"Unstable" Rearrangements of n Points

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Mathematics students learn early that there are $n!$ ways of arranging n different points; the first point can be chosen in n different ways, the second in just $n-1$ ways, and so on. If this doesn't seem familiar, have a look at the nice article by Byron Schmuland in the last issue of *π in the Sky* [3]. An interesting variation on $n!$ cropped up recently in a study of ocean mixing conducted by Kate Stansfield, Richard Dewey and me [4].

We were analyzing profiles of water density obtained in Juan de Fuca Strait using an instrument called a CTD. This records electrical conductivity (C), temperature (T) and pressure, hence depth (D). The conductivity depends on the salinity (S) as well as the temperature, so that C and T give S. In turn, T and S give the density of the water.

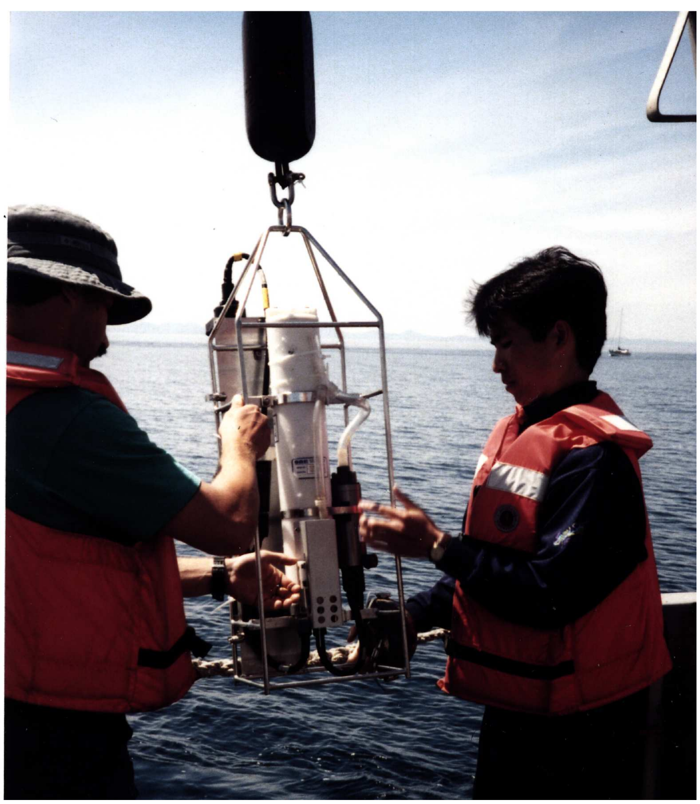

Figure 1: A CTD about to be lowered into the sea. This model records internally and is suitable for use from a small boat (in this case the University of Victoria's 16 m John Strickland). Other CTDs send signals up a conducting cable for logging on board ship.

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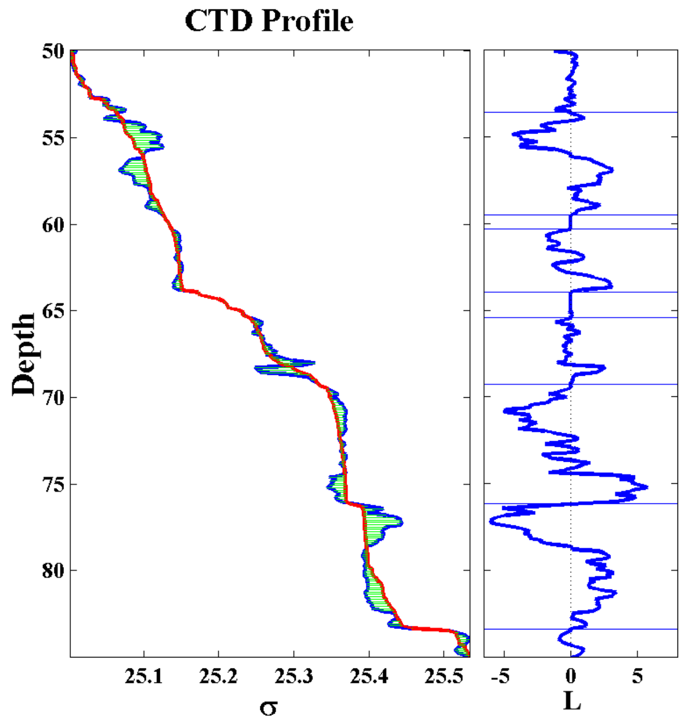

Figure 2: The blue line in the left-hand panel is the vertical profile of a quantity σ, indicating density, plotted against the depth in metres. The red line is the profile after sorting it so that σ increases with depth. The right-hand panel shows the vertical displacement L, in metres, of each data point during this sorting, and shows several distinct overturning regions. The Thorpe scale L_T is the root mean square value of L.

