Independent-Samples $t$ Test

• Chapter 14
  • Aplia week 8 (Two independent samples)

• Testing hypotheses about means of two populations
  • naturally occurring populations
    • introverts vs. extroverts
      – neuroticism
  • experimentally defined (random assignment)
    • neutral vs. aggressive model
      – child’s aggression level
Independent-Samples $t$ Test

- Testing hypotheses about means of two independent populations

- Hypotheses about population means
  - $H_0: \mu_1 = \mu_2$  $H_1: \mu_1 \neq \mu_2$  ($H_1: \mu_1 > \mu_2$  $H_1: \mu_1 < \mu_2$)
  - random sampling from two populations
  - real or hypothetical

- Draw a sample from each population and compare the means of the two samples

- Build a model of all possible outcomes, assuming the null hypothesis is true
Independent-Samples $t$ Test

- Sampling from each population is described by a distribution of sample means

$X_1$, $X_2$  

$\mu_{M_1} = \mu_{M_2}$

$\sigma_{M_1}^2 = \frac{\sigma_1^2}{n_1}$  

$\sigma_{M_2}^2 = \frac{\sigma_2^2}{n_2}$

$\mu_{M_1 - M_2} = 0$

$\sigma_{M_1 - M_2}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$

$H_0$ is true  

$\mu_1 = \mu_2$
Independent-Samples $t$ Test

- Use the distribution of differences between means to determine whether observed difference between sample means implies rejection of $H_0$

  ![Distribution of differences between means](image)

  - Need to know $\sigma_{M_1-M_2}$
  - Must estimate from $s_1^2$ and $s_2^2$
    - assumption of equal population variance ($\sigma_1^2 = \sigma_2^2$)
    - treat $s_1^2$ and $s_2^2$ as independent estimates and average them to get pooled estimate, $s_p^2$
Independent-Samples $t$ Test

• Recall the variance of the distribution of differences between means

$$\sigma_{M_1-M_2}^2 = \frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}$$

• So the standard deviation is

$$\sigma_{M_1-M_2} = \sqrt{\frac{\sigma_1^2}{n_1} + \frac{\sigma_2^2}{n_2}}$$

• Replace $\sigma_1^2$ and $\sigma_2^2$ with the estimate $s_p^2$

• yielding an estimate of $\sigma_{M_1-M_2}$

Standard deviation of difference between means

$$s_{M_1-M_2} = \sqrt{s_p^2 + s_p^2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}}$$
Independent-Samples $t$ Test

- Implications of using *estimated* standard error of difference between means

<table>
<thead>
<tr>
<th>Experimental</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M$</td>
<td>130</td>
</tr>
<tr>
<td>$n$</td>
<td>40</td>
</tr>
<tr>
<td>$s^2$</td>
<td>340</td>
</tr>
</tbody>
</table>

$s_p^2 = \frac{340 + 300}{2} = 320$

$s_{M_1-M_2} = \sqrt{s_p^2/n_1 + s_p^2/n_2} = \sqrt{\frac{320}{40} + \frac{320}{40}} = 4$

$M_1-M_2 = 130-120 = 10$

What is the model of possible outcomes?
Independent-Samples $t$ Test

- Distribution of differences between means
  - normal, with mean = 0 if $H_0$ is true

But where is a difference of 10 points in this distribution?

- Standardize the values in the distribution of differences by dividing each one by its own $s_{M_1-M_2}$

This produces a $t$ distribution with $df = n_1 + n_2 - 2$
Independent-Samples $t$ Test

<table>
<thead>
<tr>
<th></th>
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</tr>
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<tbody>
<tr>
<td>$M$</td>
<td>130</td>
<td>120</td>
</tr>
<tr>
<td>$n$</td>
<td>40</td>
<td>40</td>
</tr>
<tr>
<td>$s^2$</td>
<td>340</td>
<td>300</td>
</tr>
</tbody>
</table>

$M_1 - M_2 = 130 - 120 = 10$

$s_{M_1 - M_2} = 4$

- Standardize the observed difference between means

$t = \frac{M_1 - M_2}{s_{M_1 - M_2}} = \frac{10}{4} = 2.50$

Relevant $t$ distribution:

![t-distribution graph with critical values and degrees of freedom](image)

$df = 78$
Independent-Samples \( t \) Test

- What if unequal \( n \)?
  - recall how to compute sample variance
    \[
    s^2 = \frac{\sum (X - M)^2}{N - 1} = \frac{SS}{df}
    \]

