# NOTES AND CORRESPONDENCE

## On the Body-Force Term in Internal-Tide Generation

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ABSTRACT

The generation of internal tides can be ascribed to the action of a buoyancy force caused by the flow of the barotropic tide over topographic features. It is commonly assumed that the barotropic flow can be taken as hydrostatic, but it is shown here that this leads to a linearized governing equation for the baroclinic tide that is only valid if the baroclinic tide is also hydrostatic. A governing equation for the baroclinic tide, valid for any situation, is derived here and is shown to be exactly equivalent to a simple transformation of the governing equation for the combined barotropic and baroclinic tides.

### 1. Introduction

Internal tides in the ocean are forced by barotropic tidal flow over topography. The forcing can be prescribed either via the boundary conditions (e.g., Vlasenko et al. 2005) or via a buoyancy force, in the vertical momentum equation, arising from the vertical displacement of isopycnals by the barotropic flow (e.g., Baines 1982). Baines assumes the barotropic flow to be hydrostatic. This seems reasonable if the horizontal scale of the topography is large in comparison with the water depth, so it has been assumed that his governing equation for the baroclinic tide is valid even if this baroclinic response is not, itself, hydrostatic.

It is easy to demonstrate, however, that in nonhydrostatic conditions the Baines formulation is not consistent with the exact formulation in terms of boundary forcing. This seems not to have been noticed before. The purpose of this note is to resolve this inconsistency.

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### 2. The barotropic/baroclinic decomposition

We start with the linearized nonhydrostatic equations on the $f$ plane, under the Boussinesq approximation, assuming uniformity in one of the horizontal directions ($\partial/\partial y = 0$):

$$u_t - fv = -p_x, \tag{1}$$

$$v_t + fu = 0, \tag{2}$$

$$w_t = -p_z + b, \tag{3}$$

$$u_x + w_z = 0, \quad \text{and} \tag{4}$$

$$b_t + N^2 w = 0. \tag{5}$$

Here $(u, v, w)$ denotes the total (i.e., barotropic plus baroclinic) velocity field in the $(x, y, z)$ directions, $p$ is the associated pressure perturbation divided by a constant reference density $\rho_*$, and $b = -g\rho/\rho_*$ is the buoyancy perturbation, where $\rho$ is the density perturbation. The buoyancy frequency $N$ may vary with $z$. The basin considered here has a rigid lid at $z = 0$ and a bottom at $z = -h(x)$. Apart from minor differences in notation, these are the governing equations used by Baines (1973, 1974, 1982) and by subsequent authors.

Following Baines, we write the fields as

$$(u, v, w, p, b) = (U, V, W, P, 0) + (u', v', w', p', B + b'), \quad (6)$$

where the first set (uppercase letters) describes the barotropic tidal flow of a homogeneous fluid. The second set (primes) describes the baroclinic response, with zero vertically integrated volume flux, together with the buoyancy $B$ induced kinematically by the barotropic flow. The barotropic flow satisfies

$$U_t - fV = -P_x, \quad (7)$$

$$V_t + fU = 0, \quad (8)$$

$$W_t = -P_z, \quad \text{and} \quad (9)$$

$$U_x + W_z = 0, \quad (10)$$

with the kinematically induced buoyancy $B$ given by

$$B_t + N^2W = 0. \quad (11)$$

Subtracting these equations for the barotropic flow from (1)–(5) gives the governing equations for the baroclinic response as

$$u'_t - fv' = -p'_x, \quad (12)$$

$$v'_t + fu' = 0, \quad (13)$$

$$w'_t = -p'_z + b' + B, \quad (14)$$

$$u'_x + w'_z = 0, \quad \text{and} \quad (15)$$

$$b'_t + N^2w' = 0, \quad (16)$$

showing how the baroclinic tide is forced by the buoyancy force created kinematically by the barotropic flow. If we introduce a streamfunction $\psi$ with $u = -\psi_z$ and $w = \psi_x$ and split it into $\Psi$ and $\psi'$ for the barotropic and baroclinic flows respectively, they satisfy

$$\psi_{xxtt} + \psi_{zztt} + f^2\psi_{zz} + N^2\psi_{xx} = 0, \quad (17)$$

$$\Psi_{xxtt} + \Psi_{zztt} + f^2\Psi_{zz} = 0, \quad \text{and} \quad (18)$$

$$\psi'_{xxtt} + \psi'_{zztt} + f^2\psi'_{zz} + N^2\psi'_{xx} = -N^2\Psi_{xx}. \quad (19)$$

The boundary conditions are $\psi = \Psi = \psi' = 0$ at $z = 0$ along with $\psi = \Psi = Q\exp(-i\omega t)$ and $\psi' = 0$ at $z = -h$ for a horizontally–uniform periodic volume flux of magnitude $Q$ at frequency $\omega$. (The real part of the variables is implied.)

