Optimization Techniques

Managerial economics is concerned with the ways in which managers should make decisions in order to maximize the effectiveness or performance of the organization they manage.

To understand how this can be done, you must understand basic **optimization techniques**.

►Marginal analysis
►Differential calculus

Functional Relationships

How can economic relationships be expressed?

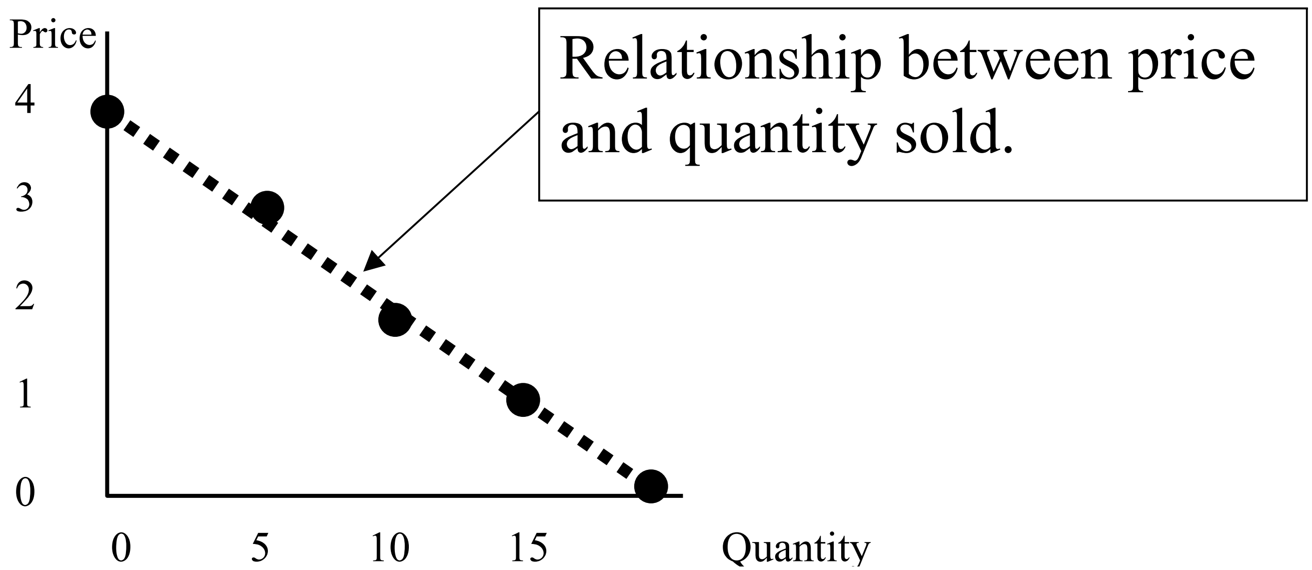
Three Ways:

►**Table**

Relationship Between the Price and Quantity Sold

$ Price per Unit	Number of Units Sold per Day
1	**15**
2	**10**
3	**5**
4	**0**

▸ **<u>Graph:</u>**


Price
4
3
2
1
0
0
5
10
15
Quantity
Relationship between price and quantity sold.

►**Equations**

The relationship between the number of units sold and the price from the table can be expressed in the following functional form:

$$\mathbf{Q}=f(\mathbf{P})$$

where Q is the number of units sold and P is the price.

The number of units sold is a function of price, which means that the number of units sold ***depends*** on price.

A more specific representation of this relationship is

$$\mathbf{Q=20-5P}$$

If your compare this equation with the data in the table you can verify that these data conform to this equation.

If price equals 2, the number of units sold should be:

$$20-5(2)=10$$

Marginal Analysis

Marginal analysis enables managers to use economic relationships more effectively.

The marginal value of a dependent variable is defined as the change in the **dependent variable** associated with a one unit change in a particular **independent variable**.

Example: Consider the next table, which illustrates the total profit of a company if the number of units produced equal various amounts. In this case, total profit is the dependent variable and output is the independent variable.

The marginal value of profit, referred to as **marginal profit** , is the change in total profit due to a one unit change in output.

The first two columns illustrate the total profit of a company if the number of units produced equals various amounts.

Total profit is the dependent variable and output is the independent variable.

The marginal value of profit is in the third column (, the change in total profit associated with a one unit change in output).

With a marginal relationship, the dependent variable, total profit, is maximized when its marginal value **shifts** from positive to negative.

