Topic 4: The Production Function

Chapter 8   

1) The Production Function

2) Properties of Production Technologies
   →No free lunch, non-reversibility, free disposability

3) Changing Factors of Production in the Short and Long runs: diminishing returns, MRTS

4) The Short-Run Production Function
   Total Product, Average Product, Marginal Product

<u>Objective:</u> To examine how firm and industry <u>*supply curves*</u> are derived.

<u>Introduction:</u> Up to this point we have examined how the market demand function is derived. Next, we will examine the **<u>supply</u>** side of the market. We will explore how firms **<u>minimize costs</u>** and **<u>maximize productive efficiency</u>** in order to produce goods and services. By effectively combining labour and capital, the firm develops a production process with the objective of efficient resource allocation and cost minimization. The firm is assumed to produce a given output at minimum cost.

The Production Function

Definitions:

- ***Production function:*** the relationship that describes how inputs like capital and labor are transformed into output.

- Mathematically, $Q = F(K, L)$

<u>Factors of Production</u>: are factors used to produce output.

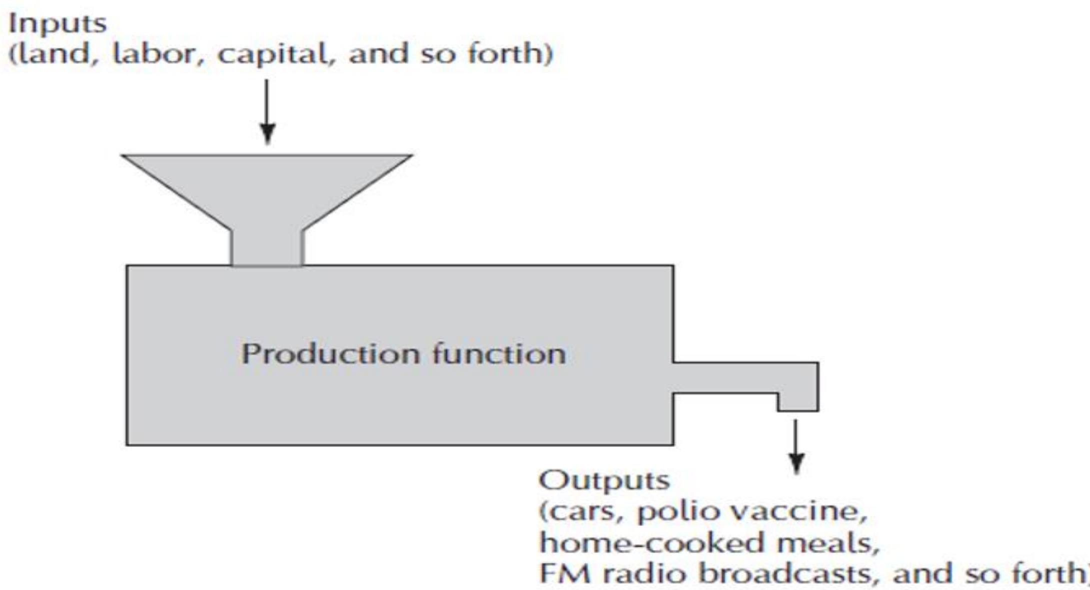
Example: Labour

Capital - machines

-buildings

Land

Natural resources



State of technology: consists of existing knowledge about method of production.

The quantity that a firm can produce with its factors of production depends on the state of technology.

The relationship between factors of production and the output that is created is referred to as the **production function**.

"The production function describes the maximum quantity of output that can be produced with each combination of factors of production given the state of technology."

For any product, the production function is a table, a graph or an equation showing the **maximum** output rate of the product that can be achieved from any specified set of usage rates of inputs.

The production function summarizes the characteristics of existing technology at a given time; it shows the technological **constraints** that the firm must deal with.

Some common **assumptions** in regard to technology:

- No free lunch: without inputs there are no outputs
- Non-reversibility: Cannot run the production process in reverse.
- Free disposability: Can throw away the excess without any cost or using more inputs.
- Convexity: ?

<u>Model Assumption:</u> The quantity produced per period is **'q'**, and the two factors of production are **<u>labour</u>** and **<u>capital</u>**.

- Mathematically, $Q = F(K, L)$

  K = Capital

  L = Labor

<u>Notation:</u> The Production Function of the Firm:

$$\mathbf{q} = \boldsymbol{f}(\mathbf{L}, \mathbf{K})$$

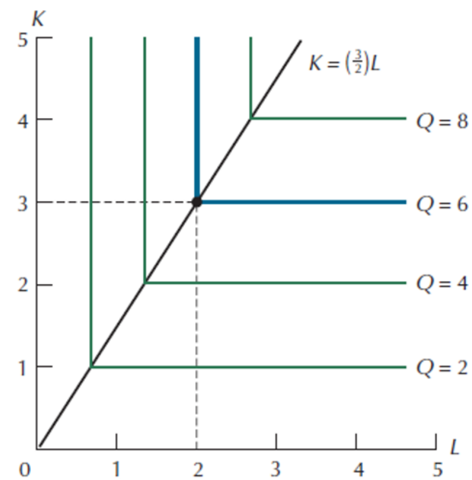
where *f* , the function, describes the relationship between the inputs L, K and the output each different combination produces per period.

Some important production functions:

1) The Leontief technology (fixed-proportions technology)

$Q(X_1,X_2) = \min(aX_1, bX_2)$
$Q(K,L) = \min(1/6L,K)$

Figure 8A.7: Isoquant Map for the Leontief Production Function Q = min (2*K*,3*L*)



2) The Cobb-Douglas technology

- The Cobb-Douglas Production Function

which in the two-input case takes the form

$$Q = mK^{\alpha}L^{\beta}, \tag{8A.3}$$

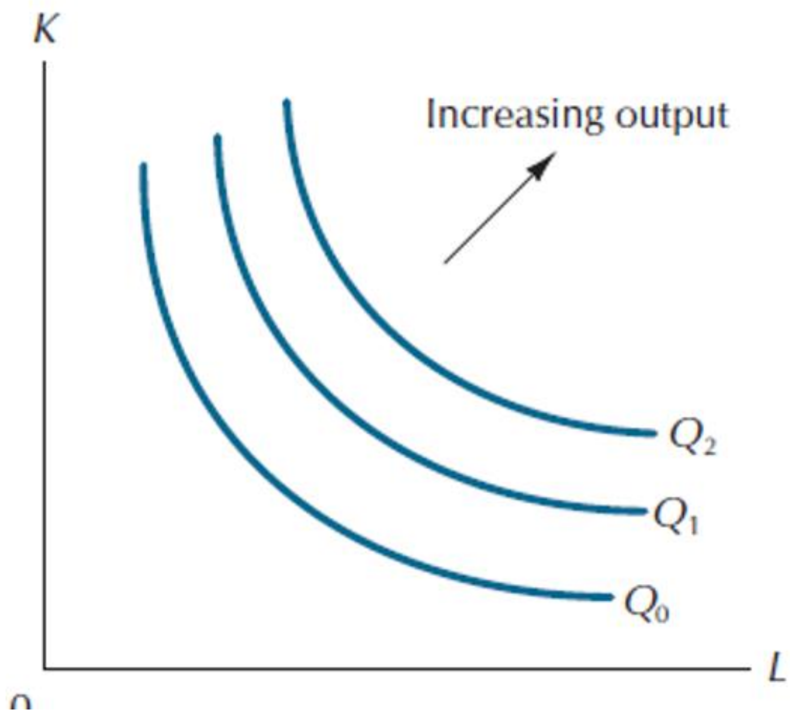
where α and β are numbers between zero and 1, and m can be any positive number.