If the density increases with depth, the water is hydrostatically stable. If, on the other hand, there are sections of the profile with dense water on top of light water (Figure 2), these are hydrostatically unstable and cannot persist. They must have been produced by some sort of overturning motion (perhaps an internal equivalent of a breaking wave at the sea surface) and will subsequently collapse. These overturns are an indication that mixing is occurring, with possibly profound influences on the physics and biology of the sea.

With some handwaving about the physics, it seems that the vertical mixing rate (which is what we were interested in) can be related to the distances that water parcels have to be moved vertically to produce a reordered stable profile in which the density increases steadily downwards (Figure 2). The root mean square (rms) distance that the water parcels are moved in this reordering is denoted L_T and known as the *Thorpe scale* after Steve Thorpe, who first used this approach while studying mixing in the fresh water of Loch Ness (where the water density depends only on temperature). The turbulent mixing rate actually has the units of diffusivity (say $m^2\,s^{-1}$), with the so-called *eddy diffusivity* found empirically to be given approximately by $0.1 N L_T^2$. Here N is a measure of the strength of the stratification of the reordered density profile, with the units of frequency (s^{-1}). It is the frequency with which a water parcel would oscillate if displaced vertically in the reordered profile. The collapse time of the unstable regions observed before reordering is proportional to N^{-1}.

So much for the physics background. We wanted to go beyond just looking at the rms displacements of water parcels and look at the probability distribution of individual displacements, i.e. did the rms value come from lots of small displacements or just a few big ones? We determined this from our data, but wanted some theoretical ideas for comparison. An obvious one was to see what would happen if all redistributions were equally likely. For a start on this, we needed to know how many reorderings we had to consider. This is less than the simple answer of $n!$ as some of the profiles were never unstable to start with, at least not in the sense that one has to reorder the whole set. For example, with just two points, labelled 1 and 2, with density increasing with the numerical value, then profile $\begin{pmatrix}2\\1\end{pmatrix}$ is unstable, but $\begin{pmatrix}1\\2\end{pmatrix}$ is not. Thus we have only 1 case to consider instead of $2! = 2$. With 3 points, $\begin{pmatrix}3\\1\\2\end{pmatrix}$, $\begin{pmatrix}3\\2\\1\end{pmatrix}$ and $\begin{pmatrix}2\\3\\1\end{pmatrix}$ qualify as "complete" overturns in which the whole array is involved in a reordering to the stable profile $\begin{pmatrix}1\\2\\3\end{pmatrix}$, even though 2 stays put in the second case. On the other hand, the profile $\begin{pmatrix}1\\2\\3\end{pmatrix}$ itself obviously does not need any reordering, and the profiles $\begin{pmatrix}1\\3\\2\end{pmatrix}$ and $\begin{pmatrix}2\\1\\3\end{pmatrix}$ can be reordered without involving the first and last points respectively. Thus we have 3 cases to consider instead of $3! = 6$.

With 4 points there are 24 cases to evaluate in the same way. To give a couple of examples, $\begin{pmatrix}4\\2\\1\\3\end{pmatrix}$ qualifies as a complete overturn, even though 2 is not moved, but $\begin{pmatrix}2\\1\\4\\3\end{pmatrix}$ does not qualify, as reordering involves just the reversal of the top and bottom pairs; it is thus a superposition of two second-order overturns rather than a fourth-order one. The reader may check that for 4 points we end up with 13 cases instead of 24 and for 5 we have 71 instead of 120, but this is getting tedious, and a simple computer program needs to be written to go much further. Running even this becomes time-consuming as n gets large; already for $n = 8$ we have to evaluate $8! = 40\,320$ cases.