As long as the value of marginal profit is positive, the company can raise its total profit by increasing output.

When marginal profit shifts from positive to negative, total profit will fall with any further increase in output.

Units of output per day (Q) (1)	Total profit (2)	Marginal profit\ (3)	Average profit (4)=(2)/(1)
0	**0**		
		10	
1	**10**		**10**
		15	
2	**25**		**12.5**
		35	
3	**60**		**20**
		40	
4	**100**		**25**
		35	
5	**135**		**27**
		15	
6	**150**		**25**
		5	
7	**155**		**22.1**
		-5	
8	**150**		**18.2**

(Independent) (Dependent)

Since managers strive to maximize profit, this is a very important concept.

It highlights the importance of marginal analysis and the danger of using average values instead.
Average profit is the total profit divided by output.

It is illustrated in column 4.

Although it would seem reasonable to produce an output where average profit is highest, this would be an error. If managers wish to maximize profit, they must choose the output where marginal profit shifts from positive to negative.

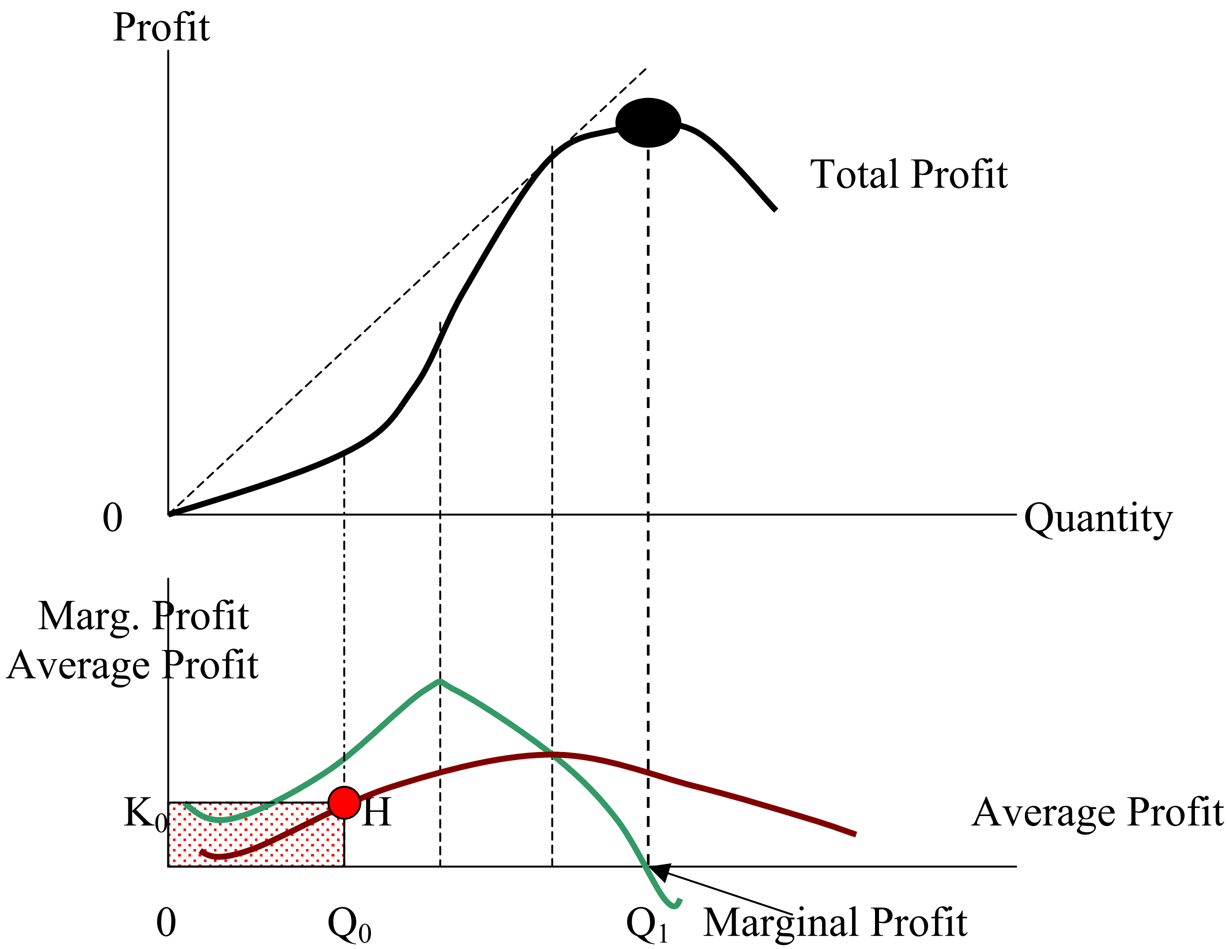
It is important to understand the relationship between average and marginal values.