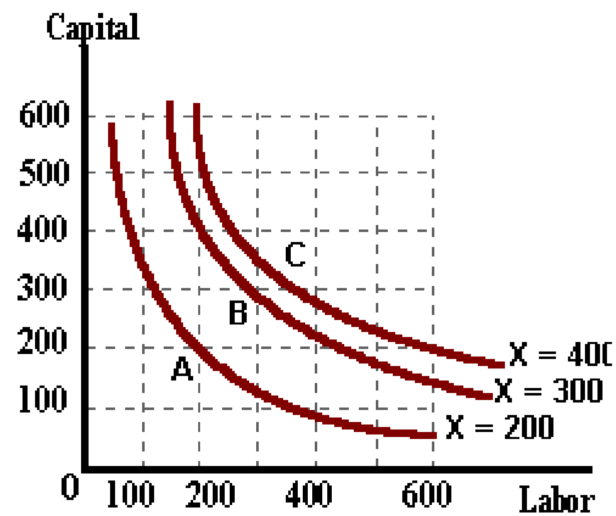
To generate an equation for the Q_0 isoquant, we fix Q at Q_0 and then solve for K in terms of L. In the Cobb-Douglas case, this yields

$$K = \left(\frac{m}{Q_0}\right)^{-1/\alpha}(L)^{-\beta/\alpha}. \tag{8A.4}$$

For the particular Cobb-Douglas function $Q = K^{1/2}L^{1/2}$, the Q_0 isoquant will be

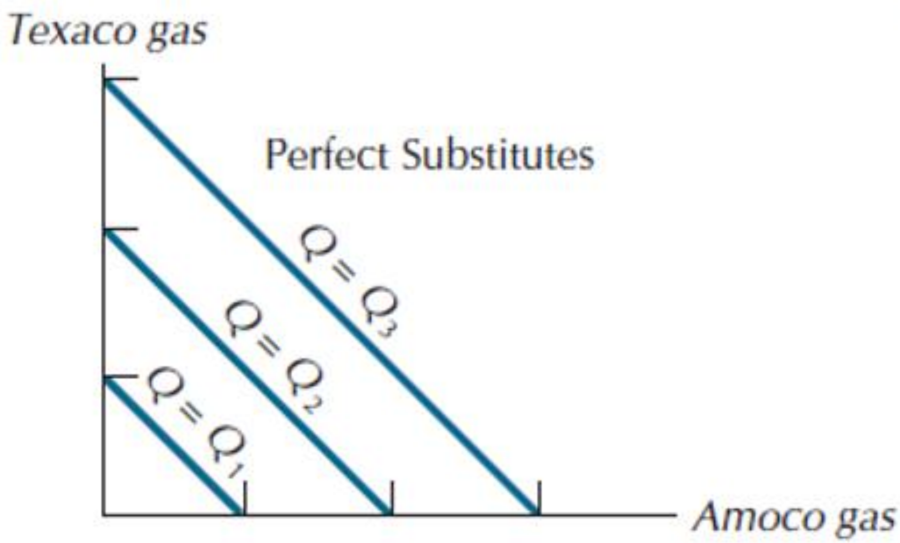
$$K = \frac{Q_0^2}{L}. \tag{8A.5}$$

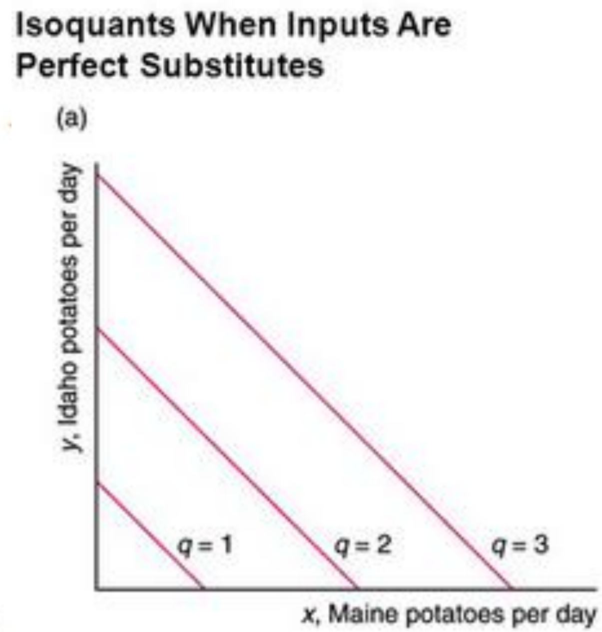

K
Increasing output
Q_2
Q_1
Q_0
L


Capital
600
500
400
300
200
100
0
C
B
A
X = 40
X = 300
X = 200
100
200
400
600
Labor

3) Perfectly Substitutable input production functions:

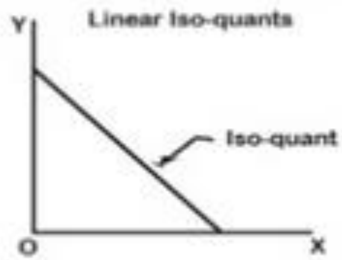
$$Q(K,L) = aK + bL$$



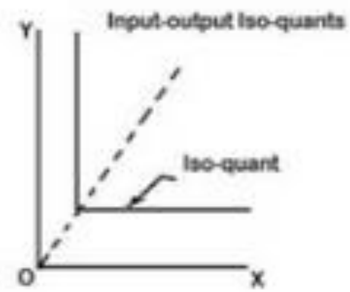


PRODUCTION FUNCTION WITH TWO VARIABLES

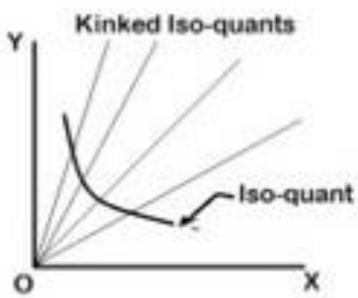
- **Linear Isoquants:**
It represents perfect substitutability of factors of production.



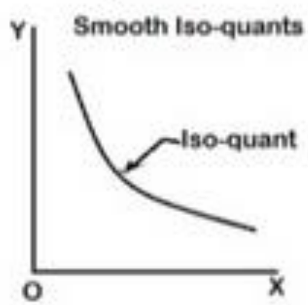
- **Input-Output Isoquants**
zero substitutability



- **Kinked Isoquants**
If the factors of production show limited substitutability



- **Smooth Isoquants**
continuous substitutability of factors of production



Changing Factors of Production in the Short and Long Runs

We must make a distinction between the **short** and **long run**.

