Bayesian Analysis

• Logic of hypothesis testing

• Assume some hypothesis is true
  *Madam X has no special power*

• Establish alternative hypothesis
  *Madam X can predict better than chance*

• Collect a sample of data and determine whether the observed result is unlikely under the assumed hypothesis (*null hypothesis*; $H_0$)
  • if result is unlikely, then reject the null hypothesis and accept the *alternative hypothesis* ($H_1$)
  • otherwise fail to reject null hypothesis
Bayesian Analysis

• Null hypothesis

• coin flip predictions – series of 12 coin flips
  • distribution of outcomes (number of correct predictions) for a person who is just guessing

What kind of result would indicate special powers?

\[0.0002 + 0.0029 + 0.0161 = 0.0192\]
Bayesian Analysis

• Null hypothesis significance testing
  • provides the probability of the observed outcome (or one that is more extreme): \( p(\text{Data} \mid H_0) \)
  • coin flip predictions: \( p(\geq 10 \text{ correct} \mid H_0) < .05 \)

• Interpretation of \( p \) values
  • natural tendency to interpret \( p \) value as an indication of how likely it is that the null hypothesis is true
  • notice that \( p(\text{Data} \mid H_0) \neq p(H_0 \mid \text{Data}) \)

What we get

What we want
Bayesian Analysis

- Understanding conditional probabilities
    - prosecutors emphasized history of abuse
    - defense contended this was irrelevant
      - FBI data: of 4 million battered women, only 0.04% (1 in 2,500) are killed by male partner

\[ p(\text{abuser kills her} \mid \text{woman is battered}) = 1/2500 = .0004 \]

- what *is* relevant here?
  - when an abused woman is killed, who is the more likely killer – her partner or someone else?

\[ p(\text{killed by abuser} \mid \text{battered woman is murdered}) = .90 \]
Bayesian Analysis

• From $p(\text{Data} \mid H_0)$, we try to infer the status of $H_0$
  • remember that $p(\text{Data} \mid H_0) \neq p(H_0 \mid \text{Data})$
  • we accept $H_1$ if the data disfavors $H_0$
  • or we fail to reject $H_0$, but then what?

• The plague of null effects: under null hypothesis testing, the null hypothesis cannot be accepted
  • even when data favoring the null hypothesis constitute a theoretically interesting outcome, hypothesis testing logic does not allow researchers to make effective use of such a result
Bayesian Analysis

• A **Bayesian** approach helps move us closer to what we want

• Bayes' theorem

\[
p(H \mid D) = \frac{p(D \mid H) \cdot p(H)}{p(D)}
\]

• Let's see how this works

• medical testing scenario: a test for cancer
Bayesian Analysis

• For every 10,000 people in a population, suppose the following facts hold for the rate of cancer of some type and a medical test for that type of cancer

<table>
<thead>
<tr>
<th>Test result</th>
<th>Actual health status</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>No cancer</td>
</tr>
<tr>
<td>Positive</td>
<td>990</td>
</tr>
<tr>
<td>Negative</td>
<td>8910</td>
</tr>
</tbody>
</table>

\[
p(C \mid Pos) = \frac{p(\text{Pos} \mid C) \cdot p(C)}{p(\text{Pos})} = \frac{.8 \cdot .01}{.107} = .075
\]
Bayesian Analysis

- Rather than an emphasis on rejecting $H_0$, a Bayesian *model selection* method can be used.
- $H_0$ and $H_1$ are characterized as opposing models of the data.
- *Bayesian* approach evaluates how likely the data are under the null model vs. the alternative model.
- The result is a quantitative estimate of the plausibility of one model over the other.
- No cutoff (e.g., $p < .05$) for rejecting an hypothesis.
Bayesian Analysis

- Bayes' theorem
  \[ p(H \mid D) = \frac{p(D \mid H) \cdot p(H)}{p(D)} \]

posterior probability of Hypothesis given Data
Bayesian Analysis

• Bayes' theorem

$$p(H \mid D) = \frac{p(D \mid H) \cdot p(H)}{p(D)}$$

• Define *relative* posterior probabilities of null and alternative hypotheses with this formulation (odds)

$$\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{\frac{p(D \mid H_0) \cdot p(H_0)}{p(H)}}{\frac{p(D \mid H_1) \cdot p(H_1)}{p(H)}}$$
Bayesian Analysis

\[
\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} \cdot \frac{p(H_0)}{p(H_1)}
\]

- posterior odds  Bayes factor  prior odds

• Bayes factor reflects change in prior odds based on new data
• strength of evidence for $H_0$ relative to $H_1$
• if we assume prior odds of 1, that is, $p(H_0) = p(H_1)$, then posterior odds are equal to the Bayes factor
Bayesian Analysis

• Example of computing Bayes factor
  • Madam X predicting 20 coin flip outcomes
    \[ H_0: \ p_r(\text{correct}) = 0.5 \]
    \[ H_1: \ p_r(\text{correct}) = 0.7 \]

\[ BF_{01} = \frac{p(D | H_0)}{p(D | H_1)} = \frac{0.17620}{0.03082} = 5.717 \]
Bayesian Analysis

• Getting from Bayes factor to $p(H \mid D)$, assuming prior odds are 1 [i.e., $p(H_0) = p(H_1)$]

$$\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} \cdot \frac{p(H_0)}{p(H_1)}$$

$$\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} = 5.717$$

So for the case of Madam X we have: $\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = 5.717$

• The data favor the null hypothesis (Madam X has no special powers)
Bayesian Analysis

• But wait… maybe Madam X’s powers are not necessarily equivalent to $p_r = .7$

• what about .6 or even .9?

