Economics 203: Intermediate Microeconomics

Calculus Review

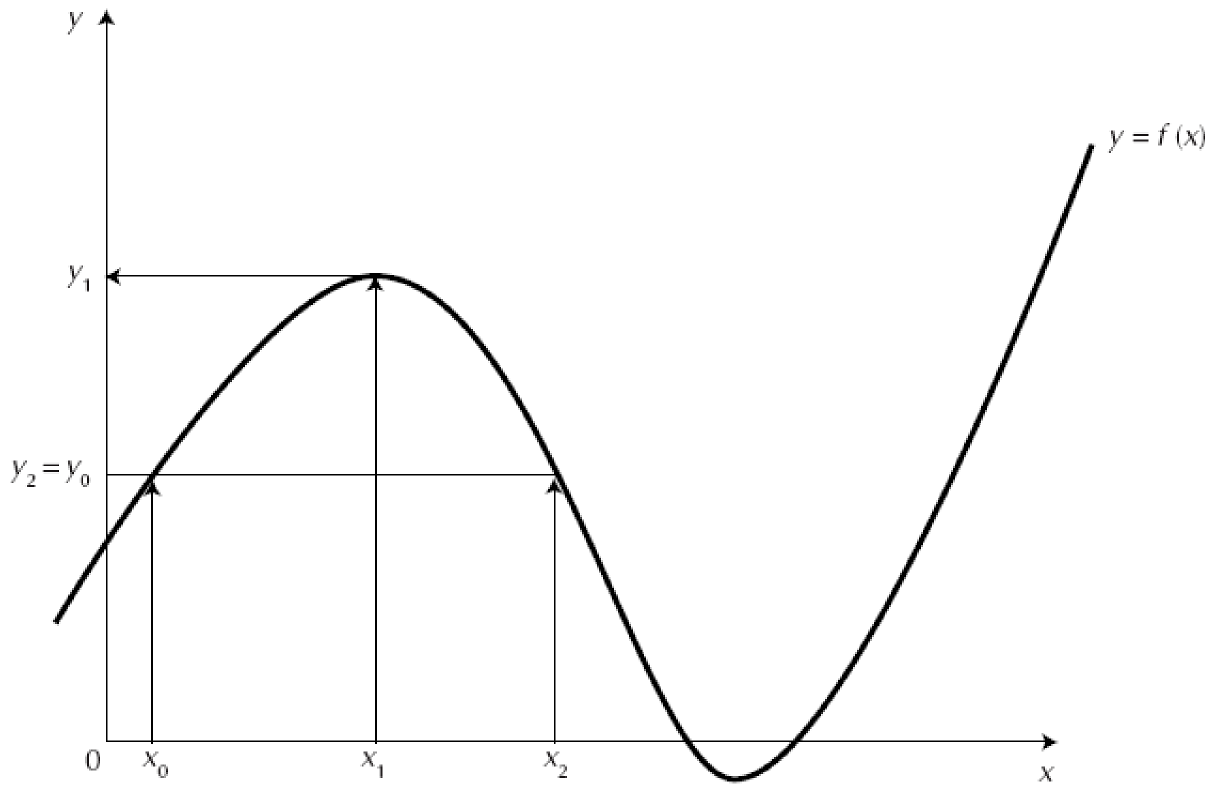
Functions, Graphs and Coordinates

Functions:

► A function f, is a rule assigning a value y for each value x.

The following illustration shows the graph of the function

$y = f(x)$.



If $x=x_0$, then we follow the arrow from x_0 up the graph of the function to get the value of y, specifically y_0, which is given by the height of the function at x_0.

Graphs and Coordinates:

► Typically, economists use 2-dimensional graphs to depict functional relationships.

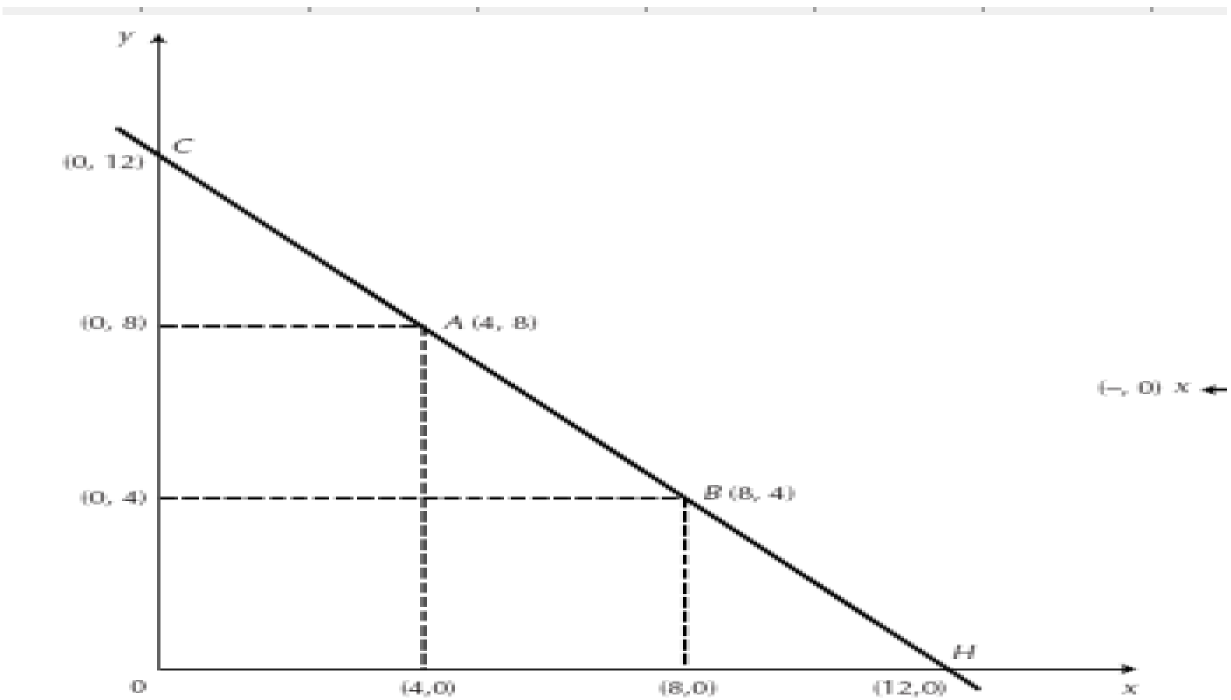
►Graphs gives us a visual format to easily grasp the information regarding a functional relationship.