In the short run, a firm is able to change some of the factors of production, but at least one factor is **fixed**.

In the long run, ***all*** factors of production can be varied.

Fixed and Variable Inputs

- ***Long run:*** the shortest period of time required to alter the amounts of all inputs used in a production process.

- ***Short run:*** the longest period of time during which at least one of the inputs used in a production process cannot be varied.

- ***Variable input:*** an input that can be varied in the short run.

- ***Fixed input:*** an input that cannot vary in the short run.

Production in the Short Run

- Three properties:
  1. It passes through the origin
  2. Initially the addition of variable inputs augments output an increasing rate
  3. beyond some point additional units of the variable input give rise to smaller and smaller increments in output.

The Short-Run Production Function

Model: consider the simplest case where there is one input whose quantity is fixed and one input whose quantity is variable.

> Suppose that the fixed input is the number of machines (capital) and the variable input is labour.

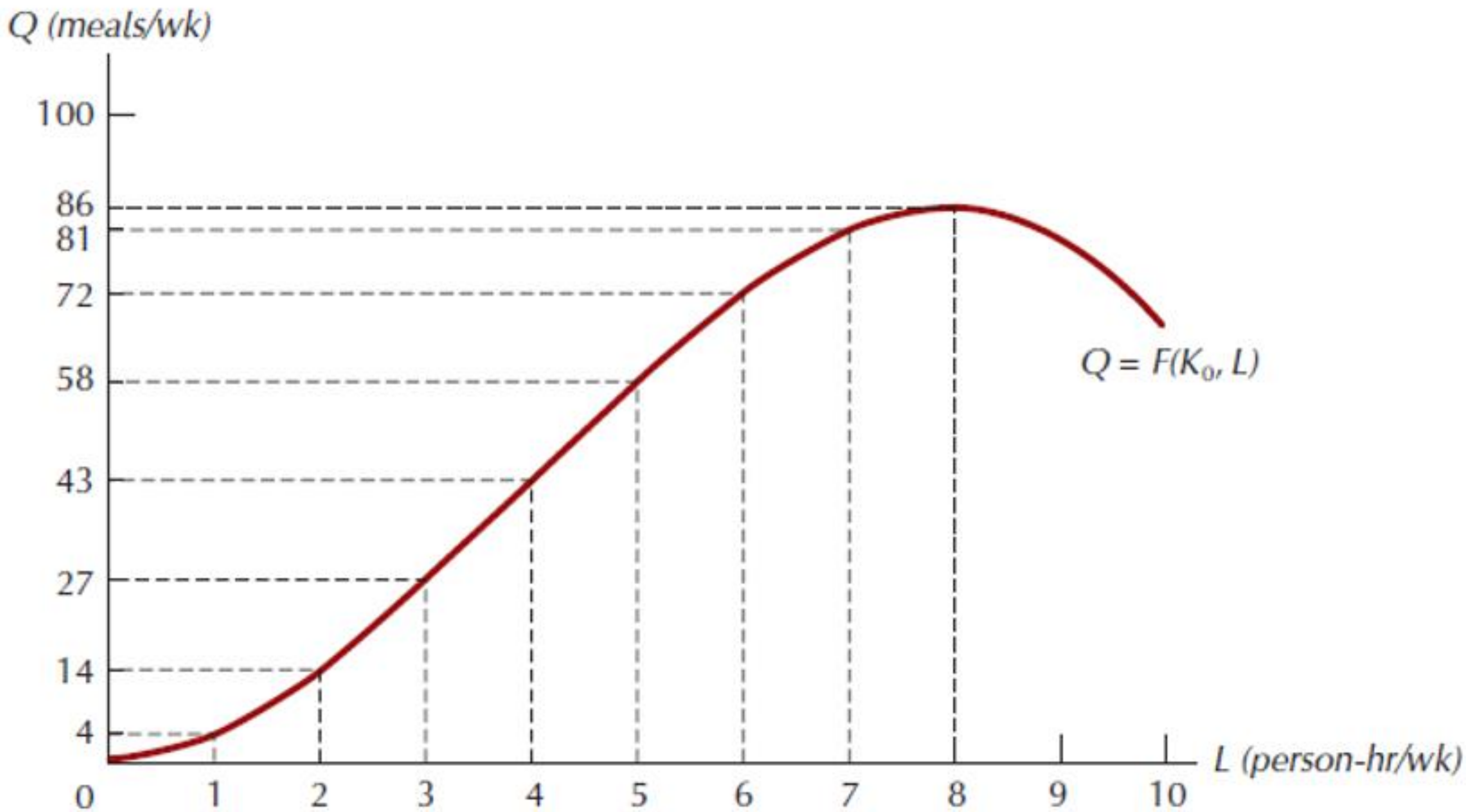
- In the short run the firm cannot change the number of machines quickly without incurring a high cost.

- With one fixed input the short-run production function shows how total output changes as the **variable** factor changes.

The Total, Average and Marginal Product of Labour

Total Product Function: expresses the relationship between the variable input and the total output.

→**The total product function of labour**: TP_L: shows the various amounts of output that is produced when the amount of labour is varied with a given fixed amount of capital.



The diagram illustrates what occurs when the amount of the **<u>variable</u>** input increases.

The total product (output) increases when the amount of labour increases, holding the amount of capital fixed at K_0.

Quantity increases initially at an increasing rate, but eventually quantity increases at a decreasing rate when more labour is employed.

Algebraically: $$\frac{\partial^2 TP_L(L,K^*)}{\partial L^2} > 0$$ at first;

and then eventually, $$\frac{\partial^2 TP_L(L,K^*)}{\partial L^2} < 0$$ (becomes negative).

At some point, adding more labour units no longer increases output.

We can derive the average and marginal product function of labour from the total product function.

The **average product function, AP_L,** measures output per unit of labour:

$$\text{Average product of labour} = \frac{\text{Total product of labour}}{\text{Number of labour units}}$$

$$AP_L(L, K_0) = \frac{TP_L(L, K_0)}{L}$$

Average product of labour is the measure of ***productivity*** of labour.

Average product of labour, AP_L, measures output per unit of labour.

It is the **slope** of a ray drawn from the origin to any point on the TP_L function.

The **average product of labour** for any given level of employment is equal to the slope of a straight line drawn from the origin to the total product function at that employment level.

Generally, the AP_L increases at first as labour is increased. I.e. the output per worker increases initially.

Further increases in labour reduce AP_L.

- AP_L declines when employment increases.

The **marginal product** of an input is the addition to total output resulting from the addition of the last unit of the input when the amount of other inputs used is held constant.

