to denote the friction coefficient for the mean flow, thus replacing F_1 in (4), we find from the same method as Jeffreys (1970) that

$$F_0 = 1 + \frac{3}{2}\varepsilon_1^2 \quad \text{and} \quad \overline{F} = 1.5\left(1 + \frac{1}{6}\varepsilon_1^2\right), \qquad (50)$$

to $O(\varepsilon_1^2)$. These expressions may be compared with those for F_0 and F_1 in (11) which can actually be derived from (50) by considering a quasi-steady current, replacing ε_1 by $\varepsilon_1 \cos\omega_1 t$. The second-order term in the expression for F_0 in (50) can then be replaced by $(3/2)\varepsilon_1^2 \overline{\cos^2\omega_1 t} = (3/4)\varepsilon_1^2$, in agreement with the expression for F_0 in (11). The use of the expression for $\overline{F}$ in (50) is a bit more subtle. The second-order term tells us that there is a contribution $1.5(1/6)\varepsilon_1^3$ to the friction felt by the mean flow. Again replacing ε_1 by $\varepsilon_1 \cos\omega_1 t$, we have a contribution involving $\cos^3\omega_1 t$, which has a Fourier component of amplitude 3/4 at frequency ω_1. Hence, the coefficient 1/6 must be replaced by 1/8, in agreement with the expression for F_1 in (11).

For the combination of a strong tide and a weaker mean flow, we expect the dependence of F_0 and $\overline{F}$ on ε_1 and angle ϕ to be different from that for F_0 and F_1 in the case of two tides. The results are shown in Fig. 7, which may be compared with Fig. 6. For $\phi = 0$ and $\varepsilon_1 = 1$, the approximate solution in (50) gives $F_0 = 2.5$ and $\overline{F} = 1.75$. The exact solution of (49) gives

$$F_0 = \frac{3}{8}\int_0^{2\pi} (\cos\theta + 1)^2 \cos\theta \, d\theta = \frac{3\pi}{8} \times 2 = 2.36 \quad \text{and}$$

$$\overline{F} = \frac{3}{16}\int_0^{2\pi} (\cos\theta + 1)^2 \, d\theta = \frac{3\pi}{8} \times \frac{3}{2} = 1.77, \tag{51}$$

but the factors $3\pi/8$ here really just serve to cancel the factors $8/(3\pi)$ in (48).

Additionally for $\phi = 0$, $\overline{F} > F_0$ for $\varepsilon_1 > 0.65$; the two currents have essentially exchanged roles, with the mean flow now having a greater influence than the oscillatory tide. This effect occurs less for larger values of ϕ. The largest deflection, $\overline{\Omega}$, of the friction on the mean flow occurs for $\varepsilon_1 \ll 1$, and is 19.5° at $\phi = 55°$, as for the tidal case.

Last, we extend these results to allow for nonzero ellipticity of the tidal current. We take

$$\mathbf{u} = u_0(\sin\phi \cos\omega_0 t - b \cos\phi \sin\omega_0 t, \cos\phi \cos\omega_0 t + b \sin\phi \sin\omega_0 t + \varepsilon_1), \tag{52}$$

representing a tidal current of ellipticity b with its major axis inclined at an angle ϕ to the mean flow. We take $0 < b < 1$ and $\varepsilon_1 = \overline{v}/u_0 < 1$, though, as remarked before, results could easily be obtained for $\varepsilon_1 > 1$. The Fourier expansion of $\mathbf{u}|\mathbf{u}|$ for the steady flow may be written as

$$u|\mathbf{u}| = \frac{8u_0^2}{3\pi}\{\mathrm{Re}[F_0^{(x)} e^{-i\omega_0 t}] + \varepsilon_1 \overline{F^{(x)}} + \cdots\} \quad \text{and}$$

$$v|\mathbf{u}| = \frac{8u_0^2}{3\pi}\{\mathrm{Re}[F_0^{(y)} e^{-i\omega_0 t}] + \varepsilon_1 \overline{F^{(y)}} + \cdots\}. \tag{53}$$