However, graphs may be less precise than algebraic representation of functions, and may even be ***misleading****.*

Figure (a) illustrates some basics.

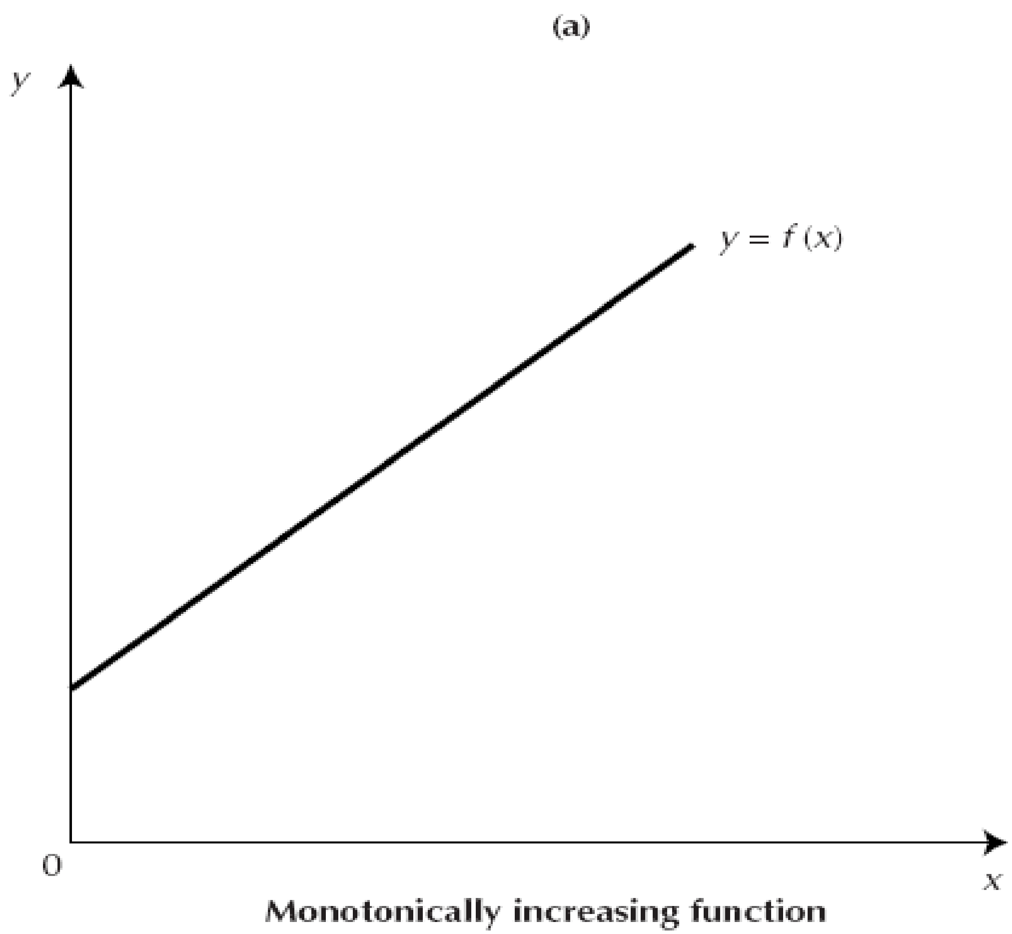
We designate the horizontal axis as the *x* axis and the vertical axis as the *y* axis.

Each point in the (x,y) plane is characterized by a pair of numbers(x_0,y_0) with x always coming first.