Since the marginal value represents the change in the total, the average value must increase if the marginal value is greater than the average value.

The average value must decrease if the marginal value is less than the average value.

Relationships Between Total, Marginal and Average Values

The following figure illustrates the relationships between total, average, and marginal profit:



Let the symbol Π represent profit, and Q represent quantity produced.

The figure contains two panels.

► The upper panel shows the relationship between total profit and output.

► The lower panel shows the relationship between average profit and marginal profit as output increases.

At a particular output, average profit equals the slope of the straight line from the origin to the point on the total profit curve corresponding to the output, or $\frac{\Pi}{Q_i}$, (total profit divided by quantity).

To determine the relationship between average profit and output from the relationship between total profit and output, we repeat this procedure for each level of output.

The resulting average profit curve is shown in the second panel.

How about the relationship between marginal profit and output?

This can be derived from the relationship between total profit and output in the first panel.

At any particular output, marginal profit equals the slope of the tangent to the total profit curve in the first panel, at the point of a particular output. That is, marginal profit equals the slope of the line which is tangent to the total profit curve.

To determine the relationship between marginal profit and output from the relationship between total profit and output, we repeat this procedure for each level of output. The resulting marginal profit curve is shown in the second panel.

Note: Total profit = average profit times output.
If output equals Q_0, total profit equals K_0 times Q_0 (the shaded rectangle $0K_0HQ_0$ in the second panel.)

The average profit curve must be rising if it is below the marginal profit curve.

The average profit curve must be falling if it above the marginal profit curve.

Derivatives

Often tables illustrating the relationship between the firm's output and profit are too complicated or inaccurate to be used to find the profit-maximizing output level of a firm.

Alternatively, an equation is used to represent the relationship between the variable we are trying to maximize (i.e. profit) and the variables under the control of the decision marker (i.e. output).

Differential calculus can be employed to find the optimal solutions to the decision maker's problem.

We previously defined the marginal value as the change in the dependent variable resulting from a one unit change in an independent variable.

If we let Y be the dependent variable and X be the independent variable, then Y = *f* (X).

Let Δ (delta) to denote change.

Hence, the marginal value of Y can be estimated by:

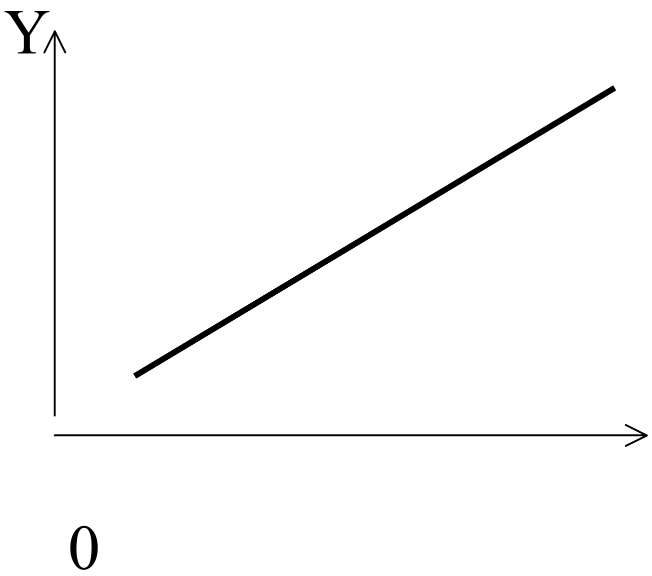
$$\frac{Change\ in\ Y}{Change\ in\ X} = \frac{\Delta Y}{\Delta X}$$

Example:

If a 4 unit increase in X results in a 1 unit increase in Y, Δ X=4 and Δ Y=1; The marginal value of Y is about ¼.

