Topic 5: Discrete Random Variables & Expectations

Reference Chapter 5

In Chapter 4 we studied rules for associating a probability value with a single event or with a subset of events in an experiment.

Now we will expand our analysis to consider all possible events in an experiment.

A **random variable** is a ____ which assigns _________ values to each possible outcome of an experiment.

Note: May need to distinguish between the ____ of a random variable, and its value.

Example: {X=x} means "the ______ variable X takes the value 'x.'"

Example: Experiment: choose student; ask if he/she is planning to major in business.

Two discrete outcomes: Yes; No.

Define random variables:

$$\left.\begin{array}{l} x = 1;\ \text{Yes} \\ x = 0;\ \text{No} \end{array}\right\} \qquad x = 0, 1 \Big\}\text{indicator variable}$$

<u>Example:</u> Midterm Test. Students may get a grade A to F
Define:

<u>x</u>	
X=5;	A
X=4;	B
X=3;	C
X=2;	D
X=1;	F

Depending on the context, sample space may be ________ or **continuous**.

If discrete, it may be finite or (countably) ___________.

Probability Distributions

Once an experiment and its outcomes have been clearly defined and the random variable of interest has been defined, then the **probability** of the occurrence of any _____ of the random variable can be specified.

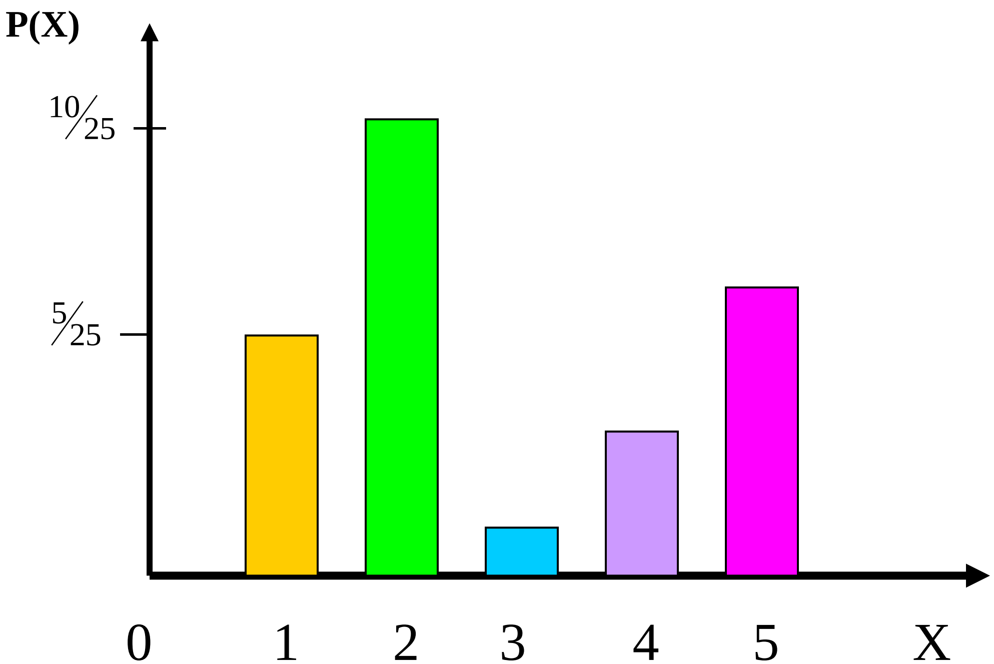
I.e. Can now specify the probability of each value of the random variable occurring.

Example: There are 5 projects to be done. Twenty-five government workers are to be allocated to "teams" (varying size) to work on them:

Outcome Project	# of Employees	Value of X (x)	Prob(X=x)
1		1	5/25
2		2	10/25
3		3	1/25
4		4	3/25
5		5	6/25
	25		1.0

Probability Distribution

Can also depict graphically:



The table or graph represent the **probability distribution of X**: Each value of X and its ___________of occurrence.

⇨Similar to our earlier notion of "frequency distribution," but now relates to **all possible situations** that can occur.