Thus our little problem was to determine the number of cases, denoted by $F(n)$, say, as a function of n. Maybe the mathematical readers of this newsletter can immediately write down the answer, and I could pretend that we did. The truth is that we cheated by generating the series up to $n = 8$ on a computer and then checking the wonderful compilation by Sloane and Pouffe [2]. There turned out to be more than one candidate solution all the way up to $n = 7$, but uniqueness seemed to emerge by $n = 8$. The answer is the recurrence relation

$$F(n) = n! - \sum_{k=1}^{n-1} k!F(n-k).$$

This actually arose in a completely different problem [1], but led us to slap our foreheads when we saw it. We had spent a little time trying for a closed form for $F(n)$, whereas the recurrence formula is really rather obvious: presented with the $n!$ rearrangements of n points we have to exclude the ones that start with the unaltered point 1 followed by the $F(n-1)$ complete overturns of the remaining $n-1$ points, also the $2!$ arrangements of the first two points times the $F(n-2)$ complete overturns of the remaining $n-2$ points, and so on. I'd add an exclamation mark at this point were it not for the risk of the reader confusing it with the factorial sign! (Oops.) The only detail remaining was that we had to choose $F(1) = 1$ to get this recurrence relation to start properly and give $F(2) = 1$, even though we could not really define a reordering of a single point. The difference between $F(n)$ and $n!$ is actually a small fraction of $n!$ as n becomes large, and maybe a clever reader can work out a good approximate formula (known as an "asymptotic" approximation) for $F(n)$ for large n.

Anyway, after establishing how many cases we needed to consider, we could return to the first problem we had set for ourselves. This was to find the probability distribution (which is just the likelihood of occurrence) of displacements of a given magnitude within any overturn of, say, n points, assuming that each permitted rearrangement is equally likely. First considering all $n!$ possible rearrangements (even though we have ruled out some of them), we see that each point can move any number of positions from 0 to whatever takes it to get to the farthest end of the overturn. A zero displacement only occurs for 1 of these n possibilities, regardless of where the point starts, and so has a probability of $1/n$. A move of 1 position can be in either direction, unless the point is already at the top or bottom of the overturn, and so on. I leave the reader to establish that a move of m has a probability of $2(n-m)/n^2$ for m from 1 to n. Overall, then the probability is as shown in Figure 3, with small displacements more likely than large ones.

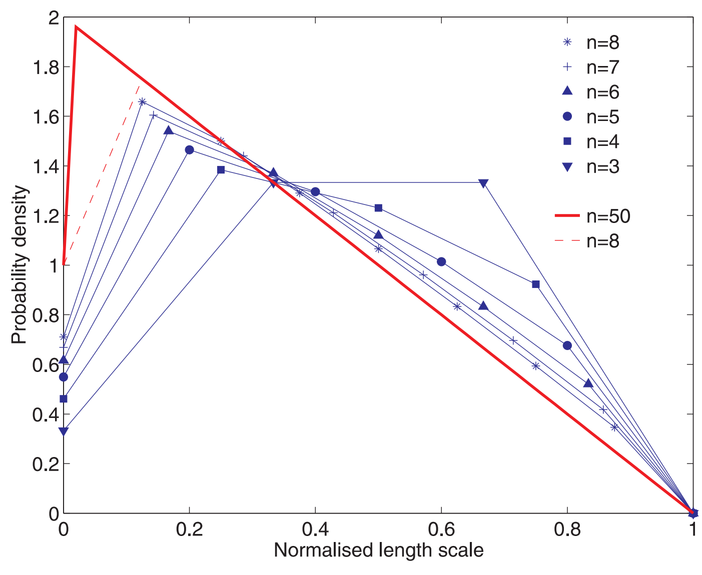

Figure 3: The lines joining the points give the probability of a particle displacement as a function of the fractional displacement (the normalised length scale) m/n. The solid red line is for $n = 50$, considering all $n!$ cases. For other values of n the solid red line still applies for a fractional displacement of $1/n$ and larger, but connects to 1 for $m = 0$, as shown by the dashed red line for $n = 8$. The blue lines connect the values for $n = 3$ to 8, now only allowing for complete overturns. Note that the probability has also been multiplied by n, so that the probability values times $1/n$ add up to 1.

We now, as before, exclude the cases that do not constitute complete overturns. This changes the picture somewhat, increasing the likelihood of larger displacements (Figure 3), though the difference decreases as n increases. Details of the mathematics are described in our paper [4].

Actual data (Figure 4) showed a greater probability of small displacements than given by our model, for reasons that may be associated with molecular diffusion, blurring extreme values. There is more to be done, but this is straying back into physical oceanography.