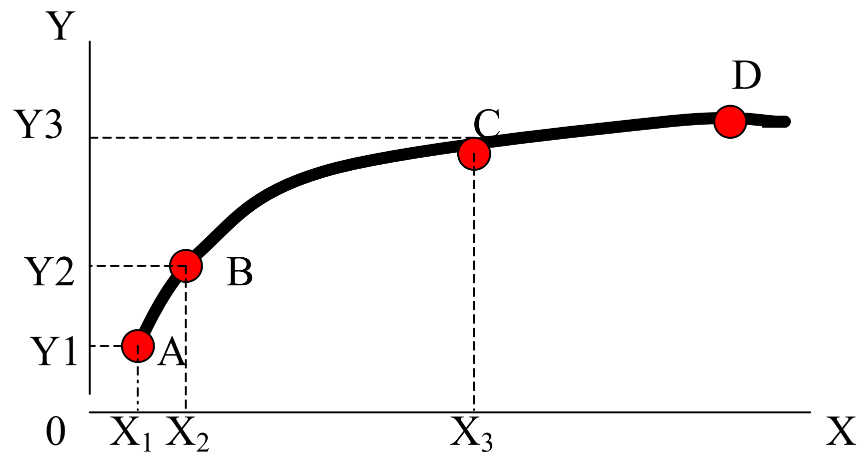
Unless the relationship between Y and X is linear, the value of $\frac{\Delta Y}{\Delta X}$ is **not** constant.

For example: **Constant slope**



Linear relationship between X and Y

Example: **Non Linear Relationship**



The value of $\frac{\Delta Y}{\Delta X}$ is related to the steepness or flatness of the curve.

If the curve is relatively steep, this indicates a small change in X results in a large change in Y. Hence, $\frac{\Delta Y}{\Delta X}$ is relatively large.

*Points between A and B $\longrightarrow$ larger derivative

If the curve is relatively flat, this indicates that a large change in X results in a small change in Y, and $\frac{\Delta Y}{\Delta X}$ is relatively small.

The derivative of Y with respect to X is defined as the ***limit of*** $\frac{\Delta Y}{\Delta X}$ ***as*** ΔX ***approaches zero.***

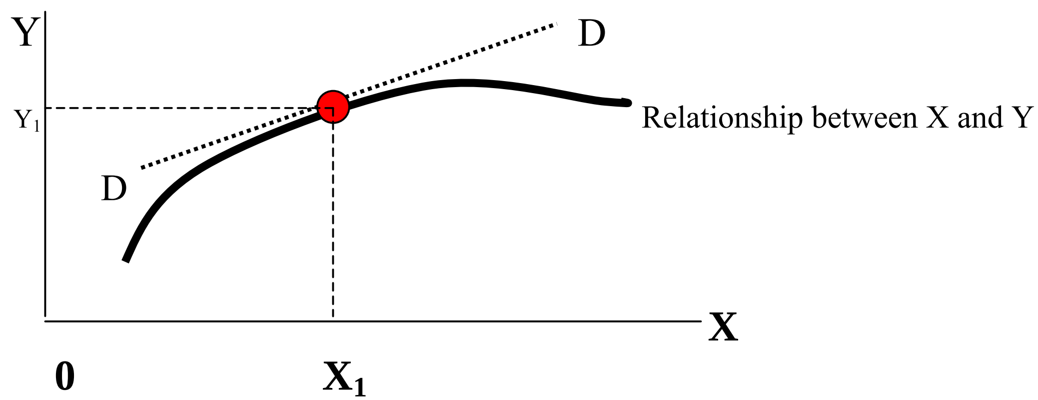
Since the derivative of Y with respect to X is denoted by $\frac{\partial Y}{\partial X}$, this definition can be restated as:

$$\frac{\partial Y}{\partial X} = \underset{\Delta X \to 0}{\text{limit}} \frac{\Delta Y}{\Delta X}$$

which is read "the derivative of Y with respect to X equals the limit of the ratio $\frac{\Delta Y}{\Delta X}$ as ΔX approaches zero."

Graphically, the derivative of Y with respect to X equals the slope of the curve showing Y on the vertical axis as a function of X, on the horizontal axis.

In the limit, as ΔX approaches zero, the ratio $\frac{\Delta Y}{\Delta X}$ is equal to the slope of the line DD, which is drawn tangent to the curve at point A.



