Table 6.5 Methods employed in geochronology. The quantities refer to the variables of equation 6.28. ^{40}K has two modes of decay but only the mode shown in table is utilized.

Parent nucleus	Progeny nucleus	Half-life	Stable nucleus
(P)	(D)	(10^9 years)	(D')
^{238}U	^{206}Pb	4.47	^{204}Pb
^{235}U	^{207}Pb	0.70	^{204}Pb
^{232}Th	^{208}Pb	14.0	^{204}Pb
^{87}Rb	^{87}Sr	48.8	^{86}Sr
^{40}K	^{40}Ar	11.9	^{36}Ar
^{147}Sm	^{143}Nd	106.0	^{144}Nd

relatively recent ages, its results can be compared, in several cases, with historical registers or with counting tree rings. These comparisons usually show very good agreement.

In recent times, the equilibrium between the carbon isotopes in the atmosphere has been upset by the indirect action of humanity. The burning of coal and petroleum has increased the proportion of stable carbon isotopes in the atmosphere, since the fossil fuels are very old to contain ^{14}C. On the other hand, tests with nuclear weapons have increased the amount of ^{14}C in the atmosphere in an uncontrolled way. These facts create obvious difficulties for the application of the method in the future.

6.8 Properties of Unstable States—Level Width

In this and the next section we shall examine some properties of unstable nuclear states, namely, the states responsible for the activity of the radioactive substances.

A quantum system, described by a wavefunction that is a Hamiltonian eigenfunction, is in a well-defined energy state and, if it does not suffer external influences, it will remain indefinitely in that state. But this ideal situation does not prevail in excited nuclei, or in the ground state of unstable nuclei. Interactions of several types can add a perturbation to the Hamiltonian and the pure energy eigenstates no longer exist. In this situation a transition to a lower energy level of the same or of another nucleus can occur.

An unstable state normally lives a long time compared to the fastest nuclear processes, for example, the time spent by a particle with velocity near that of light in crossing a nuclear diameter. In this way we can admit that a nuclear state is approximately stationary, and to write for its wavefunction

$$\Psi(\mathbf{r}, t) = \psi(\mathbf{r}) e^{-iWt/\hbar}. \tag{6.29}$$

$|\Psi(\mathbf{r},t)|^2 dV$ is the probability of finding the nucleus in the volume dV, and, if the state described by Ψ decays with a decay constant λ, it is reasonable to write

$$|\Psi(\mathbf{r},t)|^2 = |\Psi(\mathbf{r},0)|^2 e^{-\lambda t}. \tag{6.30}$$

To obey (6.29) and (6.30) simultaneously, W must be a complex quantity with imaginary part $-\lambda\hbar/2$. We can write:

$$W = E_0 - \frac{\hbar\lambda}{2} i, \tag{6.31}$$

which shows that the wavefunction (6.29) does not represent a well-defined stationary state, since the exponential contains a real part $-\lambda t/2$. However, we can write the exponent of eq. 6.29 as a superposition of values corresponding to well-defined energies E (for $t \geq 0$):

$$e^{-(iE_0/\hbar+\lambda/2)t} = \int_{-\infty}^{+\infty} A(E) e^{-iEt/\hbar}\, dE. \tag{6.32}$$

Functions connected by a Fourier transform relate to each other as

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} g(\omega) e^{-i\omega t}\, d\omega,$$
$$g(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t) e^{i\omega t}\, dt; \tag{6.33}$$

this allows us to establish the form of the amplitude $A(E)$:

$$A(E) = \frac{1}{2\pi\hbar} \int_{0}^{+\infty} e^{[i(E-E_0)/\hbar - \lambda/2]t}\, dt, \tag{6.34}$$

where the lower limit indicates that the stationary state was created at the time $t = 0$. The integral (6.34) has an easy solution, giving

$$A(E) = \frac{1}{h\lambda/2 - 2\pi i(E - E_0)}.$$

The probability of finding a value between E and $E + dE$ in an energy measurement is given by the product

$$A^*(E)A(E) = \frac{1}{h^2\lambda^2/4 + 4\pi^2(E - E_0)^2}, \tag{6.35}$$

and this function of energy has the form of a Lorentzian, with the aspect shown in figure (6.4). Its width at half-maximum is $\Gamma = \hbar\lambda = \hbar/\tau$. The relationship

$$\tau\Gamma = \hbar \qquad \tau\Gamma = \hbar \tag{6.36}$$

between the half-life and the width of a state is directly connected to the uncertainty principle and shows that the longer a state survives the greater is the precision with which its energy can be determined. In particular, only to stable states can one attribute a single value for the energy.