The **marginal product function of labour**, MP_L, measures the change in quantity due to a change in the labour input, or the slope of the total product function of labour:

$$\text{Marginal product of labour} = \frac{\Delta \text{ in total product of labour}}{\Delta \text{ in number of labour units}}$$

$$\mathrm{MP_L}(\mathrm{L},\mathrm{K_0}) = \frac{\Delta \mathrm{TP_L}(\mathrm{L},\mathrm{K_0})}{\Delta \mathrm{L}} = \frac{\partial TP_L}{\partial KL}$$

There is a distinct relationship between marginal product and average product:

When: MP>AP, AP is increasing
MP<AP, AP is decreasing
MP=AP, AP is constant and at a maximum

Relationships Among Total, Marginal and Average Product Curves

- *When the marginal product curve lies above the average product curve, the average product curve must be rising*
- *When the marginal product curve lies below the average product curve, the average product curve must be falling.*
- *The two curves intersect at the maximum value of the average product curve.*

The **law of diminishing returns** describes the eventual decline in the marginal product of the variable factor as the variable factor increases with other factors held constant.

The law of diminishing returns applies only to situations where one factor is increasing and the other factors are fixed.

"The law of **diminishing marginal** returns: if equal increments of an input are added, and the quantities of other inputs are held constant, the resulting increments of product will decrease beyond some point; that is, the marginal product of the input will diminish."

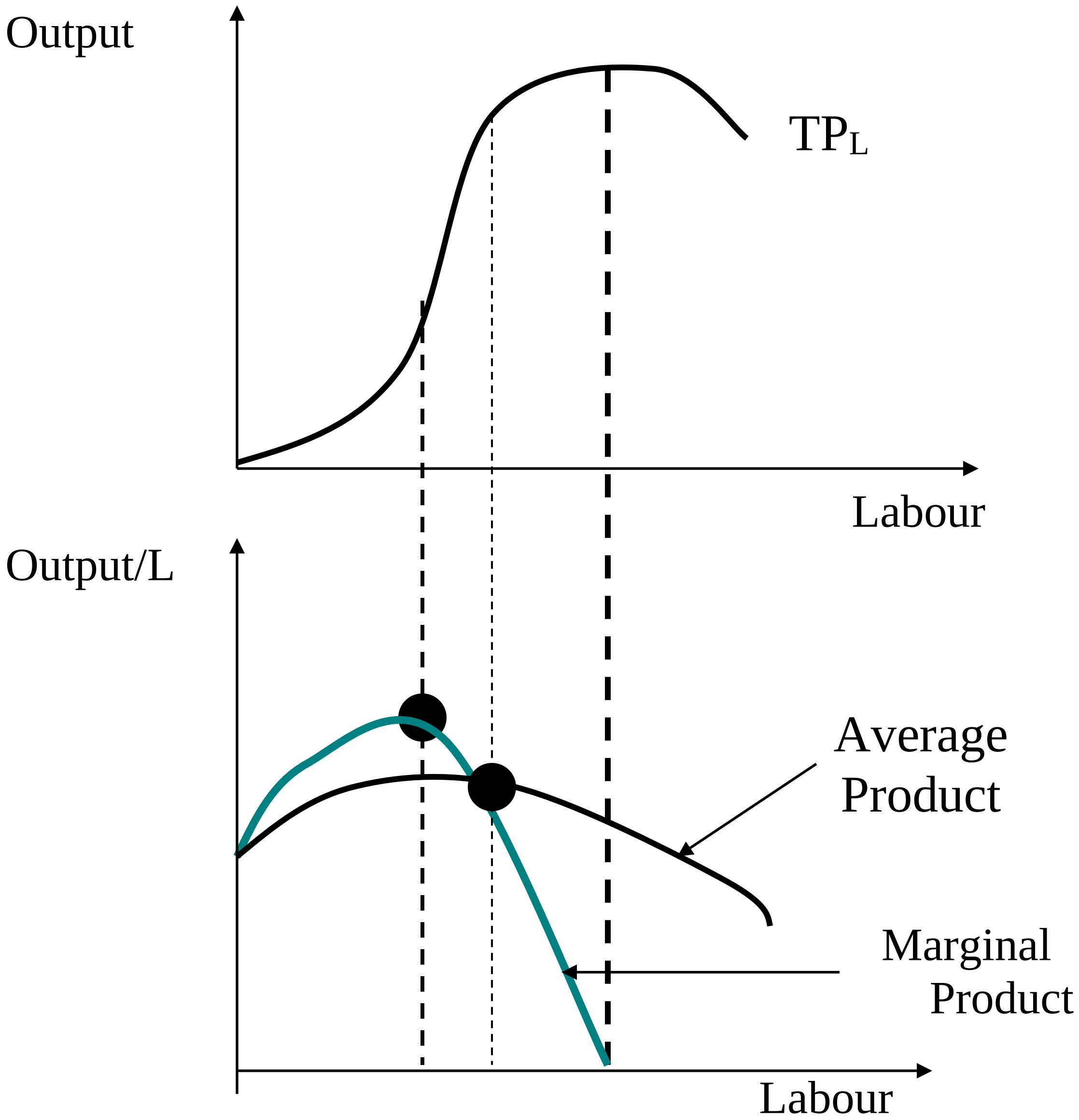
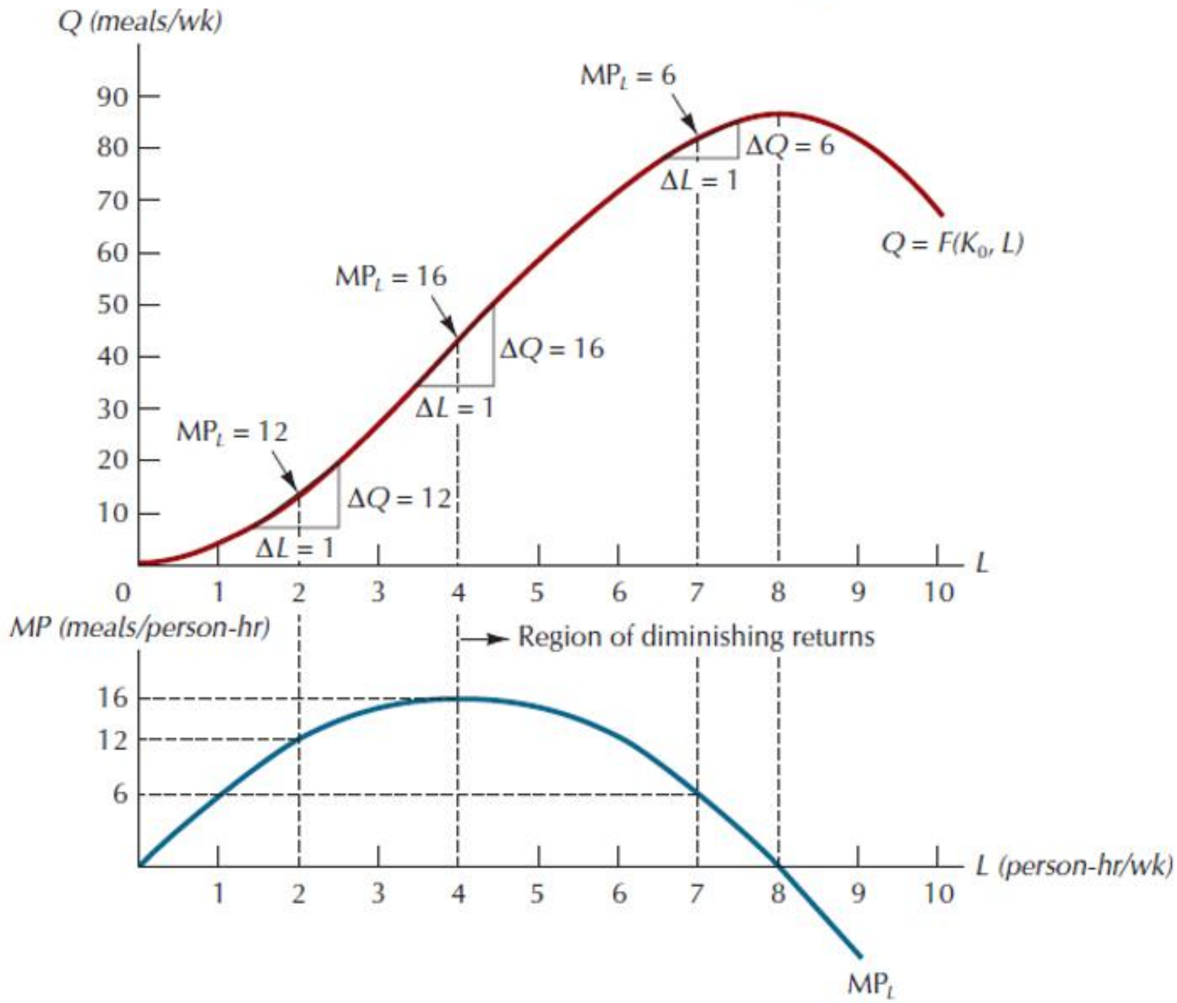