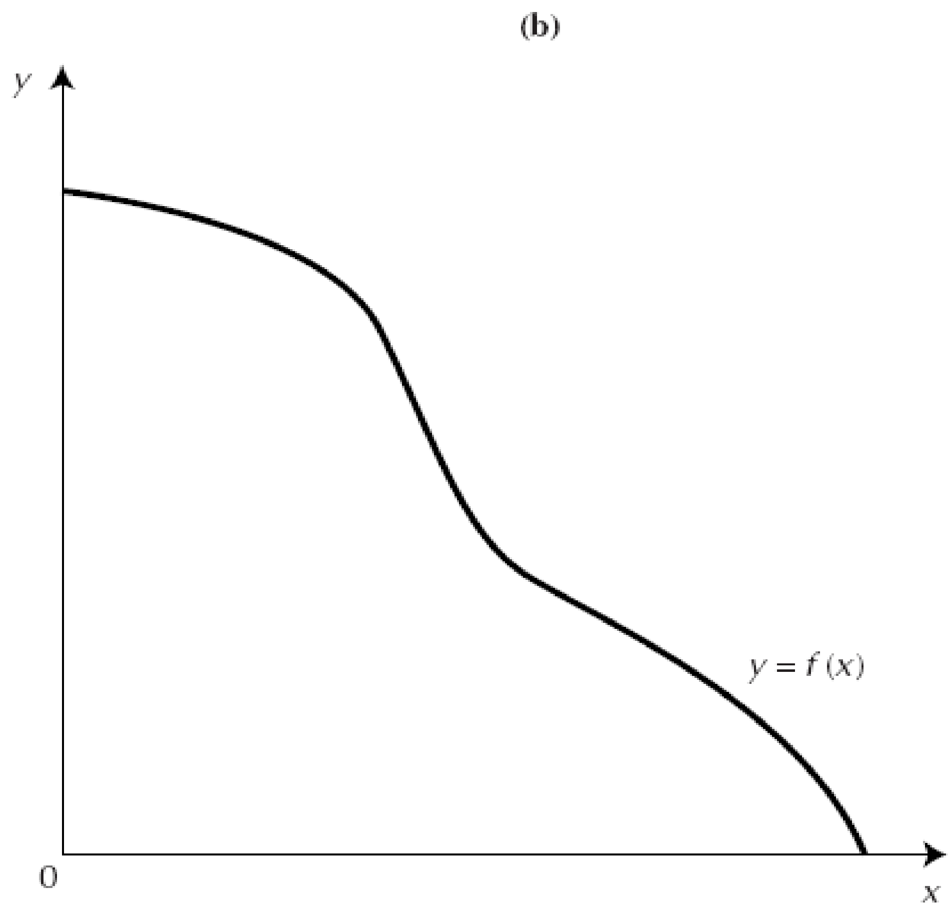


(a)

Types of Functions:



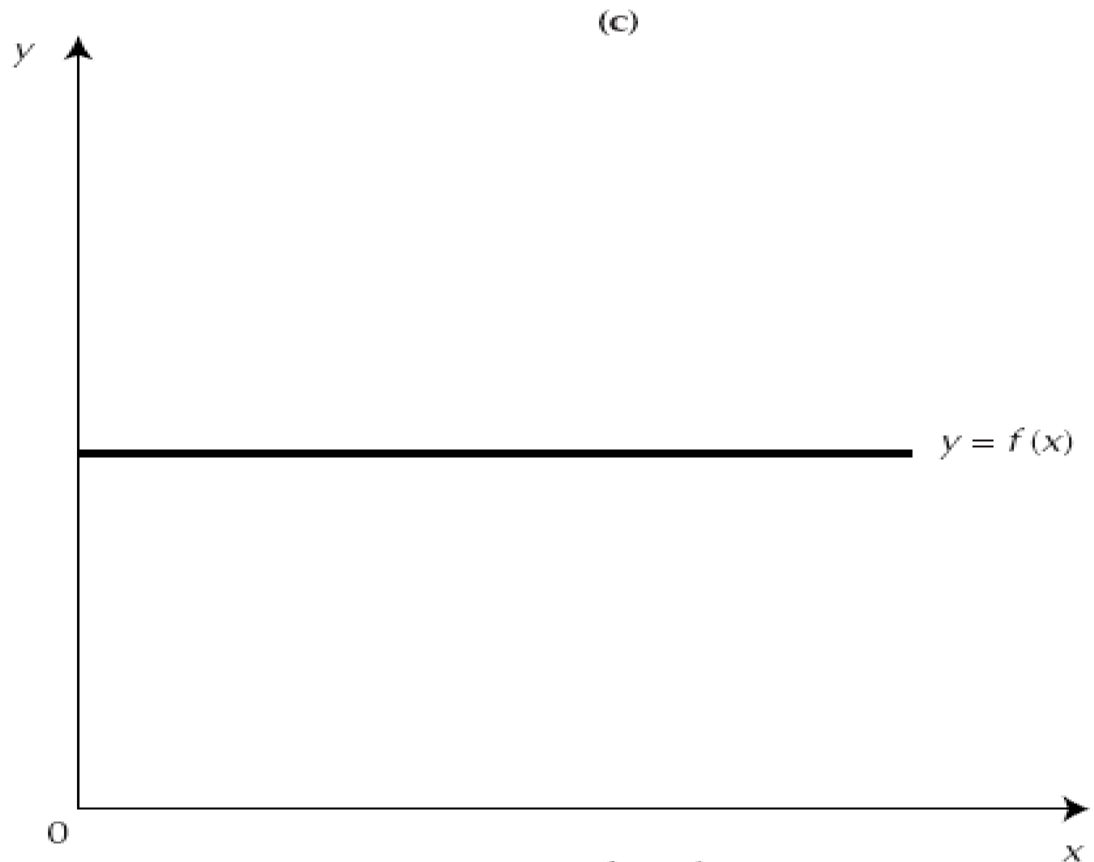
Function's characteristics: linear and positive slope throughout.