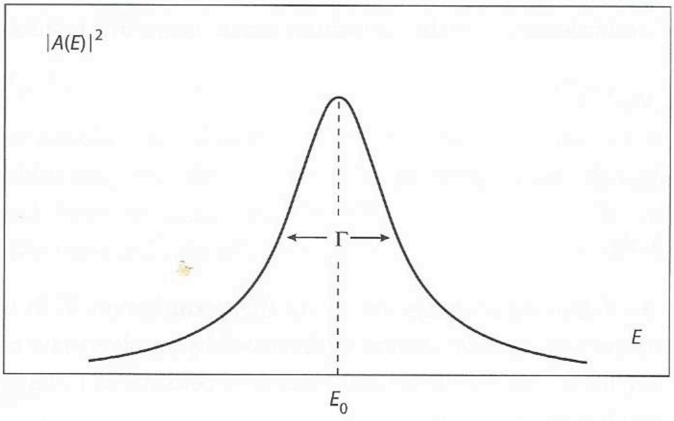

Figure 6.4 Form of the distribution (6.35).

6.9 Transition Probability—Golden Rule

The decay constant λ was presented at the beginning of the chapter as the probability per unit time of occurrence of a transition between quantum states, and its values were supposed known from experimental data. In this section we shall show that a formula to evaluate the decay constant can be obtained from the postulates of perturbation theory.

The previous section has described an unstable state by the addition of a perturbation to an stationary state. Formally we can write

$$H = H_0 + V, \tag{6.37}$$

where H is the Hamiltonian of the unstable state, composed of the nonperturbed Hamiltonian H_0 and a small perturbation V. The Hamiltonian H_0 satisfies an eigenvalue equation

$$H_0\psi_n = E_n\psi_n, \tag{6.38}$$

whose eigenfunctions form a complete basis in which the total wavefunction Ψ, obeying

$$H\Psi = i\hbar\frac{\partial\Psi}{\partial t}, \tag{6.39}$$

can be expanded:

$$\Psi = \sum_n a_n(t)\psi_n e^{-iE_nt/\hbar}. \tag{6.40}$$

Using (6.37) and (6.40) in (6.39), and with the aid of (6.38), we obtain

$$i\hbar\sum_n \dot{a}_n\psi_n e^{-iE_nt/\hbar} = \sum_n Va_n\psi_n e^{-iE_nt/\hbar}, \tag{6.41}$$

with $\dot{a}_n \equiv \partial a_n(t)/\partial t$. Using the orthogonalization properties of the ψ_n, let us multiply (6.41) to the left by ψ_k^* and integrate it in the coordinate space. From this results

$$\dot{a}_k = -\frac{i}{\hbar}\sum_n a_n V_{kn} e^{i\frac{E_k-E_n}{\hbar}t}, \tag{6.42}$$

where we introduce the matrix element

$$V_{kn} = \int \psi_k^* V \psi_n \, d\tau. \tag{6.43}$$

Let us make the following assumptions about the perturbation V. It begins to act at time $t = 0$, when the unperturbed system is described by an eigenstate ψ_m. It stays at a very low value, and after a short time interval, becomes zero at $t = T$. These assumptions allow us to say that the conditions

$$\begin{cases} a_m = 1, \\ a_n = 0 \text{ if } n \neq m, \end{cases} \tag{6.44}$$

are rigorously verified for $t < 0$ and also work approximately for $t > 0$. Thus, (6.42) has only one term, and the value of the amplitude is obtained from

$$a_k = -\frac{i}{\hbar}\int_0^T V_{km} e^{i\frac{E_k-E_m}{\hbar}t} \, dt, \tag{6.45}$$

whose value must be necessarily small by the assumption that followed (6.44). The above approach is also known as first order perturbation theory. The integral of (6.45) gives

$$a_k = \frac{V_{km}\left(1 - e^{i\frac{E_k-E_m}{\hbar}T}\right)}{E_k - E_m}. \tag{6.46}$$