Output
TP$_L$
Labour
Output/L
Average Product
Marginal Product
Labour

Figure 8.5: The Marginal Product of a Variable Input



diminishing marginal returns

An economic theory that states as additional inputs are put into production, the additional return will be in successively smaller increments. This can be due to crowding, adding less appropriate resources or increasing inputs of lower quality.

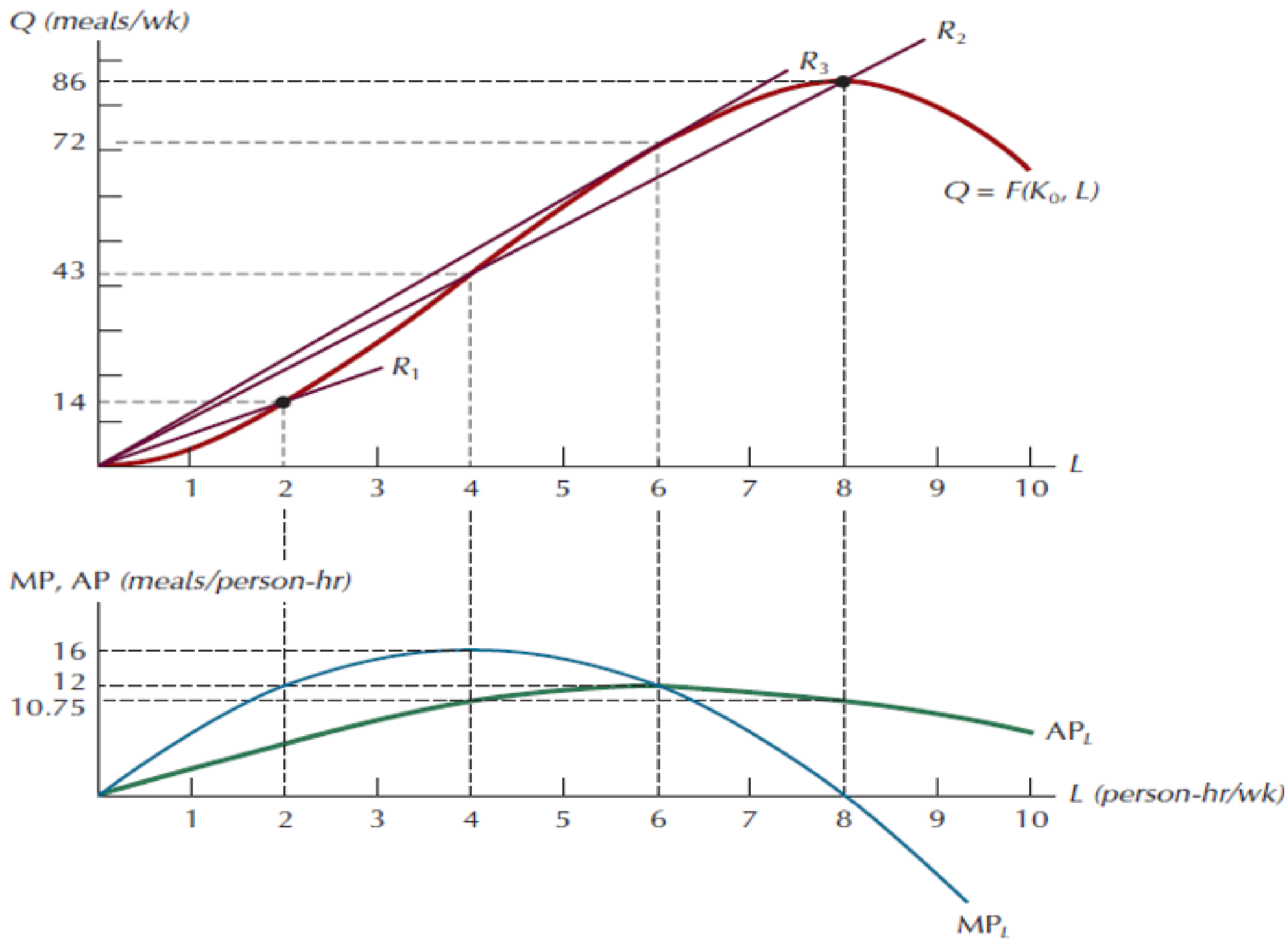
In More Laymen Terms

As the saying goes, "Too Many Cooks Spoil the Broth," in any production there is a point of diminishing returns where just adding more inputs will not give the same income as it once did. Although many industrial firms strive to reach 'scale,' where their size gives them a cost advantage at higher production levels, no matter what industry a firm finds itself there will always be a point where the additional gain from added input is reduced.

Example of Diminishing Marginal Returns

Widget Resturant	Year 1	Year 2	Year 3	Year 4
Number of Employees	5	12	25	45
Revenue	$15,000.00	$25,000.00	$35,000.00	$45,000.00
Marginal Return		$ 1,428.57	$ 769.23	$ 500.00

Figure 8.6: Total, Marginal, and Average Product Curves



The Long-Run Production Function

- In the long run, all factors of production are variable.

Substitution Among Factors

Similar to the notion of substituting between goods to maintain constant utility along an indifference curve, firms usually can produce the same output quantity by substituting between factors of production.

The important question that needs to be addressed is:

"What combination of factors should be used to produce this output?"

This question is difficult to answer because there is more than one way to produce the product.

This can be illustrated with the aid of **<u>isoquant</u>** analysis.