Monotonically decreasing function

Function's characteristics:

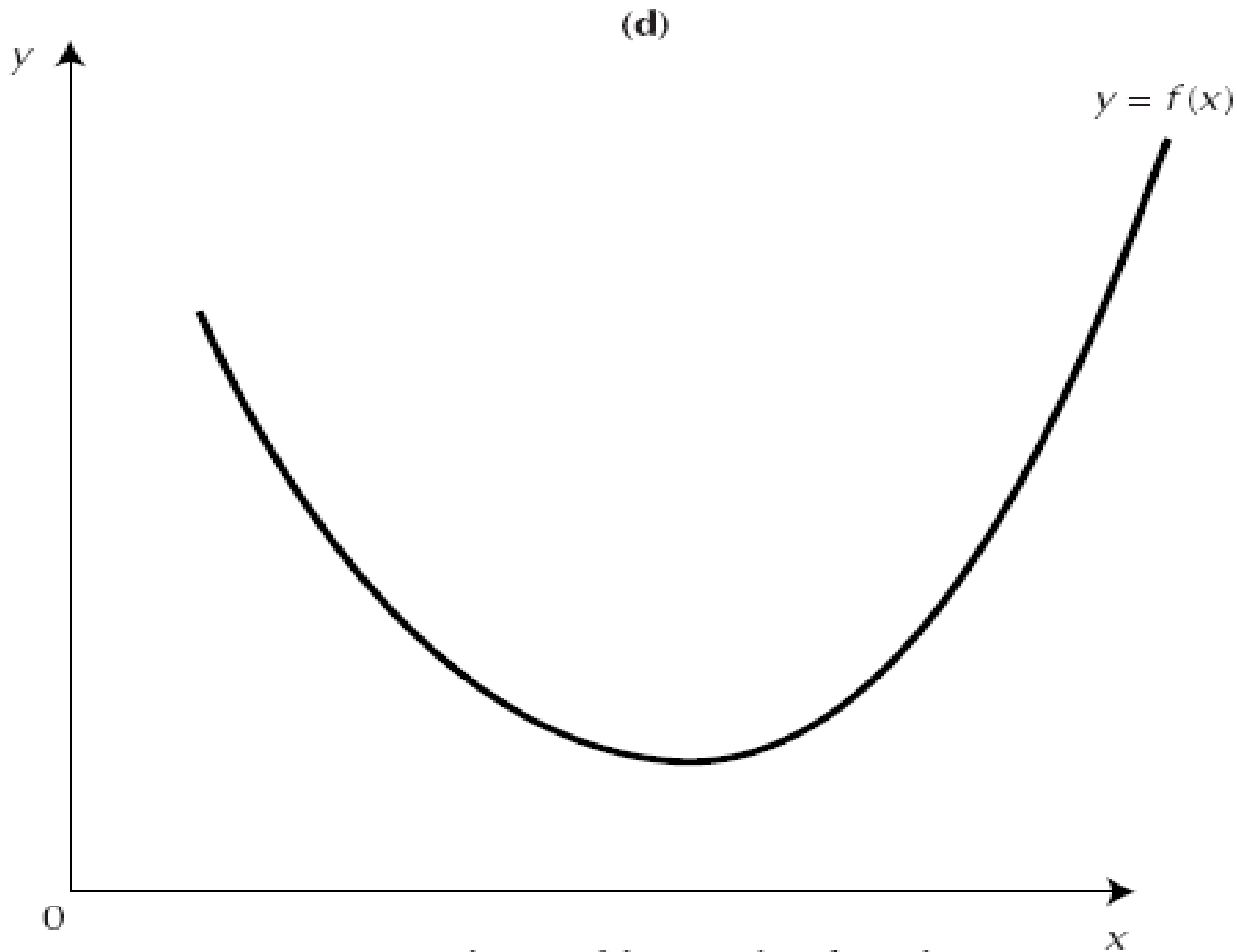
→ non-linear and decreasing throughout.



Constant function

<u>Functions' characteristics</u>: linear, non-decreasing, constant or horizontal.

Simultaneous non-increasing and non-decreasing.



Decreasing-and-increasing function

Function's characteristics: decreasing and increasing, non-linear function.

<u>Note:</u> Linear functions are represented in slope-intercept form:

$$y = a + bx,$$

where $'a'$ represents the vertical intercept and $'b'$ represents the slope of the function.

►Such parameters can take on any positive, negative or zero value.

►Nonlinear functions can take on a much wider range of forms.

For example:

$$\boxed{\begin{array}{l} y = x^3 \\ \\ y = \log(x) \quad \text{(logarithmic function)} \\ \\ y = e^x \quad \text{(exponential function)} \end{array}}$$

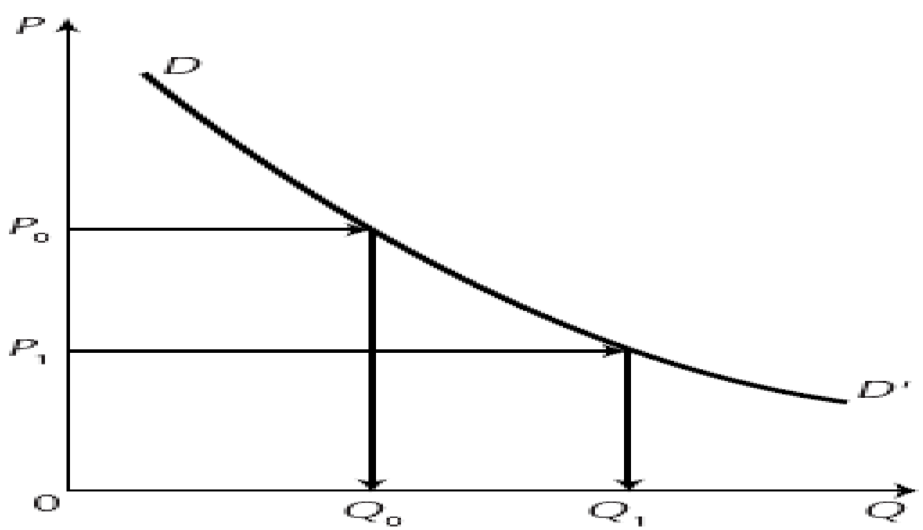
►In all of theses functions, the slope of the function changes as *x* changes.