We need now to interpret the meaning of the amplitude a_k. The quantity $a_k^* a_k$ measures the probability of finding the system in the state k. This characterizes a transition occurring from the initial state m to the state k, and the value of $a_k^* a_k$ divided by the interval T should be a measure of the decay constant λ_k relative to the state k. The total decay constant is obtained by the sum over all states:

$$\lambda = \sum_{k \neq m} \lambda_k = \frac{\sum |a_k|^2}{T}. \tag{6.47}$$

Let us now suppose that there are a large number of available states k. We can, in this case, replace the summation in (6.47) by an integral. Defining $\rho(E)$ as the density of available states around the energy E_k, we write

$$\lambda = \frac{1}{T}\int_{-\infty}^{+\infty} |a_k|^2 \rho(E_k)\, dE_k = \frac{4}{T}\int_{-\infty}^{+\infty} |V_{km}|^2 \frac{\sin^2\left[\left(\frac{E_k - E_m}{2\hbar}\right)T\right]}{(E_k - E_m)^2}\rho(E_k)\, dE_k. \tag{6.48}$$

The function $\sin^2 x/x^2$ only has significant amplitude near the origin. In the case of (6.48), if we suppose that V_{km} and ρ do not vary strongly in a small interval of the energy

E_k around E_m, both these quantities can be taken outside the integral, and we obtain the final expression

$$\lambda = \frac{2\pi}{\hbar} |V_{km}|^2 \rho(E_k) \tag{6.49}$$

that we have been looking for. Equation (6.49) is known as *golden rule no.* 2 (also known as the *Fermi golden rule*), and allows us to determine the decay constant if we know the wavefunctions of the initial and final states. The result (6.49) can also be used to obtain, to first order, the cross sections of processes induced by particle interaction through a potential.

6.10 Exercises

1. The activity of a given material decreases by a factor of 8 in a time interval of 30 days. What is its half-life, mean lifetime, and decay constant? If the sample initially had 10^{20} atoms, how many disintegrations have occurred in its second month of life?

2. The theories of grand unification predict that the proton is not a stable particle, although it has a long half-life. For a half-life of 10^{33} years, how many proton decays can we expect in one year in a mass of 10^3 tons of water?

3. A radioactive element decays to a stable nucleus. A counting of decays in a sample of this element was done during intervals of 1 minute in each hour. The values found were 93, 60, 49, 41, 27, 28, 20, 18, 11, ... Plot a "semilog" graph of count versus time and obtain from it an estimate of the source half-life. Recalling that the error in N counts is equal to $\sqrt{N}$, do the data seem reasonable?

4. Natural uranium is a mixture of 99.3% ^{238}U and 0.7% ^{235}U. The half-life for α-emission by the first nucleus is 4.5 x 10^9 years and of the second one is 7 x 10^8 years. a) How long ago were the amounts of the two isotopes in a sample the same? b) If a sample initially had 10 g of natural uranium, what is the mass of He gas produced since that time by the two isotopes?

5. A nucleus with decay constant λ exists at time $t = 0$. What is the probability that it disintegrates between t and $t + \Delta t$?

6. $^{252}_{98}$Cf has a half-life of 2.64 years. It disintegrates by α-emission in 96.9% of the events and by spontaneous fission in 3.1%. a) What is its mean lifetime? b) What is the ratio between the number of α-particles and number of fissions produced? c) What half-life would this isotope have if it did not have spontaneous fission?

7. Show from (6.12) that: a) The maximum of the activity $A_2(t)$ from substance 2 occurs at the crossing of the curves of $A_1(t)$ and $A_2(t)$. b) When $\lambda_1 \cong \lambda_2$ this crossing occurs at