- We can use probability distribution to determine probabilities of interesting events:

For Example:

$$P(x \geq 2) = P(\text{assigned to project 2 or 3 or 4 or 5})$$

$$= \mathrm{P}(\mathrm{x} = 2) + \mathrm{P}(\mathrm{x} = 3) + \mathrm{P}(\mathrm{x} = 4) + \mathrm{P}(\mathrm{x} = 5)$$

$$= \frac{10}{25} + \frac{1}{25} + \frac{3}{25} + \frac{6}{25} = \frac{20}{25} = \frac{4}{5}$$

<u>Definitions:</u>

If the individual values assigned to the random variable can be ________, it is called a **<u>discrete random variable</u>**.

● In this case, its probability distribution is called a **probability mass function, ______**.

Note: Defining Characteristics of P.M.F.:

$$(i) \quad 0 \le P(X = x) \le 1$$

$$(ii) \quad \sum_{x} P(X = x) = 1$$

In the same manner, we can define the cumulative ____ function (c.m.f.):

$$\boxed{F(X^*) = P(X \le x^*) = \sum_{x \le x^*} P(x)}$$

The value of c.m.f. at any point x is usually denoted F(x*), where it is the ___ of all values of the p.m.f. for all the values of the random variable x that are $\le$ **x*.**

Example:

Long-term experience suggests the following p.m.f. for sales of flu medication in January:

Type	X	#	P(X)	F(X)
Aspirin for Flu	**1**		**0.10**	**0.10**
Tylenol Flu	**2**		**0.50**	**0.60**
Benadryl	**3**		**0.40**	**1.00**
		1000	**1.00**	

F(2) =P(x=1 or x=2)=P(Aspirin or Tylenol)

If a customer buys flu medicine in January, the probability it is not Benadryl is 0.60.

$$P(x>1)=1-P(x\leq 1)=1-P(x=1)=1 - 0.10 = 0.9$$

$$P(1<x\leq 3)=P(x=3)+P(x=2)=0.9$$
$$=F(3)-F(1)=1-0.10=0.9$$

$$P(2<x\leq 3)=P(x=3)=0.4$$
$$=F(3)-F(2)=1-0.60=0.4$$

When we previously considered data distributions, we looked at "summary measures" such as the mean, variance, etc..

►A ____________ **distribution** is just a special type of data distribution – same motivation here to evaluate such measures.

To do this, we define:

The **E________ Value**: of the random variable X is:

$$\mu = E(x) = \sum_{x} x \times P(x) \begin{cases} \text{a weighted average} \\ \text{of all values with} \\ \text{probabilities as} \\ \text{weights.} \end{cases}$$

The **expected value** of a ________ random variable X is found by multiplying each value of the random variable by its probability and then summing all these products.

The letter **E** usually denotes an **e________ value**.

The expected value of X is the **balancing point** for the p.m.f..

$$
\begin{aligned}
E[X] &= \int_{-\infty}^{\infty} x f_X(x)dx \\
&= \int_{-\infty}^{1} x f_X(x)dx + \int_{1}^{3} x f_X(x)dx + \int_{3}^{\infty} x f_X(x)dx \\
&= \int_{-\infty}^{1} x0dx + \int_{1}^{3} x\frac{1}{2}dx + \int_{3}^{\infty} x0dx \\
&= 0 + \left[\frac{1}{4}x^2\right]_1^3 + 0 \\
&= \frac{1}{4}3^2 - \frac{1}{4}1^2 = \frac{9-1}{4} = \frac{8}{4} = 2
\end{aligned}
$$

Expected Value

Example: What is the expected value of getting heads from three coin tosses? Note that there are 3 tosses (slots) and each toss has 2 possibilities (heads or tails) so there are 2 * 2 * 2 or 8 total possibilities.

Event (Heads) (x)	Probability(p)	Product
0 (TTT)	1/8	0 * 1/8 = 0
1 (HTT, THT, TTH)	3/8	1 * 3/8 = 3/8
2 (HHT, HTH, THH)	3/8	2 * 3/8 = 6/8
3 (HHH)	1/8	3 * 1/8 = 3/8
	8/8 =1	**Sum of products**=12/8=1.5

After many trials of three tosses of a coin, you would expect to get, on average, 1.5 heads.