Here we have to allow $F_0^{(x)}$ and $F_0^{(y)}$ to be complex as the forces in the two directions have terms in both $\cos\omega_0 t$ and $\sin\omega_0 t$. However, the friction coefficients for the steady flow in the integral form are easily written with $\theta = \omega_0 t$ as

$$\overline{F^{(x)}} = \frac{3}{16\varepsilon_1}\int_0^{2\pi} (\cos^2\theta + b^2\sin^2\theta + 2\varepsilon_1 \cos\phi\cos\theta + 2\varepsilon_1 b \sin\phi \sin\theta + \varepsilon_1^2)^{1/2} (\sin\phi\cos\theta - b\cos\phi\sin\theta)\, d\theta \quad \text{and}$$

$$\overline{F^{(y)}} = \frac{3}{16\varepsilon_1}\int_0^{2\pi} (\cos^2\theta + b^2\sin^2\theta + 2\varepsilon_1 \cos\phi\cos\theta + 2\varepsilon_1 b \sin\phi \sin\theta + \varepsilon_1^2)^{1/2} (\cos\phi\cos\theta + b\sin\phi\sin\theta + \varepsilon_1)\, d\theta. \tag{54}$$

Figure 8 shows results for $\phi = 20°$, 50°, and 80°. For a given ε_1, $\overline{F}$ increases with increasing b for any value of ϕ. This is not surprising since an increase in b is associated with an increase in the current magnitude. Also, for a given ε_1, the deflection angle $\overline{\Omega}$ decreases with increasing b. This is also as expected; as $b \to 1$ the problem becomes one that can be treated as if $\phi = 0$ for which there is no deflection. For $b = 1$ and $\varepsilon_1 = 1$, the coefficients are derived from (54) as

$$\begin{aligned}
\overline{F^{(x)}} &= \frac{3}{16}\int_0^{2\pi} (2 + 2\cos\phi\cos\theta + 2\sin\phi\sin\theta)^{1/2} (\sin\phi\cos\theta - \cos\phi\sin\theta)\, d\theta \\
&= -\frac{3\times 2^{1/2}}{16}\int_\phi^{2\pi+\phi} (1 + \cos\eta_2)^{1/2} \sin\eta_2 \, d\eta_2 = 0 \quad \text{and} \\
\overline{F^{(y)}} &= \frac{3}{16}\int_0^{2\pi} (2 + 2\cos\phi\cos\theta + 2\sin\phi\sin\theta)^{1/2} (\cos\phi\cos\theta + \sin\phi\sin\theta + 1)\, d\theta \\
&= \frac{3\times 2^{1/2}}{16}\int_\phi^{2\pi+\phi} (1 + \cos\eta_2)^{3/2}\, d\eta_2 = \frac{3\times 2^{1/2}}{16} \times \frac{32}{3\times 2^{1/2}} = 2,
\end{aligned} \tag{55}$$