Conventions:

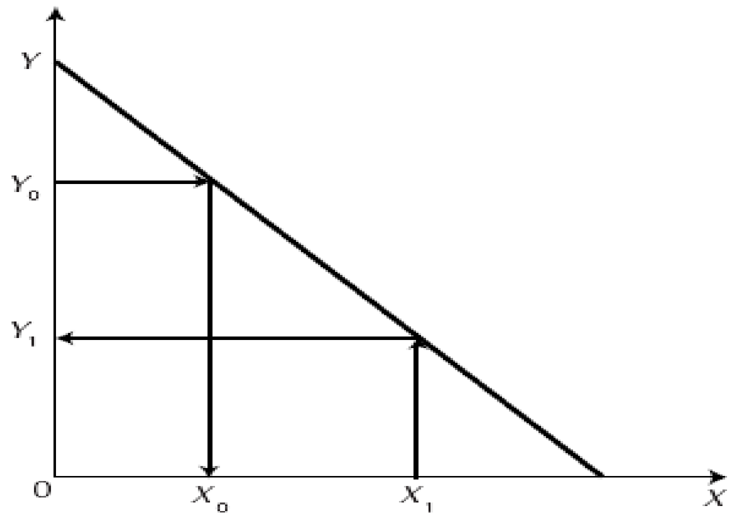
►Typically the horizontal axis of a diagram depicts the independent (x) variable and the vertical axis depicts the "dependent" (y) variable.

►However, it has become conventional to draw market demand and supply curves with the quantity on the horizontal axis and price of the vertical axis even though we are assuming that the quantity demanded of a good is a function of its price: $\boxed{\mathbf{Q_D = \mathit{f}(p).}}$

(b)



The ***budget constraint*** for an individual who has a budget of M dollars and spends it all on only two goods, X and Y, which have fixed prices, is illustrated next:



$$M = P_x X + P_y Y$$

or

$$Y = \frac{M}{P_y} - \left(\frac{P_x}{P_y}\right) X$$

The equation for the budget line can be written in a number of ways:

The intercept equals $\dfrac{M}{P_y}$; the slope equals $-\left(\dfrac{P_x}{P_y}\right)$.

Functions of Several Variables:

Many of the important functions we will study are functions of **several** variables. For example:

1) A utility function, representing a consumer's preferences can be represented by: $U = f(X_1, X_2, \ldots X_n)$,

   where U stands for the utility level received by consuming various combination of the *n* goods.

2) A production function relates combined quantities of inputs to the quantity of output each combination produces. For a two input model, the production function could be a function of capital and labour:

$$Q = f(L, K)$$

Linear Equations:

A straight line is determined by any two points on it, or by any point on it and its slope. If we define slope as ‘m’, between any two points on a line (x_1,y_1) and (x_2,y_2), the slope is derived by the equation

$$m = \frac{(y_2 - y_1)}{(x_2 - x_1)} = \frac{\Delta y}{\Delta x}.$$

►This is often referred as “rise over run.”

►The slope is a ratio of changes in two variables.

⇨If the slope is positive, an increase in one variable is associated with an increase in the other.

⇨If the slope is negative, an increase in one variable is associated with a decrease in the other.

⇨If the linear function is horizontal its slope is zero.

⇨If the linear function is vertical its slope is infinite or undefined.

Solving Linear Equation Systems:

■ The objective in solving any equation system, including the linear systems with which we are concerned here, is to find the set of values for all the variables that satisfies every equation in the system.

■ With nonlinear systems, it is possible to have multiple equilibria, but with linear systems, the solution will be unique.

Mathematical Refresher

Powers and Exponents:

1) $$\boxed{a^n = a \times a \times a \times a \times a \ldots \times a}$$

Example: $5^3 = 5 \times 5 \times 5 = 125$

2) $$\boxed{a^m \times a^n \times a^p = a^{m+n+p}}$$

Example: $5^3 \times 5^2 \times 5^7 = 5^{3+2+7} = 5^{12} = 244{,}140{,}625$

3) $$\boxed{a^m / a^n = a^{m-n}}$$

Example: $5^3 / 5^2 = 5^{3-2} = 5$

$5^7 / 5^2 = 5^{7-2} = 5^5$

4) $\boxed{(a^m)^n = a^{mn}}$ *Example*: $(5^3)^2 = 5^6$

5) $\boxed{(a^m)^{1/n} = a^{m/n}}$

Example: $(5^4)^{1/2} = 5^{4/2} = 5^2 = 25$

Logarithms and Exponential Functions

If $b^x = a$, then x is the logarithm of a to base b, that is, the power to which b must be raised to give a: $\log_b a = x$.

► The most common bases for logarithms are 10 and *e*.