• consider this version of $H_1$: $p_r = .6$ or $.7$ or $.8$ or $.9$

• how plausible is each of these values?
  • suppose they are equally plausible
Bayesian Analysis

• With this more complex $H_1$, $p(D \mid H_1)$ needs to be computed with each possible value of $p_r$ that we are considering.

• set up the distribution of outcomes for each $p_r$ value, and from that, obtain the probability of $D$ (10 correct out of 20 predictions) in each case.
Bayesian Analysis

$p_r = .6$

$p = .117$

$p_r = .7$

$p = .031$

$p_r = .8$

$p = .002$

$p_r = .9$

$p = .000$
Bayesian Analysis

• Considering these four, equally likely possibilities ($p_r = .6$ or $.7$ or $.8$ or $.9$), how likely is Madam X to get 10 correct predictions out of 20 tries?

• we apply the multiplication and addition rules of probability

$$p(D | H_1) = .25(.117) + .25(.031) + .25(.002) + .25(.000) = .0375$$

$$\frac{p(H_0 | D)}{p(H_1 | D)} = \frac{p(D | H_0)}{p(H_1 | D)} \cdot \frac{p(H_0)}{p(H_1)}$$

$$= \frac{.1762}{.0375} = 4.699 \quad BF_{01} = 4.699$$
Bayesian Analysis

\[
\frac{p(H_0 \mid D)}{p(H_1 \mid D)} = \frac{p(D \mid H_0)}{p(D \mid H_1)} \cdot \frac{p(H_0)}{p(H_1)}
\]

- Complexity in computing \( p(D \mid H_1) \)

Possible \( H_1 \) for the case of Madam X:

(Beta distribution)

- Integrate across all reasonable, continuous values of effect size
Bayesian Analysis

• With a beta prior for $H_1$, the outcome of 10 correct predictions over 20 trials yields $BF_{01} = 4.74$

• If the outcome had been 15 correct out of 20 trials, we would have $BF_{01} = 0.141$ and $BF_{10} = \frac{1}{0.141} = 7.09$

• General convention: $BF = 3 - 20$, positive evidence
  $BF = 20 - 150$, strong evidence
  $BF = > 150$, very strong

$BF_{01}$ values courtesy of J. Rouder’s (U. Missouri) web site:
  http://pcl.missouri.edu/bf-binomial
Bayesian Analysis

• Some serious computation is required to produce Bayesian analyses when applied to designs with which we are familiar

• one-sample $t$ test, related-samples $t$ test

• **BayesFactor** package in R can be used
  • produces Bayes factor for comparing null hypothesis (e.g., $\mu = 52$ or $\mu_1 - \mu_2 = 0$) to alternative hypothesis
  • what is the nature of the alternative hypothesis?
Bayesian Analysis

- $H_1$: prior distribution of standardized effect size is a $t$ distribution with 1 $df$, called a Cauchy distribution

- Emphasizes reasonable effect sizes
  - effect size in standard deviation units
Bayesian Analysis

- Example with a repeated-measures design
- Simon task
Bayesian Analysis

• From data file called *Simon.txt*

```
> data
   Comp Incom
 1  318  330
 2  428  436
 3  413  406
 4  408  427
 5  368  384
 6  394  418
 7  405  414
...
```

\[ n = 20 \]
\[ \text{Mean} \]

<table>
<thead>
<tr>
<th></th>
<th>Compatible</th>
<th>Incompatible</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mean</td>
<td>399.2</td>
<td>409.5</td>
</tr>
</tbody>
</table>

Observed \( d \) for difference scores = \( \frac{10.35}{12.22} = 0.85 \)

\[ t (19) = 3.79, \, p = .0012 \]
Bayesian Analysis

• **BayesFactor** package in R

```
> install.packages("BayesFactor")
> library(BayesFactor)
> simon = read.table(file.choose(new=T),header=T)
> describe(simon)
> ttestBF(simon$Comp, simon$Incom, paired=T)
```
Bayesian Analysis

• **BayesFactor** package in R

```r
> ttestBF(simon$Comp, simon$Incom, paired=T)
```

Bayes factor analysis

```
[1] Alt., r=0.707 : 30.20019 ±0%
```

Against denominator:
  Null, mu = 0

---

Bayes factor type: BFoneSample, JZS

\[
BF_{10} = \frac{30.20}{1} \quad BF_{01} = \frac{1}{30.20} = \frac{0.0331}{1}
\]
Bayesian Analysis

- Bayesian approach resolves various problems with \( p \) values under the NHST system
  - it provides what researchers want: \( p(H \mid D) \)
  - effective evaluation of validity of the null hypothesis
- Easy to apply in practice (\texttt{BayesFactor} package in \texttt{R})