Chi-Square Tests

• Reading: Chapter 19

• Inferential tests used when dependent variable is measured on a nominal scale (scores are not quantitative)
  • dependent measure is a frequency count, not a quantitative variable allowing arithmetic procedures
  • consumer brand preference
  • number of left-handers vs. right-handers in a sample of reading disabled children
  • distribution of male vs. female faculty members across disciplines (sciences, social sciences, humanities)
Chi-Square Tests

• General nature of the chi-square ($\chi^2$) test
  • establish an expected distribution of frequencies across categories based on a null hypothesis
  • measure discrepancy between expected and observed frequencies
  • degree of discrepancy determines decision about $H_0$

• Two general situations
  • goodness-of-fit ($H_0$: no preference across 3 brands)
  • independence ($H_0$: distribution of male vs. female faculty is independent of academic discipline)
Chi-Square Tests

• Goodness-of-fit
  • expected frequency of observations across categories based on $H_0$

• Which type of social media is deemed most important?
  • texting, Twitter, Facebook, e-mail
  • $H_0$: each type equally preferred
  • $H_1$: some types are more preferred
Chi-Square Tests

• Obtain a sample of 80 subjects and ask each person to select one of the four options

• If all options are equally likely to be the most important to a particular person, or if people do not have a preference of one over the others (and so are making an arbitrary [random] choice), what is the expected distribution of choices?

<table>
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<td>Expected freq.</td>
<td>20</td>
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</tr>
<tr>
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<td>31</td>
<td>12</td>
<td>22</td>
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• Are observed frequencies unlikely under $H_0$?
  • if so, reject $H_0$
Chi-Square Tests

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- Quantify degree of discrepancy between expected and observed frequencies
- squared deviation, weighted by expected frequency

\[
\chi^2 = \sum \frac{(O - E)^2}{E}
\]

\[
\chi^2 = \frac{(31 - 20)^2}{20} + \frac{(12 - 20)^2}{20} + \frac{(22 - 20)^2}{20} + \frac{(15 - 20)^2}{20}
\]

\[
= 6.05 + 3.20 + 0.20 + 1.25 = 10.70
\]
Chi-Square Tests

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$\chi^2 = 10.70$

- How to interpret this value?
  - size of $\chi^2$ should depend on number of categories
  - a family of $\chi^2$ distributions based on null hypothesis
  - degrees of freedom: number of cells whose frequencies are free to vary, given the total number of observations ($k - 1$ for goodness-of-fit test)
Chi-Square Tests

• What is the $\chi^2$ distribution like?
Chi-Square Tests

• How is the $\chi^2$ distribution generated?
  • under $H_0$, observed frequencies are distributed normally around the expected value
  • consider simple case of two equally likely categories, such as a taste preference between two colas
    • $H_0$ claims that each person is equally likely to choose either cola (A or B)--like flipping a coin
    • if 20 people are tested, the distribution of possible outcomes, assuming $H_0$ is true, would be…
Chi-Square Tests

- Binomial distribution
  \[ \mu = np = 20(.5) = 10 \quad \sigma = \sqrt{np(1-p)} = \sqrt{20(.5)(.5)} = 2.236 \]

- An observed outcome can be expressed as a z score:
  \[ z = \frac{X - np}{\sqrt{np(1-p)}} = \frac{12 - 10}{2.236} = 0.89 \]
Chi-Square Tests

• Now consider squaring the \( z \) score

\[
    z^2 = \frac{(X - np)^2}{np(1-p)}
\]

• Notice how \( z^2 \) fits with \( \chi^2 \)

\[
    \chi^2 = \sum \frac{(O - E)^2}{E}
\]

\( X \) is observed frequency, \( np \) is expected frequency, so we have

\[
    z^2 = \frac{(X - np)^2}{np(1-p)} = \frac{(O - E)^2}{E(1-p)} = 2\left(\frac{(O - E)^2}{E}\right) = \chi^2
\]

• Thus, under \( H_0 \), computing \( \chi^2 \) is just like computing a \( z \) score from a normal distribution and squaring it
Chi-Square Tests

• What would be generated by squaring z scores randomly selected from a normal distribution?

\[ df = 1: \chi^2 = z^2 \]
\[ (3.84 = 1.96^2) \]

\[ df = 2: \chi^2 = z^2 + z^2 \]

More generally,
\[ \chi^2 = \Sigma z^2 \]
(sum of \( k - 1 \) independent squared z scores)

• For this system to work, minimum expected frequency is 5 (for reasonable approximation by the normal distribution)
Chi-Square Tests

\[ \chi^2(3) = 10.70 \quad \alpha = .05 \quad \text{critical } \chi^2(3) = 7.82, \text{ reject } H_0 \]

- What if \( \alpha = .01 \)?
Chi-Square Tests

• Assumptions underlying the $\chi^2$ test
  • an adequate expected frequency in each cell to allow the normal distribution to be an accurate model of the data
  • each observation is independent of all other observations (as in a series of independent coin flips)
  • each subject selected into the sample independently of all other subjects
  • only one observation per subject
Chi-Square Tests

• Goodness-of-fit
  • a second example: expected frequencies do not always have to be equal across categories
  • number of left- vs. right-handers in sample of 68 reading disabled children
    • expected freq. based on normal population (15% L)