Note: $\boxed{\begin{array}{l} \log_{10} 10 = 1 \\ \log_{10} 100 = 2 \\ \log_{10} 1 = 0 \\ \log_{10} 0.1 = -1 \end{array}}$

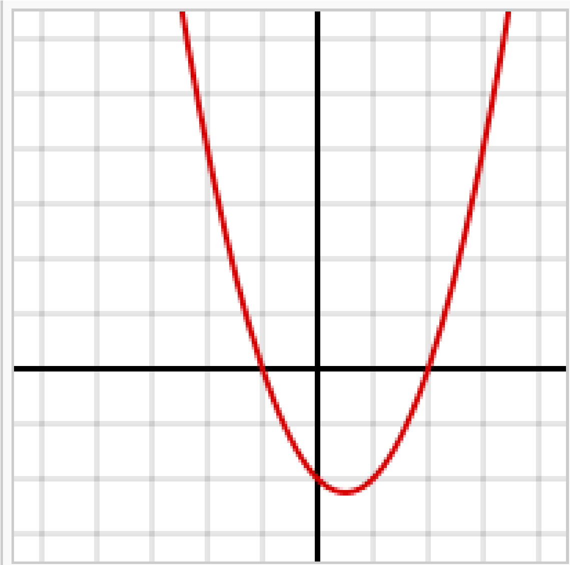
Roots of Quadratic Equations:

The roots of a quadratic equation in the form:

$$ax^2 + bx + c = 0,$$

may be calculated using the following formula:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$



For the quadratic function: $f(x) = x^2 - x - 2 = (x + 1)(x - 2)$ of a real variable x, the x-coordinates of the points where the graph intersects the x-axis, $x = -1$ and $x = 2$, are the solutions of the quadratic equation: $x^2 - x - 2 = 0$.

Derivatives

Calculus, in a two-variable case, $y = f(x)$, is basically just a method for calculating slopes or rates of change of function and areas between the function and the horizontal axis.

Partial differentiation enables us to trace the effect of a change in a single independent variable on the dependent variable, given fixed values of all the other independent variables.

If $y = f(x)$, then we can write the slope of the function several different ways:

$$y', \quad f'(x), \quad {}^{df(x)}\!/_{dx}, \quad or \quad {}^{dy}\!/_{dx}.$$

Basic Formulas for Derivatives:

	Function: $y = f(x)$	**Function: Example**	**Derivative:** $dy/dx = y' = f'(x)$	**Derivative: Example**
1.	$y = a$	$y = 4$	$y' = 0$	$y' = 0$
2.	$y = ax$	$y = 12x$	$y' = a$	$y' = 12$
3.	$y = ax^n$	$y = 4x^3$ $y = 3x^4$	$y' = nax^{n-1}$	$y' = 12x^2$ $y' = 12x^3$
4.	$y = f(x) + g(x)$	$y = 4x^3 + 3x$	$y' = f'(x) + g'(x)$	$y' = 12x^2 + 3$
5.	$y = f(x) \times g(x)$	$y = (3x^2)(2+x)$ $= 6x^2 + 3x^3$	$y' = f(x)g'(x) +$ $g(x)f'(x)\ y'$	$y' = (3x^2)(1) +$ $(2+x)(6x)$ $=12x + 9x^2$
6.	$y = \frac{f(x)}{g(x)}$	$y = \frac{(1+2x)}{(x+4)}$	$y' = [g(x)f'(x) -$ $f(x)g'(x)] / [g(x)]^2$	$y' = \frac{7}{(x+4)^2}$
7.	$y = f(u(x))$	$y = (3x+4)^2$, with $u = 3x + 4$, $y = u^2$	$y' = \frac{dy}{du} \cdot \frac{du}{dx}$	$y' = 2(3x+4)(3)$ $= 18x + 24$
8.	$y = ln\ x$	$y = ln\ x$	$y' = 1/x$	$y' = 1/x$
9.	$y = e^x$	$y = e^x$	$y' = e^x$	$y' = e^x$

Derivative of Functions:

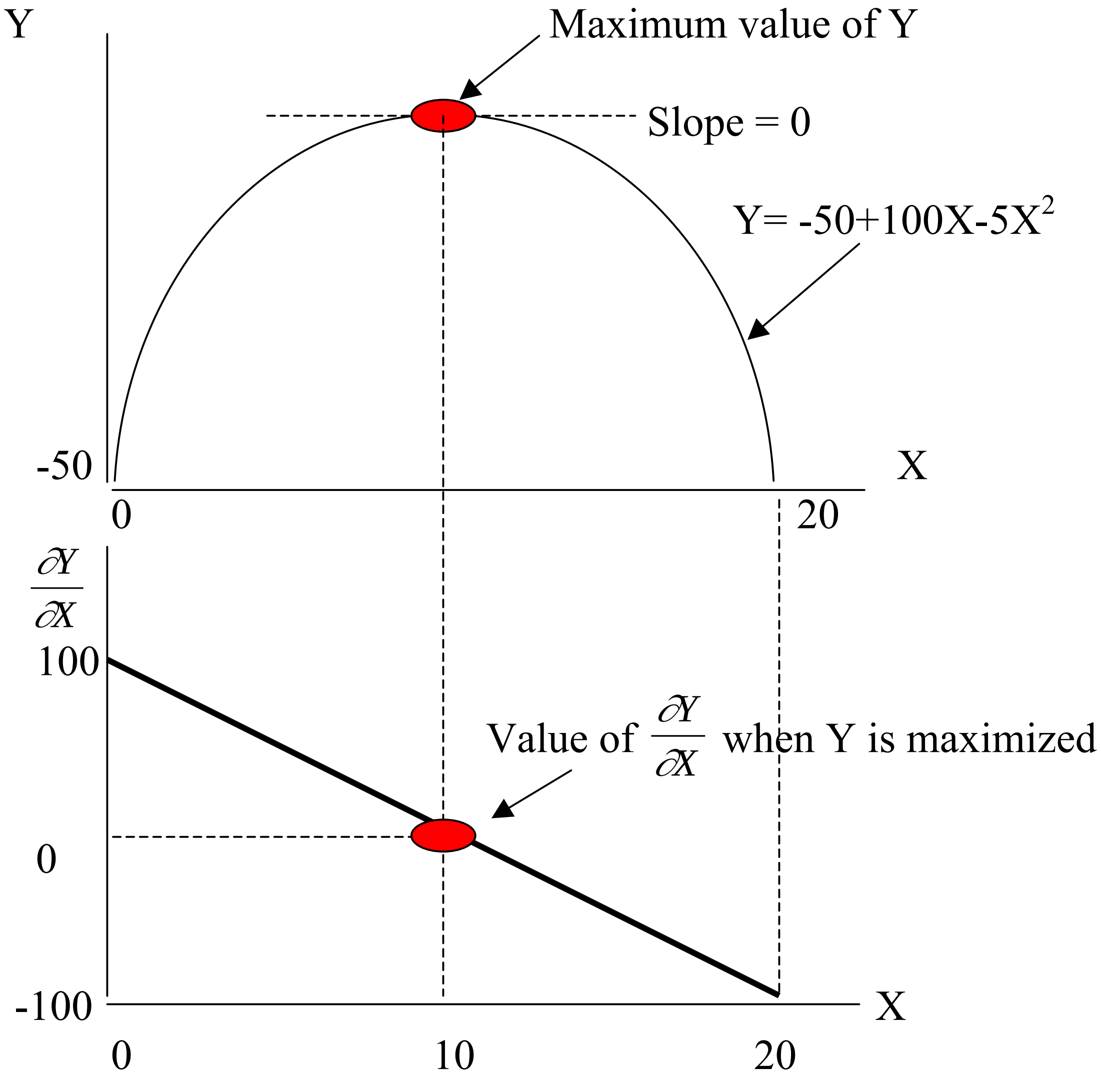
		(a)	(b)
Function	$y = f(x)$	$y = x^2 - 4x + 10$	$y = 2 + 4x - x^2$
First derivative	$y' = f'(x) = \frac{dy}{dx}$	$y' = 2x - 4$	$y' = 4 - 2x$
Second derivative	$y'' = f''(x) = \frac{d^2y}{dx^2}$	$y'' = 2 > 0$	$y'' = -2 < 0$

Often tables illustrating the relationship between the firm's output and profit are too complicated or inaccurate to be used to find the profit-maximizing output level of a firm.

Alternatively, an equation is used to represent the relationship between the variable we are trying to maximize (i.e. profit) and the variables under the control of the decision marker (i.e. output).

<u>Differential calculus</u> can be employed to find the ***optimal solutions*** to the decision maker's problem.

<u>Using Derivatives to Solve Maximization and Minimization Problems</u>



A maximum or minimum point can occur only if the slope of the curve showing Y on the vertical axis and X on the horizontal axis equals zero.

Suppose that Y equals the profit of a company and X is its output level.

The relationship is illustrated by the curve in the upper diagram.

The maximum value of Y occurs when X= 10, and at this value of X the slope of the curve equals zero.

If the relationship between Y and X is $Y = -50 + 100X - 5X^2$ then $\frac{\partial Y}{\partial X} = 100 - 10X$.

If this derivative equals zero,

$$100-10X=0$$
$$X=10.$$

This is the value of X where Y is maximized.

So again, to find the value of X that maximizes or minimizes Y, we must find the value of X where this **derivative equals zero**. The lower graph shows that this derivative equals zero when Y is maximized.

However, the fact that a derivative is zero does not distinguish between a point on the curve where Y is maximized and a point where Y is minimized.