Example: Marks obtained by students:

Marks	**X (midpoint)**	**P(x)**	**xP(x)**
0 $\leq$ x<20	**10**		**1**
20 $\leq$ x< 40	**30**		**6**
40 $\leq$ x< 60	**50**		**20**
60 $\leq$ x<80	**70**		**14**
80 $\leq$ x<100	**90**		**9**
		1.0	**μ=E(X)=50**

Sometimes we want to work out the expected value of some **f________** of a random variable.

For instance, instead of finding the mean of the random variable X, we might be interested in determining the expected ______ of X^2, or Log X, or of e^x.

If X is a random variable, then these functions of X are also ________ variables, and their expected values can be determined.

The expected value of g(x):

$$\boxed{E\left[g(x)\right] = \sum_{all\,x} g(x)\,P(x)}$$

<u>Examples:</u>

$$E\left[x^2\right] = \sum_x (x^2) P(x)$$

$$E\left[\log(x)\right] = \sum_x \log(x) P(x)$$

$$E\left[(x+2)\right] = \sum_x (x+2) P(x)$$

(Watch the units.)

Example: Expected compounded return on stocks over 3 years:

Stock	1-Yr. Return (x)	P(x)	x^3 P(x)
A	1.15		0.6084
B	1.10		0.6655
C	1.05		0.1158
			$E(x^3)=1.3897$

Expected return over 3 years is 38.97%.

The expectation "operator", E(·), can be used to help us define the **v__________** of a random variable.

Recall, the variance of a frequency distribution was:

$$\boxed{\frac{1}{N}\sum (x_i - \mu)^2 f_i} \quad \leftarrow \quad \text{(Average of } (x_i\text{-}\mu)^2 \text{ values.)}$$

⇨ So, the ________ of a random variable, X, is:

$$\boxed{V(x) = \sigma^2 = \sum (x-\mu)^2 P(x) = \left[\sum x^2 P(x) - \mu^2\right]}$$

<u>Example</u>: Volatility of Stock returns over one year:

$\mu = E(x) = 11.5\%$

Stock	Return (x)	P(x)	$(x-\mu)^2 P(x)$
A	15%		4.900
B	10%		1.125
C	5%		4.225
			10.25

Rules of "Expectation":

$$E(x) = \sum_x xP(x)$$

1) The E(c)=c, where c is a constant.

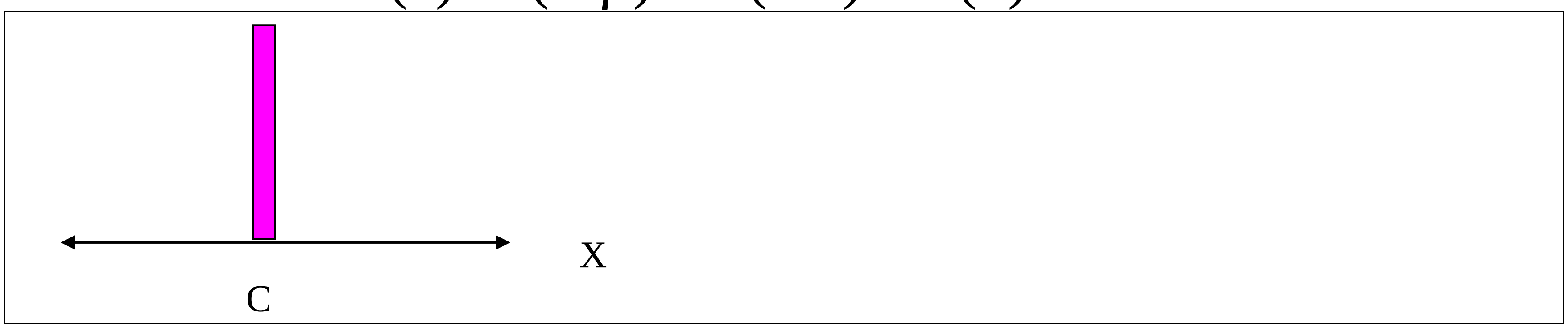
$$E(c) = \sum cP(c) = c \times 1 = c$$

eg. E(10)=_____

2) V(c)=0; The variance of a constant is zero.