- Pooling two sample estimates of variance
  - combine \( SS \) from each sample and divide by combined \( df \)
    \[
    s_p^2 = \frac{SS_1 + SS_2}{df_1 + df_2}
    \]
    \[
    s_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}
    \]
  - weighted average
Independent-Samples $t$ Test

- Full numerical example of $t$ test
- two random samples: introverts and extroverts
- scores on a neuroticism self-report scale

<table>
<thead>
<tr>
<th>Introverts</th>
<th>Extroverts</th>
</tr>
</thead>
<tbody>
<tr>
<td>$X$</td>
<td>$X - M$</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
<tr>
<td>6</td>
<td>-1</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

$M = 7.0$  
$M = 6.4$

$\Sigma (X - M)^2 = 16.0$  
$\Sigma (X - M)^2 = 17.2$

$s^2 = 4.0$  
$s^2 = 4.3$

$s_p^2 = \frac{SS_1 + SS_2}{df_1 + df_2}$

$s_p^2 = \frac{16.0 + 17.2}{4 + 4}$

$= \frac{33.2}{8} = 4.15$
Independent-Samples $t$ Test

- Compute $s_{M_1-M_2}$ for the previous example

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<td>2</td>
</tr>
<tr>
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<td>1</td>
</tr>
</tbody>
</table>

$M = 7.0$  \hspace{1cm} $M = 6.4$

$\Sigma (X-M)^2 = 16.0$  \hspace{1cm} $\Sigma (X-M)^2 = 17.2$

$s^2 = 4.0$  \hspace{1cm} $s^2 = 4.3$

$$s_{M_1-M_2} = \sqrt{\frac{s^2_p}{n_1} + \frac{s^2_p}{n_2}} = \sqrt{\frac{4.15}{5} + \frac{4.15}{5}} = \sqrt{0.83 + 0.83} = 1.29$$
Independent-Samples $t$ Test

• Continuing the example to compute a $t$ test

<table>
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<tr>
<td>6</td>
<td>-1</td>
<td>4</td>
</tr>
<tr>
<td>4</td>
<td>-3</td>
<td>9</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>5</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>8</td>
</tr>
</tbody>
</table>

$M = 7.0$  \hspace{1cm} $M = 6.4$

$\Sigma (X - M)^2 = 16.0$  \hspace{1cm} $\Sigma (X - M)^2 = 17.2$

$s^2 = 4.0$  \hspace{1cm} $s^2 = 4.3$

$s_p^2 = \frac{16.0 + 17.2}{4 + 4} = \frac{33.2}{8} = 4.15$

$s_{M_1 - M_2} = 1.29$

$t = \frac{M_1 - M_2}{s_{M_1 - M_2}} = \frac{7.0 - 6.4}{1.29} = 0.47$

$df = 8 \hspace{0.5cm} \alpha = .05$

critical $t = ??$ (nondirectional)
## Independent-Samples t Test

<table>
<thead>
<tr>
<th>df</th>
<th>.50</th>
<th>.40</th>
<th>.30</th>
<th>.20</th>
<th>.10</th>
<th>.05</th>
<th>.025</th>
<th>.01</th>
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</thead>
<tbody>
<tr>
<td>6</td>
<td>.718</td>
<td>.906</td>
<td>1.134</td>
<td>1.440</td>
<td>1.943</td>
<td>2.447</td>
<td>3.143</td>
<td></td>
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<tr>
<td>7</td>
<td>.711</td>
<td>.896</td>
<td>1.119</td>
<td>1.415</td>
<td>1.895</td>
<td>2.365</td>
<td>2.998</td>
<td></td>
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<tr>
<td>8</td>
<td>.706</td>
<td>.889</td>
<td>1.108</td>
<td>1.397</td>
<td>1.860</td>
<td>2.306</td>
<td>2.896</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>.703</td>
<td>.883</td>
<td>1.100</td>
<td>1.383</td>
<td>1.833</td>
<td>2.262</td>
<td>2.821</td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>.700</td>
<td>.879</td>
<td>1.093</td>
<td>1.372</td>
<td>1.812</td>
<td>2.228</td>
<td>2.764</td>
<td></td>
</tr>
</tbody>
</table>
Independent-Samples  \( t \) Test