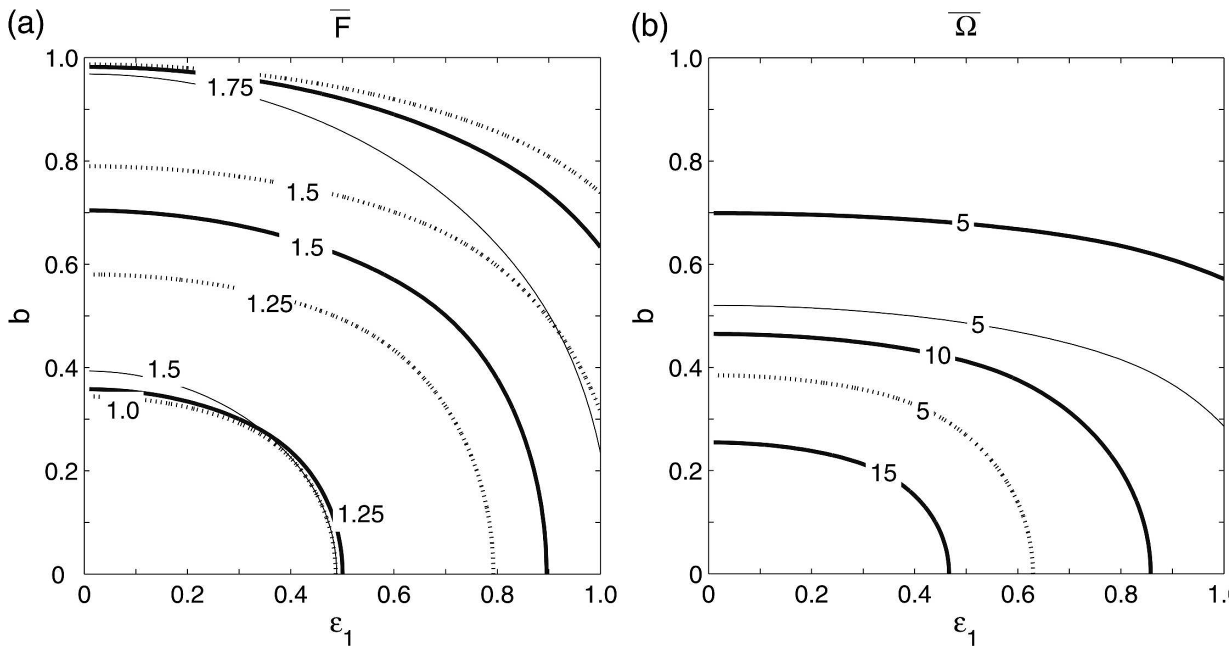

FIG. 8. The friction coefficients and rotation angles for a tide with ellipticity b and a steady flow with amplitude ratio ε_1: (a) $\overline{F}$ and (b) $\overline{\Omega}$. The thin solid line is for the relative inclination $\phi = 20°$, the thick solid line is for $\phi = 50°$, and the dotted line is for $\phi = 80°$.

where $\eta_2 = \theta - \phi$. This analytical solution also confirms that the coefficients are independent of ϕ at $b = 1$ and $\varepsilon_1 = 1$.

6. Summary

Analytical and numerical results on the Fourier representation of quadratic friction, in this and earlier studies, may be summarized as follows:

- For unidirectional currents a weak constituent experiences proportionately 50% more friction than a strong constituent if the ratio of the current amplitudes $\varepsilon_1 = u_1/u_0 \ll 1$ (Jeffreys 1970). As ε_1 increases the friction felt by both constituents increases, but the friction coefficient ratio decreases to 1 at $\varepsilon_1 = 1$ (Pingree 1983). We have introduced a new method that gives simple integrals for the coefficients for any value of ε_1 and with much less computational costs than using FFT on synthetic time series.
- As shown by Pingree (1983) and Le Provost and Fornerino (1985), and more comprehensively here with our new approach, the results may be extended to many tidal constituents, with simple additive effects if most of the constituents are weak ($\varepsilon_1, \varepsilon_2, \ldots \ll 1$).
- For unidirectional tidal currents in Juan de Fuca Strait and the Strait of Georgia, the weaker constituents experience more friction than the main constituent by an amount that varies spatially but is typically less than 50%, showing that the classic assumption should be applied with caution.
- For two-dimensional tidal currents, but with the dominant and weaker constituent having the same major axis orientation and ellipticity b, Pingree (1983) showed that the stress ellipse is flatter than the current ellipse, by a factor that is ½ for small values of b. It increases to 1 as b approaches 1. Our new approach shows that friction coefficients are separable functions of the b and ε_1. The dependence of the ratio of the friction coefficients on ε_1 is thus the same as for the unidirectional case.
- For different major axis orientations and rectilinear tides, a weak constituent orthogonal to a main constituent experiences proportionately 25% less, rather than 50% more, friction when $\varepsilon_1 \ll 1$. Intermediate values apply if the constituents are inclined at an angle (ϕ) between 0° and 90°, but in these cases the friction is rotated toward the main constituent by an angle that can be as large as 19.5°. These results correspond to Hunter (1975) and Saunders (1977) when the minor constituent is a steady flow. The extension for any value of ε_1 shows that, as the current ratio increases, the friction felt by the main constituent also rotates.
- As a special case of the different major axis orientations and different ellipticities, we examine orthogonal tides ($\phi = 90°$) with an elliptic main constituent ($0 \le b_0 \le 1$) and a rectilinear minor constituent ($b_1 = 0$). Comparison with the case of a rectilinear main constituent ($b_0 = 0$) and circular minor constituent ($b_1 = 1$) and the case with both constituents circular

($b_0 = 1$ and $b_1 = 1$) shows how the friction coefficients become larger as the amplitudes and ellipticities increase.