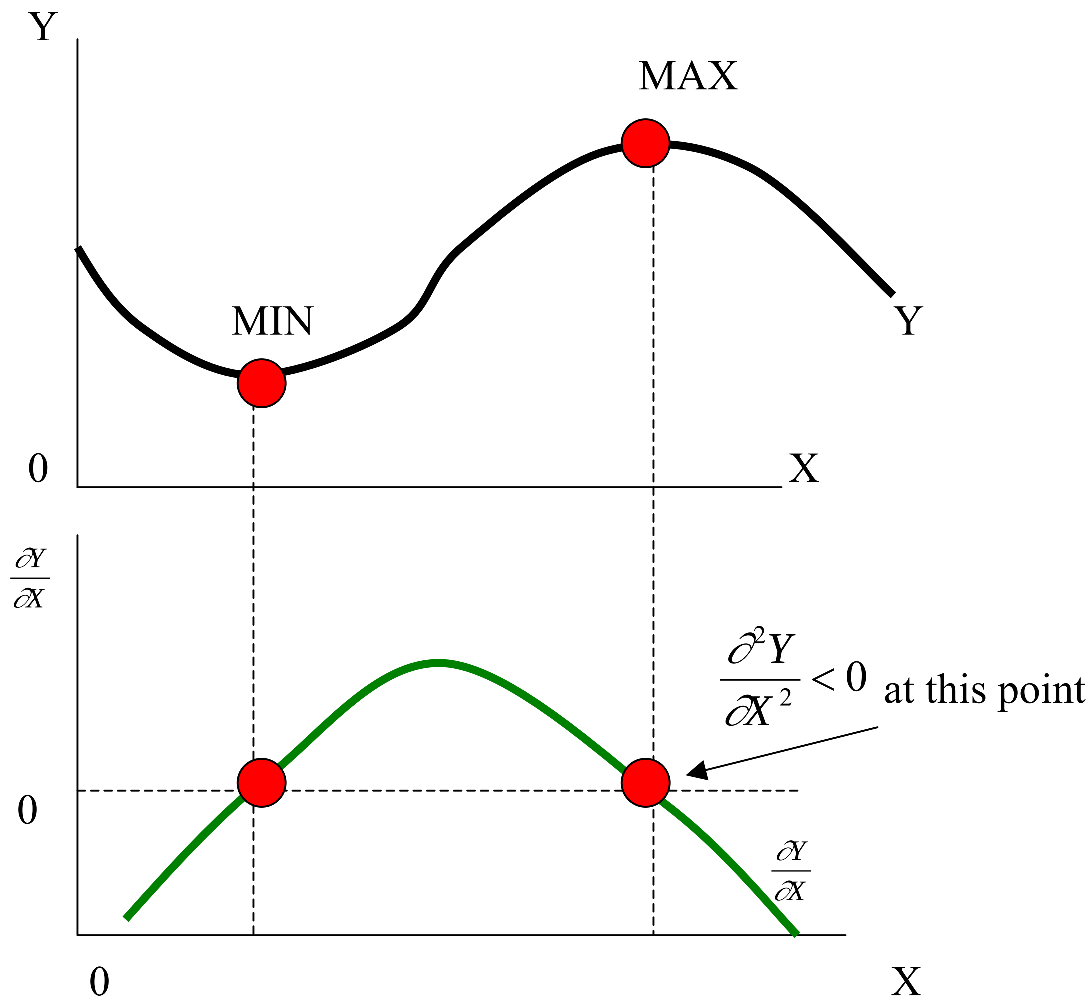
To distinguish between a maximum and minimum one must find the **second derivative** of Y with respect to X. The second derivative measures the slope of the curve showing the relationship between the first derivative and X.

The second derivative measures the slope of the first derivative curve.

The second derivative is always negative at a point of **maximization** and always positive at a point of **minimization.**

So, to distinguish between maximization and minimization points, we must simply determine the **sign** of the second derivative at each point.

To more fully comprehend why the second derivative is always negative at a maximization point and always positive at a minimization point, consider the following illustration:



When the second derivative is negative, this means that the slope of the $\frac{\partial Y}{\partial X}$ curve in the lower panel is negative.

►Since $\frac{\partial Y}{\partial X}$ equals the slope of the Y curve in the upper panel, this in turn means that the slope of the Y curve decreases as X increases.

At a maximum point, this must be the case.

►When the second derivative is positive, this means that the slope of the $\frac{\partial Y}{\partial X}$ curve is positive, which is another way of saying that the slope of the Y curve increases as X increases.

At a minimum point, this must be the case.

Example: Suppose the relationship between profit and output at the Trisping Corporation is:

$$Y = -1 + 9X - 6X^2 + X^3$$

where Y equals annual profit (in millions of dollars), and X equals annual output (in millions of units).

Due to a capacity limitation, the firm cannot produce more than 3 million units per year.

To find the values of output that maximize or minimize profit, we must determine the derivative of Y with respect to X and set it to equal zero:

$$\frac{\partial Y}{\partial X} = 9 - 12X + 3X^2 = 0$$

Solving this equation for X, we find two values of X that result in the derivative being equal to zero:

If an equation is of the general quadratic form:

$Y = aX^2 + bX + c,$

the values of X where Y is 0 are:

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the equation in the example, $a = 3,\ b = -12,$ and $c = 9.$ Hence,

$$X = \frac{-(-12) \pm \sqrt{144 - 108}}{6} = 2 \pm 1$$

Therefore, $Y = 0$ when X equals 1 or 3.

To determine whether each of these two output levels maximizes or minimizes profit, we find the value of the **second derivative** at these two values of X.

Taking the derivative of $\frac{\partial Y}{\partial X}$ we find that:

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6X$$

If **<u>X =1:</u>**

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6(1) = -6$$

Hence, since the second derivative is negative**,** profit is at a maximum when output equals one million units.

If **<u>X=3:</u>**

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6(3) = 6$$

Hence, since the second derivative is positive**,** profit is a minimum when output equals 3 million units.

Partial Derivative Practice:

Question 1: For the Bridges-to-Shores Corporation**, the relationship between output (Q) and the number of hours of specialized manual labour (S) and machine-operated labour (M) is:**

$$\mathbf{Q=650S+145M-0.45\,S^2-0.65M^2}$$

Determine the marginal product of S and M:

$$MP_S=\frac{\partial Q}{\partial S}=650-0.9S$$

{Partial derivatives

$$MP_M=\frac{\partial Q}{\partial M}=145-1.3M$$