• Continuing the example to compute a \( t \) test

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<td>2</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
</tr>
</tbody>
</table>

\[ M = 7.0 \qquad M = 6.4 \]
\[ \Sigma (X-M)^2 = 16.0 \qquad \Sigma (X-M)^2 = 17.2 \]
\[ s^2 = 4.0 \qquad s^2 = 4.3 \]

\[ s_p^2 = \frac{16.0 + 17.2}{4 + 4} = \frac{33.2}{8} = 4.15 \]
\[ s_{M_1-M_2} = 1.29 \]

\[ t = \frac{M_1 - M_2}{s_{M_1-M_2}} = \frac{7.0 - 6.4}{1.29} = 0.47 \]

\[ df = 8 \quad \alpha = 0.05 \]

Critical \( t \) = \( \pm 2.306 \) (nondirectional)
Independent-Samples $t$ Test

• Summary of steps for computing an independent-samples $t$ test

1. Obtain SS for each group from given information

**Introverts**
- $M = 7.0$
- $n = 5$
- $s^2 = 4.0$
- $SS = s^2(df)$
- $SS = 4.0(4) = 16.0$

**Extroverts**
- $M = 6.4$
- $n = 5$
- $s^2 = 4.3$
- $SS = s^2(df)$
- $SS = 4.3(4) = 17.2$

2. Pool SS and $df$ from each group to obtain pooled estimate of $\sigma^2$

$$s_p^2 = \frac{16 + 17.2}{4 + 4} = \frac{33.2}{8} = 4.15$$
Independent-Samples $t$ Test

3. Use pooled variance estimate to obtain estimated standard deviation of difference between means

$$s_{M_1-M_2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} = \sqrt{\frac{4.15}{5} + \frac{4.15}{5}} = \sqrt{0.83 + 0.83} = 1.29$$

4. Form the $t$ ratio by dividing the difference between means by the estimated standard deviation of difference between means (also called the standard error of difference)

$$t = \frac{M_1 - M_2}{s_{M_1-M_2}} = \frac{7.0 - 6.4}{1.29} = 0.47$$
Independent-Samples $t$ Test

5. Compare the obtained $t$ ratio with the critical $t$ ratio and reject the null hypothesis if the obtained value is more extreme

$\alpha = .05$, nondirectional

$df = 8$

critical $t = \pm 2.306$ (observed $t = 0.47$)

Do not reject $H_0$
Independent-Samples \( t \) Test

- Another example: attitude toward water conservation

<table>
<thead>
<tr>
<th></th>
<th>Treatment</th>
<th>Control</th>
</tr>
</thead>
<tbody>
<tr>
<td>( M )</td>
<td>18</td>
<td>14</td>
</tr>
<tr>
<td>( n )</td>
<td>17</td>
<td>25</td>
</tr>
<tr>
<td>( s^2 )</td>
<td>10</td>
<td>12</td>
</tr>
</tbody>
</table>

\[
S_p^2 = \frac{(n_1 - 1)s_1^2 + (n_2 - 1)s_2^2}{n_1 + n_2 - 2}
\]

\[
S_{M_1-M_2} = \sqrt{\frac{S_p^2}{n_1} + \frac{S_p^2}{n_2}}
\]

<table>
<thead>
<tr>
<th>df</th>
<th>.50</th>
<th>.40</th>
<th>.30</th>
<th>.20</th>
<th>.10</th>
<th>.05</th>
<th>.02</th>
<th>.01</th>
<th>.001</th>
</tr>
</thead>
<tbody>
<tr>
<td>28</td>
<td>.683</td>
<td>.855</td>
<td>1.056</td>
<td>1.313</td>
<td>1.701</td>
<td>2.048</td>
<td>2.467</td>
<td>2.763</td>
<td>3.674</td>
</tr>
<tr>
<td>29</td>
<td>.683</td>
<td>.854</td>
<td>1.055</td>
<td>1.311</td>
<td>1.699</td>
<td>2.045</td>
<td>2.462</td>
<td>2.756</td>
<td>3.659</td>
</tr>
<tr>
<td>30</td>
<td>.683</td>
<td>.854</td>
<td>1.055</td>
<td>1.310</td>
<td>1.697</td>
<td>2.042</td>
<td>2.457</td>
<td>2.750</td>
<td>3.646</td>
</tr>
<tr>
<td>40</td>
<td>.681</td>
<td>.851</td>
<td>1.050</td>
<td>1.303</td>
<td>1.684</td>
<td>2.021</td>
<td>2.423</td>
<td>2.704</td>
<td>3.551</td>
</tr>
<tr>
<td>50</td>
<td>.679</td>
<td>.849</td>
<td>1.047</td>
<td>1.299</td>
<td>1.676</td>
<td>2.009</td>
<td>2.403</td>
<td>2.678</td>
<td>3.496</td>
</tr>
</tbody>
</table>