- The combination of a rectilinear tide and mean flow is somewhat different from the case with two tidal constituents. For $\varepsilon_1 \ll 1$, the friction experienced by the mean flow as a consequence of the tide has been analyzed by Saunders (1977). We further show that, as the strength of the mean flow increases relative to the tidal current, there is an effect of the mean flow on the magnitude and direction of the tidal friction. The analysis for any value of b shows that the rotation of the tidal and mean friction decreases as the ellipticity of the tidal current increases.

The results can be applied to the analysis of tidal resonance and used to understand the behavior of tidal friction and tidal energy dissipation in frequency-domain tidal models. Although general circulation models neglect the interaction between steady flow and tides via bottom friction, the results here suggest the possibility of modified bottom water circulation. In particular, the effect of the different directions of current and friction deserves further investigation.

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APPENDIX A

Effects of Phase Lag

We have chosen to represent each constituent by a cosine with zero phase lag and need to show that there is a time origin for which this is possible. In general, we assume phase lags γ_0 and γ_1 and write

$$\begin{aligned} u &= u_0[\cos(\omega_0 t - \gamma_0) + \varepsilon_1 \cos(\omega_1 t - \gamma_1)] \\ &= u_0\{\cos[\omega_0(t - t_0) + \omega_0 t_0 - \gamma_0] + \varepsilon_1 \cos[\omega_1(t - t_0) \\ &\quad + \omega_1 t_0 - \gamma_1]\}. \end{aligned} \tag{A1}$$

To bring the phases to zero with the new time origin t_0, we require

$$\omega_0 t_0 - \gamma_0 = 2p\pi \quad \text{and} \quad \omega_1 t_0 - \gamma_1 = 2q\pi, \tag{A2}$$

where p and q are arbitrary integers. Eliminating t_0, we require

$$p + \frac{\gamma_0}{2\pi} = \left(\frac{\omega_0}{\omega_1}\right)\left(q + \frac{\gamma_1}{2\pi}\right). \tag{A3}$$

Integral values of p and q can be found to satisfy this to any desired accuracy, albeit not exactly.

For three constituents, we write

$$\begin{aligned} u &= u_0[\cos(\omega_0 t - \gamma_0) + \varepsilon_1 \cos(\omega_1 t - \gamma_1) \\ &\quad + \varepsilon_2 \cos(\omega_2 t - \gamma_2)] \\ &= u_0\{\cos[\omega_0(t - t_0) + \omega_0 t_0 - \gamma_0] + \varepsilon_1 \cos[\omega_1(t - t_0) \\ &\quad + \omega_1 t_0 - \gamma_1] + \varepsilon_2 \cos[\omega_2(t - t_0) + \omega_2 t_0 - \gamma_2]\}. \end{aligned} \tag{A4}$$

In addition to (A3), we also require

$$p + \frac{\gamma_0}{2\pi} = \left(\frac{\omega_0}{\omega_2}\right)\left(r + \frac{\gamma_2}{2\pi}\right). \tag{A5}$$

The two conditions, (A3) and (A5), may now be satisfied to any accuracy with appropriate integral values of p, q, and r.

APPENDIX B

The Friction on Other Components

The Fourier expansion of the quadratic friction $\mathbf{u}|\mathbf{u}|$ has components at odd harmonics of the frequencies present in $\mathbf{u}$ as well as at the primary frequencies, providing forcing for overtides. Here, we restrict our attention to the case of unidirectional currents and just two constituents. We take

$$u = u_0(\cos\omega_0 t + \varepsilon_1 \cos\omega_1 t), \tag{B1}$$

as in (3) and write the Fourier expansion of $u|u|$ as

$$\begin{aligned} u|u| = \frac{8u_0^2}{3\pi}[&F_0 \cos\omega_0 t + \varepsilon_1 F_1 \cos\omega_1 t + F_{3,0} \cos 3\omega_0 t + \varepsilon_1 F_{2,-1} \cos(2\omega_0 t - \omega_1 t) + \varepsilon_1 F_{2,1} \cos(2\omega_0 t + \omega_1 t) \\ &+ \varepsilon_1^2 F_{1,-2} \cos(\omega_0 t - 2\omega_1 t) + \varepsilon_1^2 F_{1,2} \cos(\omega_0 t + 2\omega_1 t) + \cdots], \end{aligned} \tag{B2}$$