In the frequency domain, (17)–(19) become

$$(N^2 - \omega^2)\psi_{xx} - (\omega^2 - f^2)\psi_{zz} = 0, \quad (20)$$

$$-\omega^2\Psi_{xx} - (\omega^2 - f^2)\Psi_{zz} = 0, \quad \text{and} \quad (21)$$

$$(N^2 - \omega^2)\psi'_{xx} - (\omega^2 - f^2)\psi'_{zz} = -N^2\Psi_{xx}. \quad (22)$$

Equation (22), along with the boundary conditions for $\psi'$, poses the internal-tide-generation problem in terms of body forcing while (20), with boundary conditions for $\psi$, poses the problem in terms of boundary forcing. The two formulations are equivalent because adding up (21) and (22) produces (20).

### *a. Hydrostatic barotropic forcing: Baines's version*

Baines uses the nonhydrostatic version (22) but then assumes the barotropic field $\Psi$ to be hydrostatic. This means that $W_t$ is neglected in (9); hence, the vertical derivative of (7) and (8) implies that $U$ is independent of $z$ and so is given by $Q/h$ [dropping the term $\exp(-i\omega t)$ for convenience]. Hence, $\Psi$ is replaced by the hydrostatic

$$\Psi^{(0)} = -zQ/h(x). \quad (23)$$

As a consequence, the vertical baroclinic velocity is now forced by $-N^2\Psi^{(0)}_x$, and this expression has often been used as a basis for predicting the internal tide generated for particular choices of $N(z)$ and $h(x)$ (e.g., Morozov 1995; Xing and Davies 1998; Sherwin et al. 2002; Azevedo et al. 2006).

With (23), (22) may be written

$$\psi'_{xx} - c^2\psi'_{zz} = \frac{N^2}{N^2 - \omega^2}Qz\left(\frac{1}{h}\right)_{xx},$$

$$\text{where } c^2 = \frac{\omega^2 - f^2}{N^2 - \omega^2}. \quad (24)$$

This, together with the boundary conditions that $\psi' = 0$ at $z = 0$ and $-h$, is the governing equation proposed by Baines. Notice that the barotropic streamfunction now satisfies $\Psi^{(0)}_{zz} = 0$ instead of (21); importantly, adding up the former and (22) does not lead to (20), which means that the connection to the boundary forcing problem has now been lost.

### *b. Hydrostatic barotropic forcing: Alternative version*

On the other hand, if we start from (20), which is the governing equation for the whole flow, and substitute

$$\psi = -Qz/h + \psi'' \quad (25)$$

we have

$$\psi''_{xx} - c^2\psi''_{zz} = Qz\left(\frac{1}{h}\right)_{xx}, \quad (26)$$

with $\psi'' = 0$ at $z = 0$ and $-h$. This must be correct because we have made no approximations in deriving it.

### c. Interpretation

There are a number of ways of interpreting the difference between the two approaches. First, we note that, as already mentioned, the hydrostatic assumption for the barotropic flow means that we omit $\Psi_{xxtt}$ from (18) and $\Psi_{xx}$ from (21). This carries through into the difference between (24) and (26). As an alternative, we could write the full, nonhydrostatic barotropic response as $\Psi = \Psi^{(0)} + \Psi^{(r)}$, where $\Psi^{(0)}$ is given by (23) and $\Psi^{(r)}$ is the remainder. The right-hand side of (22) is then $\Psi^{(0)}_{xx} + \Psi^{(r)}_{xx}$. By using

$$\omega^2[\Psi^{(0)}_{xx} + \Psi^{(r)}_{xx}] + (\omega^2 - f^2)\Psi^{(r)}_{zz} = 0 \qquad (27)$$

from (21), (22) can be rearranged to give

$$(N^2 - \omega^2)[\psi' + \Psi^{(r)}]_{xx} - (\omega^2 - f^2)[\psi' + \Psi^{(r)}]_{zz} = -(N^2 - \omega^2)\Psi^{(0)}_{xx}. \qquad (28)$$

This is just (26) with $\psi'' = \psi' + \Psi^{(r)}$, showing that, in the transformation leading to (26), the hydrostatic part of the barotropic flow, $\Psi^{(0)}$, is forcing the nonhydrostatic part of the barotropic flow as well as the baroclinic flow. We show in an appendix how $\Psi^{(r)}$ can be obtained. One could then use the exact (26) to solve for $\psi''$, and subtract $\Psi^{(r)}$ to find the purely baroclinic response $\psi' = \psi'' - \Psi^{(r)}$.

This interpretation would also apply if we changed the decomposition by removing $W_t$ from (9), thus ensuring a hydrostatic solution of (7)–(10), in which case it would appear as an extra forcing term $-W_t$ on the right-hand side of (14). (This extra forcing term is absent from Baines's equation because the hydrostaticity of the barotropic field was in his case imposed a posteriori, i.e., after (22) had been derived.)

### d. Fully hydrostatic flow

If the baroclinic, as well as the barotropic, flow is assumed to be hydrostatic, then we may ignore $w'_t$ in (14), $\psi'_{xxtt}$ in (19), and $\omega^2\psi'_{xx}$ in (22). This leads to (26), though with $c^2 = (\omega^2 - f^2)/N^2$ now, as for $\omega^2 \ll N^2$.