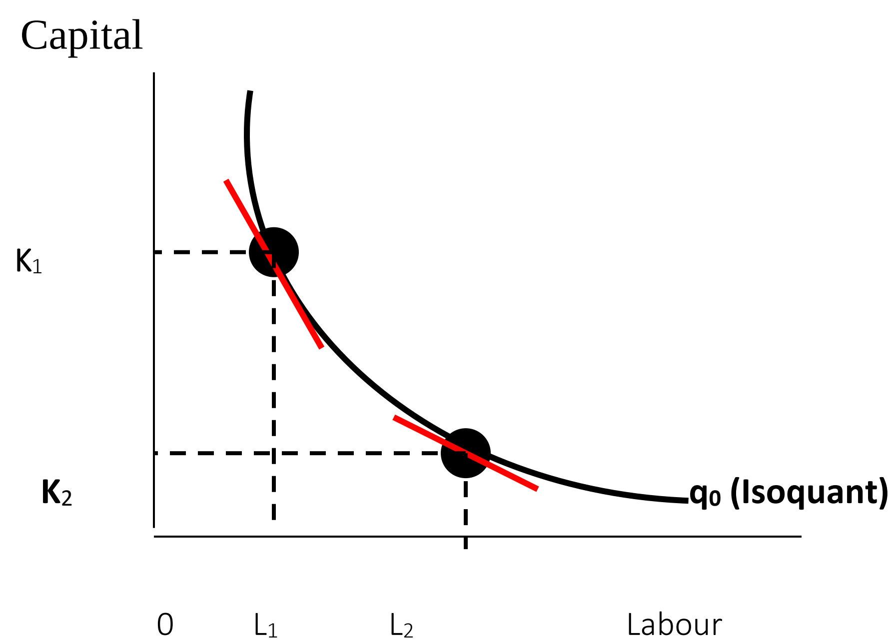
The amount of capital is on the vertical axis and number of labour units is on the horizontal axis.

The curve with an output of ‘q_0’ is called an isoquant.

- “Iso” means equal
- “quant” means quantity.

The amount produced is the **<u>same</u>** along the isoquant. The points along the isoquant q_0 represent the different factor combinations that can produce q_0 units per period.

"An isoquant shows the different combinations of factors of production that can produce a given quantity of output."



<u>The Marginal Rate of Technical Substitution</u>

The marginal rate of technical substitution (MRTS) measures the rate of substitution of one factor for another along an isoquant.

"The marginal rate of technical substitution is the rate at which a firm can substitute capital and labour for one another such that the output is constant."

$$\text{MRTS}_{\text{KL}} = \left.\frac{\Delta K}{\Delta L}\right|_{q=\text{constant}}$$

where $\Delta K/\Delta L$ is the slope between two point on an isoquant.

Note: An isoquant **<u>cannot</u>** have a positive slope.

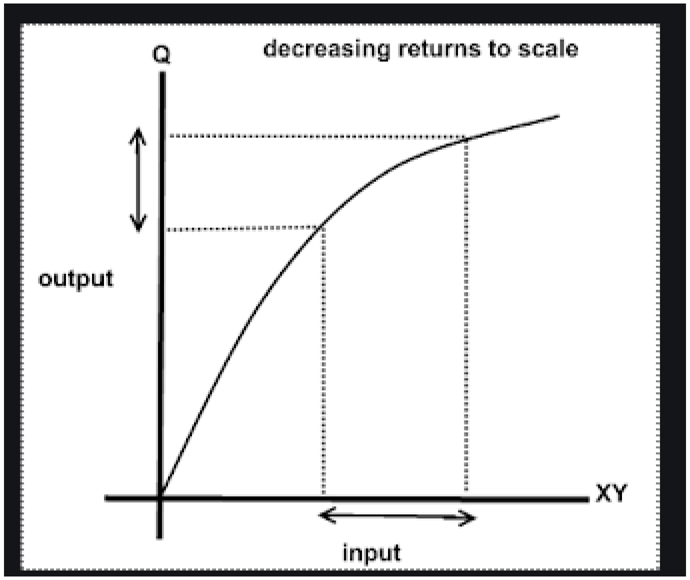
- An increase in one factor of production causes output to increase. Hence this increase in one factor must be offset by a decrease in other factor in order to keep output at the same level.

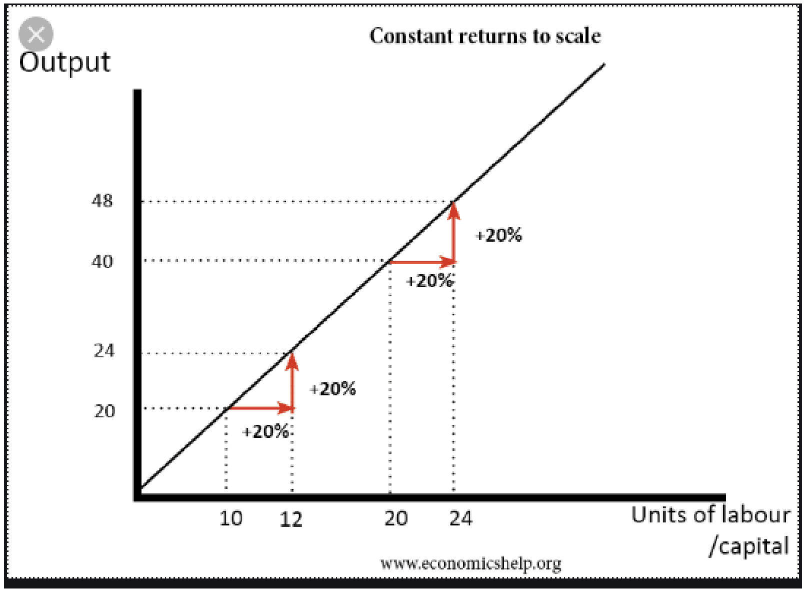
- As the firm moves along the isoquant from left to right, the slope increases. The firm substitutes labour for capital, but at a diminishing rate. When this occurs there is a ***diminishing*** marginal rate of technical substitution.

Returns to Scale

In general, the level of a firm's productivity changes as the quantity produced by the firm changes.

Returns to scale refers to the percentage change in output to a percentage change in inputs.





$$P_1 = f(xL, xK)$$

1. If P_1 increases in the same proportion as the increase in factors of production i.e., $\frac{P_1}{P} = x$, it will be constant returns to scale.

2. If P_1 increases less than proportionate increase in the factors of production *i.e.*, $\frac{P_1}{P} < x$, it will be diminishing returns to scale.

3. If P_1 increases more than proportionate increase in the factors of production, *i.e.*, $\frac{P_1}{P} > x$, it will be increasing returns to scale. Returns to scale can be shown with the help of table 8.