Recall: $V(x)=E(x-\mu)^2$;

$$V(c)=E(c-\mu)^2=E(c-c)^2=E(0)^2=0$$



3) E(aX)=aE(X): Multiply X by a constant:

$$E(ax) = \sum_x axP(x) = a\sum_x xP(x) = aE(x)$$

eg. E(10x)=10E(x)

4) E(x+b)=E(x)+b: Add a constant

$$\begin{aligned} &E(x+b) \\ &= \sum_x (x+b)P(x) \\ &= \sum_x (xb)P(x) + \sum_x (b)P(x) \\ &= E(x) + b\sum_x P(x) \\ &= E(x) + b \end{aligned}$$

since $\sum_{\mathrm{x}} P(x) = 1.$

5) $V(ax)=a^2V(x)$:

Recall $V(x)=E(x-\mu)^2$;

$$
\begin{aligned}
V(ax) &= E(ax-E(ax))^2 \\
&= E(ax-aE(x))^2 \\
&= E(a(x-E(x)))^2 \\
&= E(a^2(x-E(x))^2)
\end{aligned}
$$

$$V(ax)=a^2\,E(x-E(x))^2=a^2V(x)$$

Example: $V(120x)=$_________

6) $V(x+b)=V(x)$

$$V(x+b)=E((x+b)-E(x+b))^2$$
$$=E(x+b-E(x)-b)^2$$
$$=E(x-E(x))^2=V(x)$$

$E(ax\pm b)=aE(x)\pm b$

$V(ax\pm b)=$________

Examples:

$V(x+10)=$______

$V(x-20)=V(x)$

****

One advantage of knowing these rules is to <u>speed up</u> calculations.

<u>Example</u>: Data in U.S. $: expected value=U.S. $10 and standard deviation =U.S. $2. ⇨What about the results in $ CAN?

Suppose C$1 =U.S. $.996 (U.S. $1 =C$1.004)

$E(ax)=aE(x)$; $a = 1.004$

⇨So, expected value C$ (10 × 1.004)= C$10.04

$V(ax)= a^2\ V(x)$; $a= 1.004$

Std. deviation $(ax)=|a|$ S.D. (X)

=C$(2 × 1.004)=C$2.008

Example: F=random variable of temperature (ºF)

E(F)= 72.4ºF; V(F)=64(ºF)^2.

What is E(C) and V(C), where C is temperature (ºC)?

$$F = 32 + \tfrac{9}{5} C;$$

$$C = \frac{F-32}{\tfrac{9}{5}} = -17.778 + 0.5556F$$

And since E(ax+b)=aE(x)+b:

$$E(C) = \overbrace{-17.778}^{b} + \overbrace{0.5556}^{a} E(F) = 22.44$$

$$V(C) = \underbrace{(0.5556)^2}_{a} V(F) = 19.75$$

Note: these expectation and variance rules also enable us to standardize random variables for **comparison purposes**:

$E(x)=\mu$; $V(x)=\sigma^2$; S.D.$(x)=\sigma$.

Then let: $$Z=\left[\frac{X-\mu}{\sigma}\right]=\underbrace{\frac{1}{\sigma}}_{a}X-\underbrace{\left(\frac{\mu}{\sigma}\right)}_{b}$$

$$E(Z)=\frac{1}{\sigma}E(X)-\left(\frac{\mu}{\sigma}\right)$$

$$=\left(\frac{1}{\sigma}\right)\mu-\left(\frac{\mu}{\sigma}\right)=0$$

So: $E(Z)=0$;
$V(Z)=1$

$$V(Z)=\left(\frac{1}{\sigma}\right)^2 V(X)$$

$$=\left(\frac{1}{\sigma^2}\right)\sigma^2=\frac{\sigma^2}{\sigma^2}=1$$

Bivariate Probability Distributions:

● In many situations, an experiment may involve outcomes that are related to ____or more random variables

● This section considers probability functions that involve more than one variable: such functions are called **multivariate probability functions**

We will only deal with the case of **bivariate** (two variables) probability functions.