18
Independent-Samples $t$ Test

- Assumptions
  - random (independent) sampling
  - normal distribution of differences between sample means
    - usually by assuming normal distributions of raw scores
  - equal variance in populations of raw scores (homogeneity of variance)
    - violation of this assumption (heterogeneity of variance) if sample variances differ by approx. a ratio of 4:1 or more
  - if unequal variance: compute $t$ ratio without pooling variance estimates; test $t$ ratio using $df$ from the smaller of the two samples
Independent-Samples $t$ Test

- Example with heterogeneity of variance
- drug therapy improves concentration ability of children with ADHD

<table>
<thead>
<tr>
<th>Placebo group</th>
<th>Drug group</th>
<th>$H_0$: $\mu_1 = \mu_2$</th>
<th>$H_1$: $\mu_1 &lt; \mu_2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$M = 22.5$</td>
<td>$M = 26.6$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$n = 12$</td>
<td>$n = 8$</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$s = 8.8$</td>
<td>$s = 2.1$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

$s_{M_1-M_2} = \sqrt{\frac{s_1^2}{n_1} + \frac{s_2^2}{n_2}} = \sqrt{\frac{8.8^2}{12} + \frac{2.1^2}{8}}$

t $= \frac{M_1 - M_2}{s_{M_1-M_2}} = \frac{22.5 - 26.6}{2.65} = -1.55$

df $= 7$, $\alpha = .05$
critical $t = -1.895$
Do not reject $H_0$
Independent-Samples $t$ Test

• Application example

  • certain type of amnesic patients (e.g., HM) suffer impaired conscious recollection
  • free recall of list of words
  • hypothesize more words recalled in control group versus amnesic group
Independent-Samples *t* Test

- Hypothetical data for free recall test

<table>
<thead>
<tr>
<th>Control</th>
<th>Amnesic</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>X–M</td>
</tr>
<tr>
<td>10</td>
<td>0</td>
</tr>
<tr>
<td>11</td>
<td>1</td>
</tr>
<tr>
<td>8</td>
<td>-2</td>
</tr>
<tr>
<td>7</td>
<td>-3</td>
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<td>8</td>
<td>-2</td>
</tr>
<tr>
<td>14</td>
<td>4</td>
</tr>
</tbody>
</table>

\[ M=10 \quad \Sigma (X–M)^2=40 \quad M=5 \quad \Sigma (X–M)^2=44 \]

\[ s_p^2 = \frac{SS_1 + SS_2}{df_1 + df_2} \]

\[ s_p^2 = \frac{40 + 44}{9 + 9} = 4.67 \]

\[ s_{M_1-M_2} = \sqrt{\frac{s_p^2}{n_1} + \frac{s_p^2}{n_2}} = \sqrt{\frac{4.67}{10} + \frac{4.67}{10}} = 0.97 \]
Independent-Samples $t$ Test

• Hypothetical data for free recall test

$H_0$: $\mu_1 = \mu_2$ \hspace{1cm} $H_1$: $\mu_1 > \mu_2$

$\alpha = .05$ \hspace{1cm} $df = 9 + 9 = 18$ \hspace{1cm} critical $t = 1.734$

$s_{M_1-M_2} = 0.97$ \hspace{1cm} $t = \frac{10 - 5}{0.97} = 5.15$

• Reporting results of the $t$ test: $t(18) = 5.15, p < .05$

• $H_0$ is rejected, conclude that amnesic subjects are impaired on a test of conscious recollection
Independent-Samples Design

- `t.test` function in R

```r
> data= read.table(file.choose(new=F),header=T)
> describe(data)
> t.test(data$control,data$amnesic, var.equal=T)

Two Sample t-test

data:  dat$control and dat$amnesic
t = 5.1755, df = 18, p-value = 6.36e-05
alternative hypothesis: true difference in means is not equal to 0
95 percent confidence interval:
  2.970316  7.029684
sample estimates:
mean of x  mean of y
   10        5
```
Independent-Samples $t$ Test

• Effect size
  • standard method of quantifying the size of an effect
  • particularly useful when considering statistical power

$$d = \frac{M_1 - M_2}{s_p}$$

From example with amnesia and recall:

$$s_p^2 = 4.67 \quad s_p = \sqrt{4.67} = 2.16 \quad d = \frac{10 - 5}{2.16} = 2.31$$

• Try plotting two normal population distributions of raw scores representing this effect size