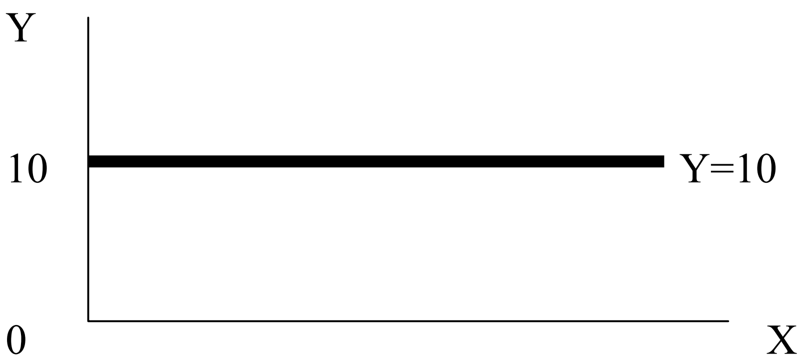
How To Find A Derivative

1) Derivatives of Constants:

If the dependent variable Y is a constant, its derivative with respect to X is always zero.

$$\frac{\partial Y}{\partial X} = 0$$

Example: Suppose Y=10. ← function



Since the value of Y does not change as X varies, $\frac{\partial Y}{\partial X}$ must be equal to zero.

Also, recall that $\frac{\Delta Y}{\Delta X}$ equals the slope of the curve showing Y as a function of X.

This slope equals zero; meaning, that $\frac{\partial Y}{\partial X}$ must equal zero.

2) **<u>Derivatives of Power Functions:</u>**

A power function can be expressed as $Y = aX^b$, where a and b are constants.

If the relationship between X and Y is of this kind, the derivative of Y with respect to X equals ‘b’ times ‘a’ multiplied by X raised to the (b-1) power:

$$\frac{\partial Y}{\partial X} = baX^{b-1}$$

<u>Example</u>: $Y=4X^3$

$$\frac{\partial Y}{\partial X} = 4 \times 3X^{3-1} = 12X^2$$

<u>Example</u>: $Y=X^2$

$$\frac{\partial Y}{\partial X} = 2X$$

<u>Example</u>: $Y=X^{-4}$

$$\frac{\partial Y}{\partial X} = -4 \times 1X^{-4-1} = -4X^{-5} = -\frac{4}{X^5}$$

3) Derivatives of Sums and Differences

Suppose that U and W are two variables which depend on X.

U=*g*(X) and W=*h*(X)

The functional relationship between U and X is denoted by g, and that between W and X is denoted by h.

Suppose that Y=U+W

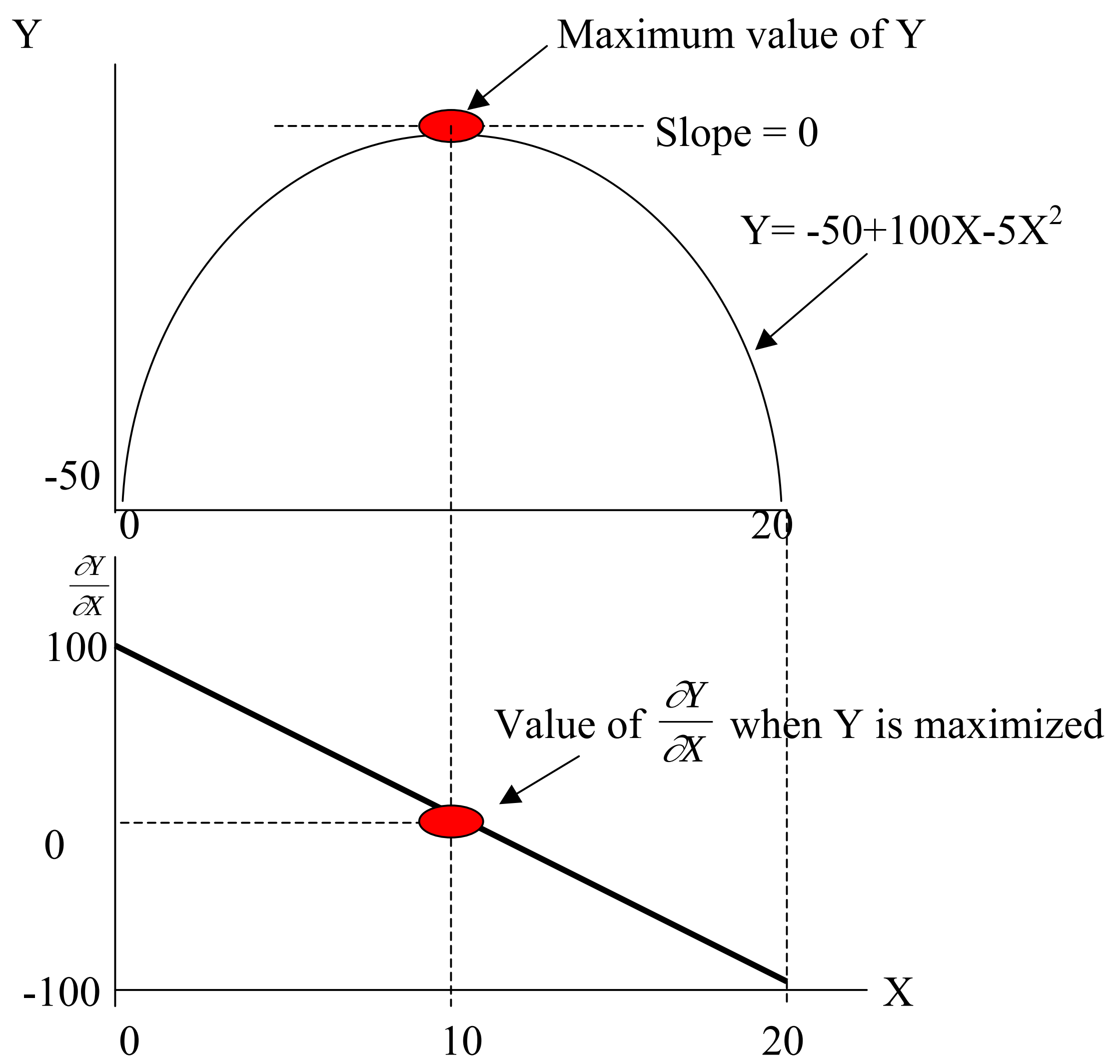
That is, Y is the sum of U and W. If so, the derivative of Y with respect to X is equal to the sum of the derivatives of the individual terms: $\frac{\partial Y}{\partial X} = \frac{\partial U}{\partial X} + \frac{\partial W}{\partial X}$.