where F_0 and F_1 are as in (4), and we define the Fourier coefficient for a harmonic at frequency $p\omega_0 + q\omega_1$ as $F_{p,q}$. The amplitude of the harmonic is taken to scale like $\varepsilon_1^{|q|}$. By considering the odd power polynomial expansion of $u|u|$, it is simple to show that only cosine terms contribute to (B2). With α_1 from (5), ζ_1 from (7), η_1 from (24) and $\theta = \omega_0 t$, the coefficient for $3\omega_0 t$ becomes

$$F_{3,0} = \frac{3}{16\pi}\int_0^{2\pi}\left[\int_0^{2\pi}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)\cos(\theta - \zeta_1)|\cos(\theta - \zeta_1)|\cos3\theta\, d\theta\right]d\alpha_1$$
$$= \frac{3}{16\pi}\int_0^{2\pi}\left[\int_{\zeta_1}^{2\pi+\zeta_1}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)\cos\eta_1|\cos\eta_1|\cos3(\eta_1 + \zeta_1)\, d\eta_1\right]d\alpha_1$$
$$= \frac{1}{10\pi}\int_0^{2\pi}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)\cos3\zeta_1\, d\alpha_1. \qquad \text{(B3)}$$

This may be expanded to fourth order in ε_1 as

$$F_{3,0} = \frac{1}{5}\left(1 - \frac{5}{4}\varepsilon_1^2 + \frac{45}{64}\varepsilon_1^4\right). \qquad \text{(B4)}$$

This approximation was obtained by Pingree (1983) using a different technique. He remarked on the decrease of $F_{3,0}$ with increasing ε_1, showing that the generation of M_6 is made less effective by the presence of another constituent as well as M_2. Our closed form expression (B3) for any value of ε_1 is new and shows that this trend continues. In particular, at $\varepsilon_1 = 1$ we have

$$F_{3,0} = \frac{2}{5\pi}\int_0^{\pi}(1 + \cos\alpha_1)\cos3\frac{\alpha_1}{2}\, d\alpha_1 = \frac{16}{75\pi} = 0.068. \qquad \text{(B5)}$$

We note that $F_{3,0}/F_0$ decreases from 0.2 to 0.04 as ε_1 increases from 0 to 1.

We next evaluate $F_{2,-1}$ using the relation, $2\omega_0 t - \omega_1 t = \theta + \alpha_1$ to obtain

$$F_{2,-1} = \frac{1}{2\varepsilon_1\pi}\int_0^{2\pi}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2} \times (\cos\alpha_1 + \varepsilon_1\cos2\alpha_1)\, d\alpha_1. \qquad \text{(B6)}$$

To fourth order in ε_1 this gives

$$F_{2,-1} = \frac{1}{2}\left(1 - \frac{3}{8}\varepsilon_1^2 + \frac{3}{64}\varepsilon_1^4\right). \qquad \text{(B7)}$$

By symmetry arguments or by using $\alpha_1 = \omega_0 t + \omega_1 t$ we find $F_{2,1} = F_{2,-1}$.

To derive $F_{1,-2}$ we use $\omega_0 t - 2\omega_1 t = 2\alpha_1 - \theta$ and have

$$F_{1,-2} = \frac{1}{2\varepsilon_1^2\pi}\int_0^{2\pi}(1 + 2\varepsilon_1\cos\alpha_1 + \varepsilon_1^2)^{1/2} \times (\cos2\alpha_1 + \varepsilon_1\cos\alpha_1)\, d\alpha_1, \qquad \text{(B8)}$$

with an expansion to second order

$$F_{1,-2} = \frac{3}{8}\left(1 - \frac{1}{12}\varepsilon_1^2\right). \qquad \text{(B9)}$$

Symmetry again gives $F_{1,2} = F_{1,-2}$.

Pingree (1983) obtained the first two terms in (B7) and the first in (B9), but our expressions are valid for any value of ε_1. In particular, at $\varepsilon_1 = 1$,

$$F_{2,-1} = F_{1,-2} = \frac{2^{1/2}}{2\pi}\int_0^{2\pi}(1 + \cos\alpha_1)^{1/2}(\cos2\alpha_1 + \cos\alpha_1)\, d\alpha_1 = \frac{16}{15\pi}. \qquad \text{(B10)}$$

These coefficients are, like $F_{3,0}$, decreasing functions of ε_1, showing a reduced tendency for tidal harmonic generation.

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