Table 8. Showing different stages of return to scale

Units of Labour	Units of capital	%age increase in Labour & Capital	Total Product	%age increase in TP	Returns to scale
1	3	–	10	–	
2	9	100%	30	200%	Increasing
3	9	50%	60	100%	
4	12	33%	80	33%	Constant
5	15	25%	100	25%	
6	18	20%	120	10%	Decreasing
7	21	16.6%	130	8.3%	

Increasing Returns to Scale

- Proportionate increase in all factor of production results in a more than proportionate increase in output.
- Increasing Returns => Output > Input
- Example :

Output		Input
100 Unit	=	3L + 3K
200 Unit	=	5L + 5K
300 Unit	=	6L + 6K

- Where L = labor and K=capital (in unit)

Returns to scale: Example 3

$$F(K,L) = K^2 + L^2$$

$$F(zK,zL) = (zK)^2 + (zL)^2$$

$$= z^2\left(K^2 + L^2\right)$$

$$= z^2 F(K,L)$$

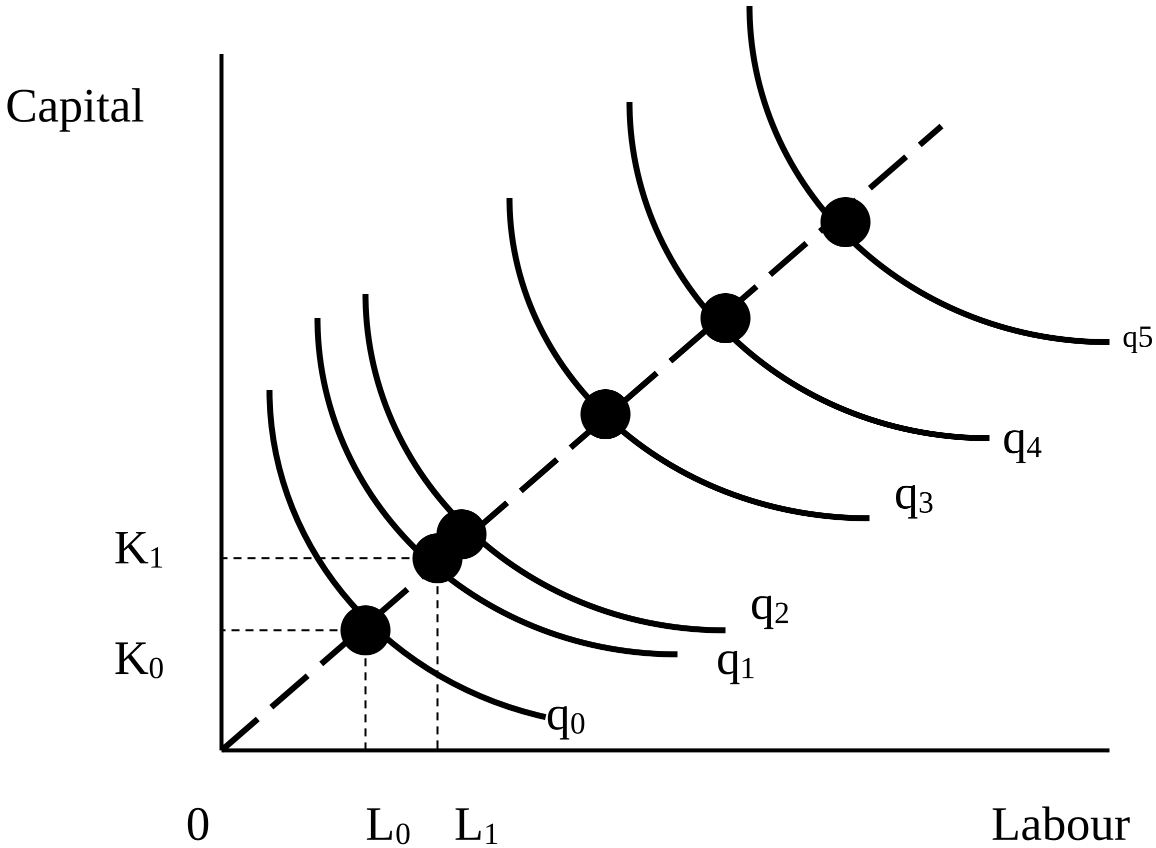
increasing returns to scale for any $z > 1$

Three Cases:

1) When the percentage increase in inputs is smaller than the percentage increase in output, there are **increasing** returns to scale.

2) When the percentage increase in inputs leads to the same percentage increase in output, there are **constant** returns to scale.

3) When the percentage increase in inputs is larger than the percentage increase in output, there are **decreasing** returns to scale.



The **returns to scale** can be measured along the ray from the origin.

Here the capital to labour ratio is given.

➢There is a relationship between returns to scale and the spacing of the isoquants.

When there are increasing returns to scale, the isoquants are bunched closer together.

When there are decreasing returns to scale, the isoquants are farther apart.

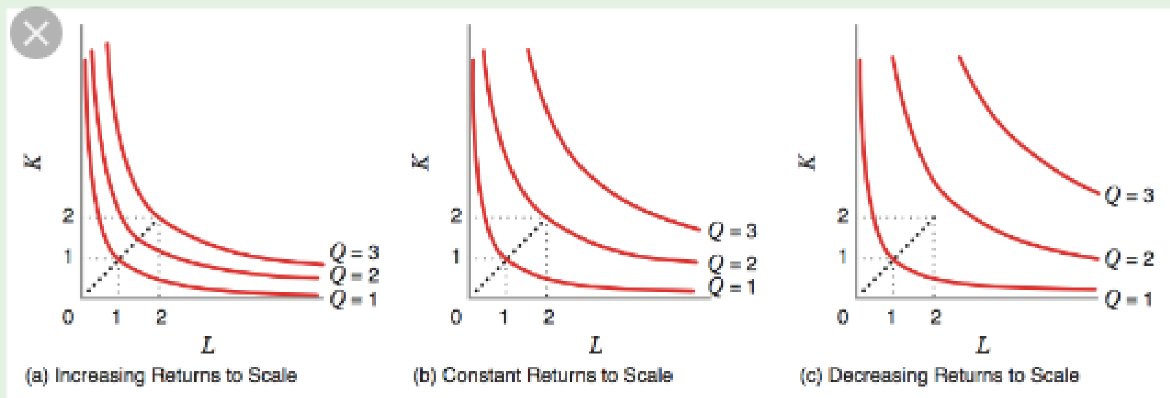
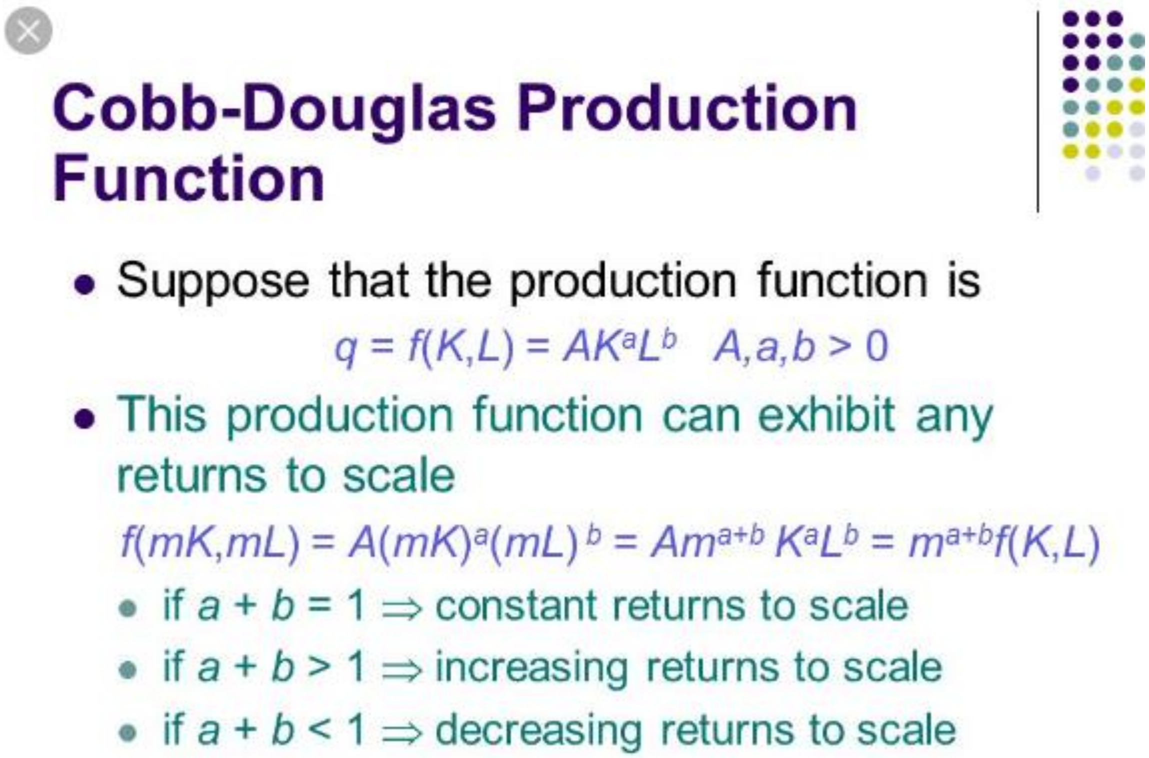