When a sample space involves two or more random variables, the function describing their combined probability is called a

⇨ **j_____probability function.**

Example: Behaviour of Households:

X=Income of the oldest member of the household
Y= Years work experience of this member

By looking at P(X=x and Y=y) for all x, y values, we get the JOINT PROBABILITY DISTRIBUTION.

			X Income ($)	
Experience (Years) Y		25,000	30,000	50,000
	_____	0.20	_____	0.01
	_____	0.05	_____	0.09
	_____	0.10	_____	0.20

Entries in the table are j_____ probabilities!

Two **properties** for Joint Probability Functions:

$$(i) \quad 0 \le P(x,y) \le 1$$

$$(ii) \quad \sum_x \sum_y P(x,y) = 1$$

where P(x,y)=P(X=x and Y=y).

↳This joint distribution also helps us with calculation of conditional probabilities.

		X Income ($)			
Experience (Years) Y		**25,000**	**30,000**	**50,000**	**P(y)**
	_____	**0.20**	**0.05**	**0.01**	_____
	_____	**0.05**	**0.20**	**0.09**	_____
	_____	**0.10**	**0.10**	**0.20**	_____
	P(x)	**0.35**	**0.35**	**0.30**	**1.00**

$$P(x) = P(X = x) = \sum_{y} P(\mathrm{X} = \mathrm{x} \text{ and } \mathrm{Y} = y)$$
$$= \text{"marginal probability of X"}$$

$$P(y) = P(Y = y) = \sum_{x} P(\mathrm{X} = \mathrm{x} \text{ and } \mathrm{Y} = y)$$
$$= \text{"marginal probability of Y"}$$

A marginal probability for any value of Y may be found by summing all the ____ probabilities involving that value of Y.

The table depicts joint probability distributions of X and Y.

- When we discussed "events," we had:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}. \begin{array}{l} \Leftarrow \text{Joint} \\ \Leftarrow \text{Marginal} \end{array}$$

Conditional

$$P(X = x|Y = y) = \frac{P(X = x \text{ and } Y = y)}{P(Y = y)}.$$

or

$$P(x|y) = \frac{P(x, y)}{P(y)}.$$

From the last example:

(i) P($25,000 and 5 years experience)=___

(ii) P($25,000)=____;
P(5Years)=0.26

(iii) P($25,000| 5 years experience)= $\frac{0.2}{0.26} = 0.7692.$

Independence

As with events, two random variables are **in__________** if:

$P(x|y) = P(x)$ for all x and y.

$P(x,y)=P(x)P(y)$; for all x and y

❖If the knowledge of a certain __________ on one variable does **not** affect the probability of the occurrence of values of the other random variable, then the two variables are **independent**.

I.e. X and Y are independent if the conditional probability of X given Y is the same as the marginal probability of X.

⇨If X and Y are **<u>independent</u>**, the **P(x,y)=___ ·____**.

An example of not independent from the last example:

P(x=$25,000 and y=5 years)=____
P(x=$25,000)= ____
P(y=5 years) =0.26

However: (0.35)(0.26) $\neq$ 0.20.

Years of experience and income are "<u>dependent</u>" random variables.

Expectations of Combined Random Variables:

When dealing with the joint probability distribution of X and Y, the rule for finding expectations is similar to the rule for finding the expected value of a function.

Basic rule: $E\left[g(x)\right] = \sum_{all\ x} g(x)P(x)$

If we combine two random variables together into h(X,Y), then:

$$E\left[h(X,Y)\right] = \sum_{x}\sum_{y} h(x,y)P(x,y)$$

Example: Selling items of different value:

$$E[XY] = \sum_x \sum_y (xy) P(x, y)$$

		X Quantity			
Price ($) Y		1	2	3	**P(y)**
	____	**0.00**	____	**0.03**	**0.05**
	____	**0.05**	____	**0.10**	**0.35**
	____	**0.10**	____	**0.30**	**0.60**
	P(x)	**0.15**	**0.42**	**0.43**	**1.00**

$$E[XY] = \sum_x \sum_y (xy) P(x, y)$$

$$= [(1 \times 4 \times 0) + (2 \times 4 \times 0.02) + (3 \times 4 \times 0.03) +$$

$$(1 \times 5 \times 0.05) + (2 \times 5 \times 0.2) + (3 \times 5 \times 0.10) +$$

$$(1 \times 6 \times 0.10) + (2 \times 6 \times 0.20) + (3 \times 6 \times 0.3) = \$12.67$$