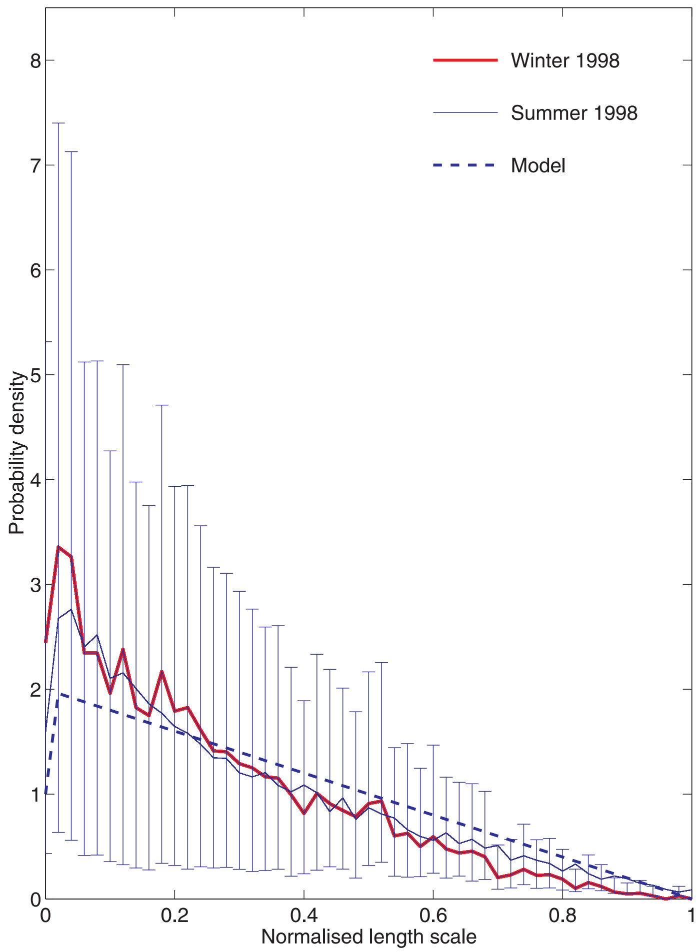

Figure 4: The solid red and blue lines show the probability distribution of actual displacements from a number of CTD profiles taken in winter (red) and summer (blue). The error bars on each value (vertical lines) for each value are large, but the trends in the results are clear. The dashed blue line shows, for comparison, the expectations from our model for $n = 50$ (a typical value). This is for the unrestricted case (with all $n!$ possible rearrangements considered), but there is little difference from the restricted case for large values of n.

So what are the messages of this article? One is that even those of us who have drifted from physics and mathematics into their application in the real world still love problems like the one described here when they crop up in our work. Another is that Monsieur Comtet surely didn't anticipate oceanographic applications of his series, where as Sloane and Plouffe might be less surprised at the range of users of their encyclopaedia. My main message to any student reader, though, is that if you enjoy mathematics and physics, and want to apply them to a research field that is fun, consider ocean physics!

References

[1] Comtet, L., 1972: Sur les coefficients de l'inverse de la série formelle $\sum n!\,t^n$. *C. R. Acad. Sci. Paris, Série A*, **275**, 569–572.

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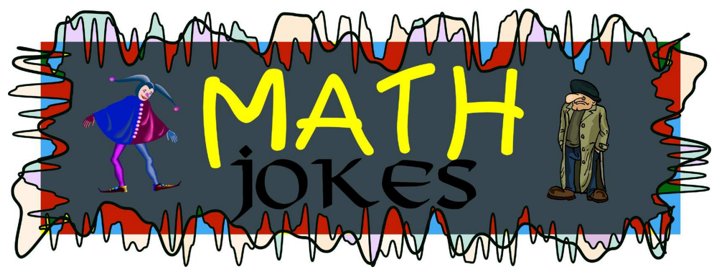
[4] Stansfield, K., C. Garrett, and R. Dewey, 2001: The probability distribution of the Thorpe displacement within overturns in Juan de Fuca Strait. *J. Phys. Oceanogr.*, **31**, 3421–3434.



Q: Why do mathematicians, after a dinner at a Chinese restaurant, always insist on taking the leftovers home?
A: Because they know the Chinese remainder theorem!

"That math prof.'s marriage, I heard, is falling apart."
"That doesn't surprise me: he's into scientific computing, and she's incalculable...."

Q: What is polite and works for the phone company?
A: A deferential operator....

Trigonometry for farmers: swine and coswine....