FIGURE 6.18 **Increasing, Constant, and Decreasing Returns to Scale**
In panel (a), doubling the quantities of capital and labor more than doubles output. In panel (b), doubling the quantities of capital and labor exactly doubles output. In panel (c), doubling the quantities of capital and labor less than doubles output.

Returns to Scale and the Cobb-Douglas Production Function



A common production function is the Cobb-Douglas production function.

Algebraically:

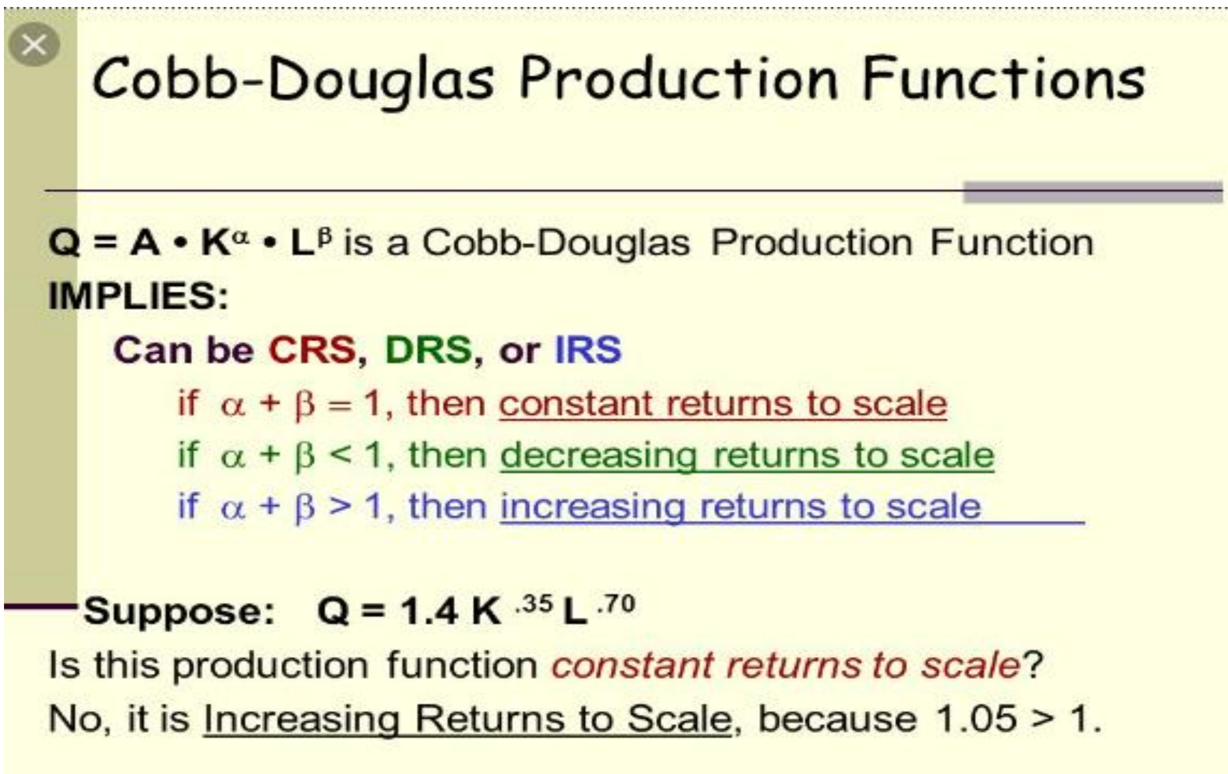
$$\mathbf{q=AL^{a}K^{b}}$$

where A, a and b are constant and greater than zero.

To determine the returns to scale for this function, we could change labour and capital by a factor ‘m’ and then determine if output changes by more than, equal to or less than ‘m’ times.

$$\mathbf{q=A\,(mL)^{a}(mK)^{b}}$$
$$\mathbf{q=Am^{a}L^{a}m^{b}\,K^{b}}$$
$$\mathbf{q=m^{a+b}[AL^{a}K^{b}]}$$

Since originally output was $q=AL^aK^b$, we can determine if output will increase by either less than m times if $a+b<1$ (because $m^{a+b}<m$), by exactly m times if $a+b=1$ or by more then m times if $a+b>1$.



The MRTS and MP of Both Factors of Production

The MRTS and marginal product of labour and capital are related.

Suppose the firm decides to increase the amount of capital it employs, holding the amount of labour constant. Output will increase by the amount Δq_K because of the increase in capital ΔK.

The increase in output is approximated by

$$\Delta q_K = MP_K \Delta K$$

If the firm holds capital constant and decreases the amount of labour it employs, output decreases by Δq_L, where the amount of labour decreases by ΔL.

The decrease in output is approximated by

$$\Delta q_L = MP_L \Delta L \; .$$

Along a given isoquant, output must be constant.

If the firm increases capital by ΔK, labour must decrease by an amount ΔL such that: $\Delta q_K + \Delta q_L = 0$ in order to remain on the same isoquant.

Substituting in these expressions, the condition becomes:

$$MP_K \Delta K + MP_L \Delta L = 0$$

Rearranging such that we have an expression for the MRTS in terms of the marginal products of the two factors:

$$MRTS_{KL} \equiv \frac{\Delta K}{\Delta L} = -\frac{MP_L}{MP_K}$$

The MRTS of K for L equals the negative of the ratio of the marginal product of L and K.