Covariance of X and Y

When we have 2 random variables it is interesting to measure the extent to which their random behaviours are **related.**

The **covariance**, C(X,Y), measures how much the two random variables ____ with each other, (how they co-vary).

Look at the variation of each random variable relative to its respective _____.

$$\boxed{Cov(X,Y) = E(X,Y) - E(X)E(Y)}$$

$$Let\ \mu_x = E(X)$$

$$\mu_y = E(Y):$$

$$Cov(X,Y) = E\left[\left(x-\mu_x\right)\left(y-\mu_y\right)\right]$$

$$= \sum_x \sum_y \left[\left(x-\mu_x\right)\left(y-\mu_y\right)\right]P(x,y)$$

Note: if X and Y are independent: P(X,Y)=P(X) * P(Y)

$$Cov(X,Y) = \sum_{x}\sum_{y}\left[(x-\mu_x)(y-\mu_y)\right]P(x,y)$$

$$= \left[\sum_{x}(x-\mu_x)P(x)\right]\left[\sum_{y}(y-\mu_y)P(y)\right]$$

$$= \left[\sum_{x}(x)P(x) - \mu_x\sum_{x}P(x)\right]\left[\sum_{y}(y)P(y) - \mu_y\sum_{y}P(y)\right]$$

$$= \left[E(x) - \mu_x\right] \bullet \left[E(y) - \mu_y\right] = 0$$

$$= (\mu_x - \mu_x = 0)\ (\mu_y - \mu_y = 0)$$

If **high** values of X (relative to μ_x) tend to be associated with **high** values of Y (relative to μ_y) then C(X,Y) will be a large _________ number.

If **low** values of one variable tend to be associated with **high** values of the other, then C(X,Y) will be a large _________ number.

Example continued:

		X			
			Quantity		
Price		1	2	3	**P(y)**
($)	4	**0.00**	_____	**0.03**	**0.05**
Y	5	**0.05**	_____	**0.10**	**0.35**
	6	**0.10**	_____	**0.30**	**0.60**
	P(x)	**0.15**	**0.42**	**0.43**	**1.00**

$$E\left[Quantity(X)\right] = \sum_{x} X \times P(X)$$

$$= [(1 \times 0.15) + (2 \times 0.42) + (3 \times 0.43) = 2.28$$

$$E\left[\text{Price}(Y)\right] = \sum_{y} \text{Price} \times P(\text{price})$$

$$= [(4 \times 0.05) + (5 \times 0.35) + (6 \times 0.60) = 5.55$$

$$Cov[X,Y] = \sum_{x} \sum_{Y} (x - \mu_x)(y - \mu_y)P(x,y)$$

$$= [(1 - 2.28)(4 - 5.55)(0)] + [(1 - 2.28)(5 - 5.55)(0.05) +$$

$$\cdots + [(3 - 2.28)(6 - 5.55)(0.3)] = 0.016.$$

or:

$$COV(x,y) = E(XY) - E(X)E(Y)$$

$$= [12.67 - (2.28)(5.55) = 0.016$$

One <u>limitation</u> of the covariance measure is that it depends on **u_____** of the data, and it can take any positive or negative value.

To compensate for both of these failings, especially when wanting to ________ across different data sets, consider the following measure:

$$\rho_{xy} = \left[\frac{Cov.(X,Y)}{\sigma_x \sigma_y} \right] = correlation$$

where

σ_x is the standard deviation of X,

$$\sigma_{\mathrm{x}} = \sqrt{\text{variance(x)}} = \sqrt{E(x-\mu_x)^2}$$

We call ρ_{xy} the simple correlation between the random variables X and Y.