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Chi-Square Tests

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<th>Right</th>
</tr>
</thead>
<tbody>
<tr>
<td>Expected freq.</td>
<td>10.2</td>
<td>57.8</td>
</tr>
<tr>
<td>Observed freq.</td>
<td>14</td>
<td>54</td>
</tr>
</tbody>
</table>

\[ \chi^2(1) = \sum \frac{(O - E)^2}{E} = \frac{(14 - 10.2)^2}{10.2} + \frac{(54 - 57.8)^2}{57.8} \]

\[ = \frac{14.44}{10.2} + \frac{14.44}{57.8} = 1.67 \]

\[ \alpha = 0.05 \]

critical \( \chi^2(1) = 3.84 \), do not reject \( H_0 \)
Chi-Square Tests

• Independence
  • determine whether distribution of frequencies across levels of a categorical variable is independent of a second variable

  $H_0$: variables are independent
  $H_1$: variables are not independent

• example: effectiveness of treatment methods for mental disorder
  • variables: treatment condition, mental health status (remission, no remission)

  $H_0$: remission rate equal for all treatments
  $H_1$: remission rate not equal for all treatments
Chi-Square Tests

- Sample of 100 subjects assigned to one of three treatment conditions
- no treatment, drug, behavioral therapy
- expected frequencies?

<table>
<thead>
<tr>
<th>Treatment</th>
<th>Remis.</th>
<th>No rem.</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>None</td>
<td>8</td>
<td>12</td>
<td>20</td>
</tr>
<tr>
<td>Drug</td>
<td>24</td>
<td>10</td>
<td>34</td>
</tr>
<tr>
<td>Behav.</td>
<td>34</td>
<td>12</td>
<td>46</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td><strong>66</strong></td>
<td><strong>34</strong></td>
<td><strong>100</strong></td>
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Chi-Square Tests

• Expected frequencies based on $H_0$ (independence)

• Probability of an observation occurring in a specific cell is the product of the probability of the relevant row and column (multiplication rule of probability)

• None-Rem. cell probability = \( \frac{20}{100} \times \frac{66}{100} = (.20)(.66) \)

\[
\begin{array}{c|cc|c}
\text{Treatment} & \text{Rem.} & \text{No rem.} & \text{Total} \\
\hline
\text{None} & 8 & 12 & 20 \\
\text{Drug} & 24 & 10 & 34 \\
\text{Behav.} & 34 & 12 & 46 \\
\text{Total} & 66 & 34 & 100 \\
\end{array}
\]

\[= .132\]
Chi-Square Tests

• None-Rem. cell probability = \( \left( \frac{20}{100} \right) \left( \frac{66}{100} \right) = (.20)(.66) \)  
  \[ = .132 \]

• to get expected frequency, multiply probability by total number of observations:  \(.132(100) = 13.2\)

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• more simply,  
  
  \[ E = \left( \frac{R}{N} \right) \left( \frac{C}{N} \right) N = \frac{RC}{N} \]

  \[ E = \frac{20(66)}{100} = 13.2 \]
### Chi-Square Tests

- Applying RC/N to each cell yields the following

<table>
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<tr>
<td>None</td>
<td>13.2</td>
<td>6.8</td>
<td></td>
<td>20</td>
</tr>
<tr>
<td>Drug</td>
<td>22.4</td>
<td>11.6</td>
<td></td>
<td>34</td>
</tr>
<tr>
<td>Behav.</td>
<td>30.4</td>
<td>15.6</td>
<td></td>
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Chi-Square Tests

• Compute $\chi^2$ as before

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<td>34</td>
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<td>Behav.</td>
</tr>
</tbody>
</table>

$\chi^2 = \sum \frac{(O - E)^2}{E} = \frac{(8 - 13.2)^2}{13.2} + \ldots + \frac{(12 - 15.6)^2}{15.6}

= 2.05 + \ldots + 0.83 = 7.62$
Chi-Square Tests

• How many \( df \) ?

\[
df = (r - 1)(c - 1) = (3 - 1)(2 - 1) = 2
\]

• number of cells free to vary, given row and column totals

\[
\begin{array}{ccc}
66 & 34 & 20 \\
34 & 46 &
\end{array}
\]
Chi-Square Tests

• $df = 2 \quad \chi^2(2) = 7.62 \quad \alpha = .05 \quad \text{critical} \quad \chi^2(2) = 5.99$

=> reject $H_0$

• treatments affect remission rate

• could follow up with pairwise comparison
  • e.g., None vs. Drug $[\chi^2(1) = 5.01 ]$

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Chi-Square Tests

• Working with proportions or percentages
  • if data are presented as proportions or percentages convert to frequencies and compute $\chi^2$ as usual
  • treatment of delinquents and subsequent arrest

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<th>Arrest</th>
<th>No arr.</th>
<th>Number</th>
</tr>
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<tbody>
<tr>
<td>Comm. serv.</td>
<td>32%</td>
<td>68%</td>
<td>81</td>
</tr>
<tr>
<td>Detention</td>
<td>58%</td>
<td>42%</td>
<td>105</td>
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<tr>
<td>Comm. serv.</td>
<td>26</td>
<td>55</td>
<td>81</td>
</tr>
<tr>
<td></td>
<td>(37.9)</td>
<td>(43.1)</td>
<td></td>
</tr>
<tr>
<td>Detention</td>
<td>61</td>
<td>44</td>
<td>105</td>
</tr>
<tr>
<td></td>
<td>(49.1)</td>
<td>(55.9)</td>
<td></td>
</tr>
<tr>
<td>Total</td>
<td>87</td>
<td>99</td>
<td>186</td>
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$$E = \frac{RC}{N} = \frac{81 \times 87}{186} = 37.9$$

*etc.*
### Chi-Square Tests

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\[
\chi^2(1) = \frac{(26 - 37.9)^2}{37.9} + \frac{(44 - 55.9)^2}{55.9}
\]

\[
= 12.44
\]

Critical \(\chi^2(1) = 3.84\)

Reject H_0

Conceptual conclusion?