Example: Consider the case in which $U=g(X)=3X^3$ and $W=h(X)=4X^2$.

If $Y=U+W=3X^3+4X^2$

$$\frac{\partial Y}{\partial X} = 9X^2 + 8X$$

Using Derivatives to Solve Maximization and Minimization Problems



A maximum or minimum point can occur only if the slope of the curve showing Y on the vertical axis and X on the horizontal axis equals zero.

Suppose that Y equals the profit of a company and X is its output level.

The relationship is illustrated by the curve in the upper diagram.

The maximum value of Y occurs when X= 10, and at this value of X the slope of the curve equals zero.

If the relationship between Y and X is $Y = -50 + 100X - 5X^2$

then $\frac{\partial Y}{\partial X} = 100 - 10X$

If this derivative equals zero,

$$100-10X=0$$
$$X=10.$$

This is the value of X where Y is maximized.

So again, to find the value of X that maximizes or minimizes Y, we must find the value of X where this **derivative equals zero**. The lower graph shows that this derivative equals zero when Y is maximized.

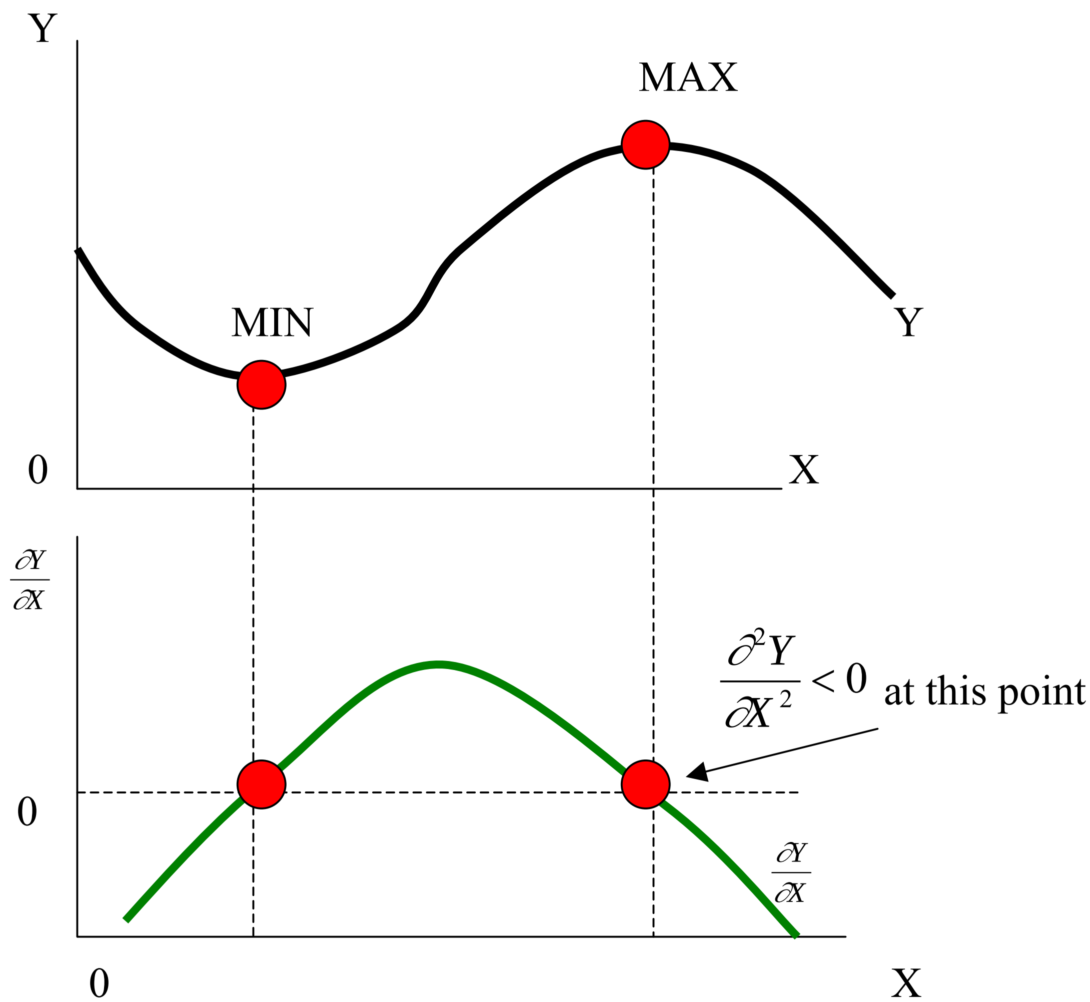
However, the fact that a derivative is zero does not distinguish between a point on the curve where Y is maximized and a point where Y is minimized.

To distinguish between a maximum and minimum one must find the **second derivative** of Y with respect to X. The second derivative measures the slope of the curve showing the relationship between the first derivative and X.

The second derivative measures the slope of the first derivative curve.

The second derivative is always negative at a point of **maximization** and always positive at a point of **minimization.**

So, to distinguish between maximization and minimization points, we must simply determine the **sign** of the second derivative at each point.
To more fully comprehend why the second derivative is always negative at a maximization point and always positive at a minimization point, consider the following illustration:


Y
MAX
MIN
Y
0
X
$\frac{\partial Y}{\partial X}$
$\frac{\partial^2 Y}{\partial X^2} < 0$ at this point
0
$\frac{\partial Y}{\partial X}$
0
X

When the second derivative is negative, this means that the slope of the $\frac{\partial Y}{\partial X}$ curve in the lower panel is negative.

Since $\frac{\partial Y}{\partial X}$ equals the slope of the Y curve in the upper panel,

this in turn means that the slope of the Y curve decreases as X increases .

At a maximum point, this must be the case.

When the second derivative is positive, this means that the slope of the $\frac{\partial Y}{\partial X}$ curve is positive, which is another way of

saying that the slope of the Y curve increases as X increases.

- At a minimum point, this must be the case.

<u>Example</u>: Suppose the relationship between profit and output at the Trisping Corporation is:

$$\mathbf{Y = -1 + 9X - 6X^2 + X^3}$$

where Y equals annual profit (in millions of dollars), and X equals annual output (in millions of units).

Due to a capacity limitation, the firm cannot produce more than 3 million units per year.

To find the values of output that maximize or minimize profit, we must determine the derivative of Y with respect to X and set it to equal zero:

$$\frac{\partial Y}{\partial X} = 9 - 12X + 3X^2 = 0$$

Solving this equation for X, we find two values of X that result in the derivative being equal to zero:

If an equation is of the general quadratic form:

$Y = aX^2 + bX + c$,

the values of X where Y is 0 are:

$$X = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

In the equation in the example, $a = 3$, $b = -12$, and $c = 9$. Hence,

$$X = \frac{-(-12) \pm \sqrt{144 - 108}}{6} = 2 \pm 1$$

Therefore, $Y = 0$ when X equals 1 or 3.

To determine whether each of these two output levels maximizes or minimizes profit, we find the value of the second derivative at these two values of X.