(i) correlation is unitless.

(ii) $-1 \leq \rho \leq 1$

		X Quantity			
Price		1	2	3	**P(y)**
($)	4	**0.00**	____	**0.03**	**0.05**
Y	5	**0.05**	____	**0.10**	**0.35**
	6	**0.10**	____	**0.30**	**0.60**
	P(x)	**0.15**	**0.42**	**0.43**	**1.00**

$$E\left[Quantity(X)\right] = \sum_{x} X \times P(X) = 2.28$$

$$E\left[\text{Price}(Y)\right] = \sum_{\text{y}} \text{Y} \times P(Y) = 5.55$$

$$COV(x,y) = E(XY) - E(X)E(Y) = 0.016$$

$$E[XY] = \sum_{x}\sum_{y}(xy)P(x,y) = \$12.67$$

$$E\left[Quantity(X^2)\right] = \sum_x X^2 \times P(X)$$

$$= [(1^2 \times 0.15) + (2^2 \times 0.42) + (3^2 \times 0.43) = 5.7$$

$$E\left[\text{Price}(Y^2)\right] = \sum_y \text{Y}^2 \times P(Y)$$

$$= [(4^2 \times 0.05) + (5^2 \times 0.35) + (6^2 \times 0.60) = 31.15$$

$$VAR(x) = E(X^2) - [E(x)]^2$$

$$= 5.7 - [2.28]^2 = 0.5016$$

$$\sigma_x = 0.7082$$

$$VAR(Y) = E(y^2) - [E(y)]^2$$

$$= 31.15 - [5.55]^2 = 0.3475$$

$$\sigma_y = 0.5895$$

$$\boxed{\rho_{x,y} = \frac{COV(x,y)}{\sigma_x \sigma_y} = \frac{0.016}{0.7082 \times 0.5895} = \frac{0.016}{0.3552} = 0.04504}$$

_____ positive correlation between x and y.

Linear Combinations of Random Variables

In general, we had:

$$E\left[h(X,Y)\right]=\sum_{x}\sum_{y}h(x,y)\,p(x,y)$$

Now consider the expectation of the weighted sum of:

$$h(X,Y)=aX+bY$$

where a and b are constants:

i)

$$E(aX + bY) = \sum_{x}\sum_{y}(aX + bY) \times \mathrm{P(x, y)}$$

$$= \sum_{x}\sum_{y}\left[(aX) \times P(x,y) + (\mathrm{bY}) \times \mathrm{P(x, y)}\right]$$

$$= a\sum_{x}\left[\mathrm{x}\sum_{y}\mathrm{P(x, y)}\right] + \mathrm{b}\sum_{y}\left[\mathrm{y}\sum_{x}\mathrm{P(x, y)}\right]$$

$$= a\sum_{x}\mathrm{xP(x)} + \mathrm{b}\sum_{y}\mathrm{yP(y)}$$

$$= a\mathrm{E(X)} + \mathrm{bE(Y)}$$

More directly:

$$E(aX + bY) = E(aX) + E(bY)$$
$$= a\,E(X) + b\,E(Y)$$

(ii)

$$\mathrm{Var}(aX + bY)$$
$$= E\left[(aX + bY) - E(aX + bY)\right]^2$$
$$= E\left[(aX - aE(X) + bY - bE(Y))\right]^2$$
$$= E\left[a(X - E(X)) + b(Y - E(Y))\right]^2$$
$$= E\left[a^2(X - E(X))^2\right] + E\left[b^2(Y - E(Y))^2\right]$$
$$+ E\left[2ab(X - E(X))(Y - E(Y))\right]$$
$$= a^2V(X) + b^2V(Y) + 2ab\mathrm{Cov.}(X, Y)$$