### e. Physics

For an ocean with constant buoyancy frequency $N$, the correct governing equation, (28), shows that the response $\psi' + \Psi^{(r)}$ is $(N^2 - \omega^2)/N^2$ times the baroclinic response $\psi'$ given by (22) if the hydrostatic approximation $\Psi^{(0)}$ is used in place of the full $\Psi$ in the forcing term on the right-hand side. Away from topographic features, $\Psi^{(r)}$ vanishes so that the amplitude of the internal tide generated will be overestimated by a factor $N^2/(N^2 - \omega^2)$ if the hydrostatic approximation for the barotropic flow is used in the forcing.

This seems physically surprising at first. The hydrostatic approximation for the barotropic flow requires that $h^2 \ll L^2$ (see the appendix), where $L$ is the horizontal length scale of the topography, and this seems distinct from the condition $\omega^2 \ll N^2$ that is required for the validity of the solution using the hydrostatic approximation in the forcing. The reconciliation comes from recognition that, as $\omega$ approaches $N$, the dispersion relation for internal waves implies that propagating waves (or modes) must have a small horizontal scale, in which case we cannot assume $h^2 \ll L^2$. In other words, large topographic scales will generate no propagating solution anyway, and topographic scales small enough to do so are too small for the hydrostatic approximation to be valid for the barotropic flow. Right at $\omega = N$, the forcing vanishes in (28), implying that $\psi' + \Psi^{(r)} = 0$ so that the baroclinic response exactly balances the nonhydrostatic part of the barotropic flow and there are no radiated waves whatever the scale of the topography.

## 3. Discussion

Our main point is that a frequently used formulation for the generation of internal tides, leading to (24), is valid only when $\omega^2 \ll N^2$, under which circumstances the baroclinic as well as the barotropic tide is hydrostatic. The correct equation for nonhydrostatic conditions is (26), which is actually simpler than (24). There does not seem to be a situation in which one can argue that the barotropic flow is hydrostatic but the baroclinic response is not.

In fact, the decomposition into barotropic and baroclinic tides is unnecessary. If one wishes to solve the internal-tide problem with homogeneous boundary conditions rather than boundary forcing of a homogeneous equation, then a forcing term does appear in the governing equation, but we have shown that this can be regarded as the hydrostatic part of the barotropic tide, with the nonhydrostatic part being incorporated into the solution for the baroclinic tide.

In practical terms, the difference between (24) and (26) is likely to be slight for internal tides, though $\omega^2/N^2$ might be as large as 0.2 if $N = 3 \times 10^{-4}\ \mathrm{s}^{-1}$, taking $\omega = 1.4 \times 10^{-4}\ \mathrm{s}^{-1}$ as for the $M_2$ tide. The issue is more important for internal-wave generation by higher-frequency motions such as overtides, but, in any event, it does seem worthwhile to start with a correct equation, especially if it is simpler than the approximation!

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## APPENDIX

### The Nonhydrostatic Barotropic Tide

The barotropic tide can be regarded as a long wave, but this does not necessarily imply that it can be treated as hydrostatic; if the flow goes over steep topography, it is the length scale of the topography that determines the aspect ratio, rather than the wavelength. It is therefore of some interest to derive the expression for the nonhydrostatic barotropic tide. We can formally solve (21) by successive approximation; that is, $\Psi$ is written as

$$\Psi = \sum_n \Psi^{(n)}(x, z) \quad \text{and}$$

$$\Psi^{(0)}_{zz} = 0; \quad \Psi^{(n)}_{zz} = \frac{\omega^2}{f^2 - \omega^2}\Psi^{(n-1)}_{xx} \quad \text{for} \quad n = 1, 2, \ldots. \tag{A1}$$

Here $\Psi^{(0)}$ represents the hydrostatic solution (23). Subsequent orders can be easily solved via the *ansatz*

$$\Psi^{(n)} = \sum_{k=0}^{n} C_{n,k}(x)z^{2k+1}; \tag{A2}$$

substitution into (A1) yields, for $n = 1, 2, \ldots,$

$$C_{n,k+1}(x) = \frac{\omega^2}{f^2 - \omega^2}\frac{C''_{n-1,k}(x)}{(2k+3)(2k+2)} \quad \text{for} \quad k = 0, \ldots, n-1. \tag{A3}$$

Finally, the remaining unknown $C_{n,0}$ is found by requiring $\Psi^{(n)}|_{z=-h} = 0$ [the other condition $\Psi^{(n)}|_{z=0} = 0$ is already satisfied]:

$$C_{n,0}(x) = -\sum_{k=1}^{n} C_{n,k}(x)[-h(x)]^{2k}. \tag{A4}$$

With this, all coefficients have been obtained, and the nonhydrostatic part $\Psi^{(r)} = \Psi^{(1)} + \Psi^{(2)} + \cdots$ has now been specified.

The form of (A1) and the solutions derived here show that, as long as $\omega$ is not close to $f$, the nonhydrostatic part of the barotropic flow is a fraction of order $h^2/L^2$ times the hydrostatic part, where $h$ is the water depth and $L$ is the horizontal scale of the topography.

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