Taking the derivative of $\frac{\partial Y}{\partial X}$ we find that:

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6X$$

If **<u>X =1:</u>**

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6(1) = -6$$

Hence, since the second derivative is negative**,** profit is at a maximum when output equals one million units.

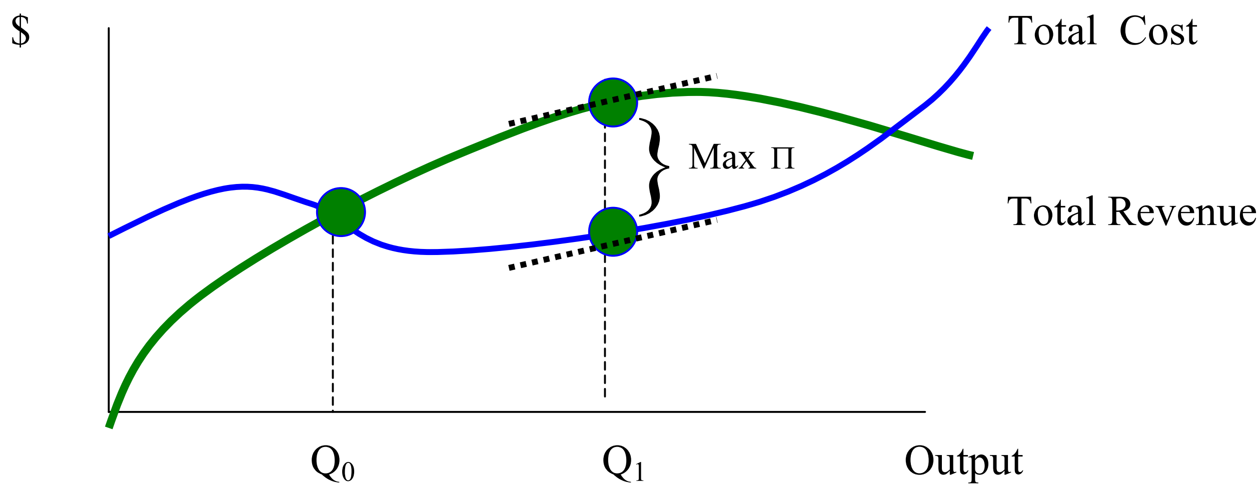
If **<u>X=3:</u>**

$$\frac{\partial^2 Y}{\partial X^2} = -12 + 6(3) = 6$$

Hence, since the second derivative is positive**,** profit is a minimum when output equals 3 million units.

The Marginal Cost Equals Marginal Revenue Rule and the Calculus of Optimization

Once we understand how calculus can be employed to solve optimization problems, it is easy to see how the fundamental rule for profit maximization – (setting MC=MR) – has its roots in the calculus of optimization. Consider the following graph:



The diagram shows a firm's total cost and total revenue functions.

Since total profit equals total revenue minus total cost, it is equal to the vertical distance between the total revenue and total cost curves at any level of output.

This distance is maximized at output Q_1, where the slopes of the total revenue and total cost curves are equal.

Since the slope of the total revenue curve is marginal revenue and the slope of the total cost curve is marginal cost, this means that profit is maximized when marginal cost equals marginal revenue.

Q_1 must be the profit maximizing output.

At output levels below Q_0, the firm will experience losses, since total cost is greater than total revenue.

As output increases beyond Q_0, total revenue rises faster than total cost, and hence profit is rising.

As long as the slope of the total revenue curve is greater than the slope of the total cost curve, profit will rise as output increases.

When these slopes become equal, profit no longer rises, but is at a maximum.

Now, let's employ calculus to see why firms maximize profit by setting marginal cost equal to marginal revenue.

Let: $$\Pi = TR - TC$$

where Π=Total Profit,
TR=Price * Output = Total Revenue
and TC = Total Cost

Taking the derivative of Π with respect to Q (output), we find that:

$$\frac{\partial \Pi}{\partial Q} = \frac{\partial TR}{\partial Q} - \frac{\partial TC}{\partial Q}.$$

For Π to be a maximum, this derivative must be zero, so:

$$\frac{\partial TR}{\partial Q} = \frac{\partial TC}{\partial Q}.$$

And since MR is defined as $\frac{\partial TR}{\partial Q}$ and marginal cost is defined as $\frac{\partial TC}{\partial Q}$, marginal revenue must equal marginal cost.