		X Quantity			
Price ($) Y		1	2	3	**P(y)**
	4	**0.00**	____	**0.03**	**0.05**
	5	**0.05**	____	**0.10**	**0.35**
	6	**0.10**	____	**0.30**	**0.60**
	P(x)	**0.15**	____	**0.43**	**1.00**

$$E\left[Quantity(X)\right] = 2.28$$

$$E\left[\text{Price}(Y)\right] = 5.55$$

$$COV(x, y) = 0.016$$

$$E[XY] = \$12.67$$

$$E\left[Quantity(X^2)\right] = 5.7$$

$$E\left[\text{Price}(Y^2)\right] = 31.15$$

$$VAR(x) = 0.5016$$

$$\sigma_x = 0.7082$$

$$VAR(Y) = 0.3475$$

$$\sigma_y = 0.5895$$

$$\rho_{x,y} = 0.04504$$

Let R=3X+7Y. What is the expected value and the standard deviation of R?

E(R)=3E(x) + 7E(y)

3(2.28)+7(5.55)=45.69

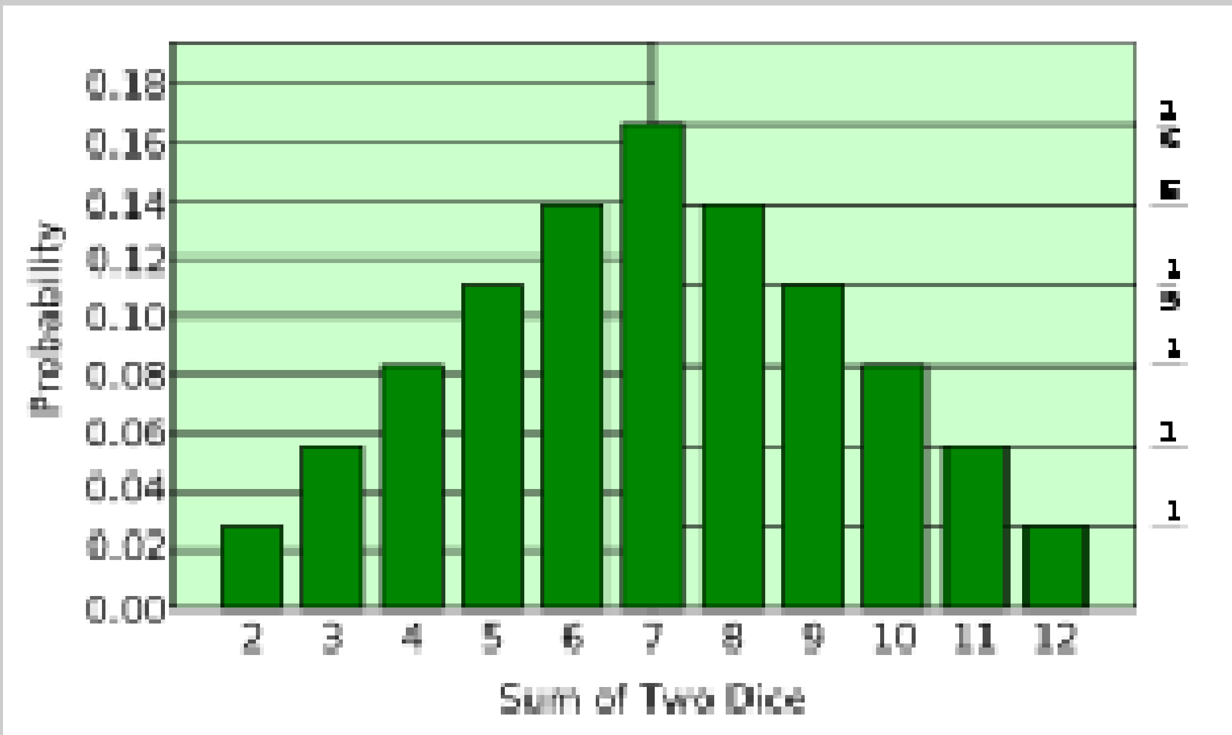
V(R)=V(3x+7y)

=9V(x)+49V(y)+2(3)(7)Cov(x,y)

=(9)(0.5016)+(49)(0.3475)+(2)(3)(7)(0.016)

=4.5144+17.0275+0.672=22.2139

Std. Dev.=_________



Discrete probability distribution for the sum of two dice.

Discrete Probability Distributions

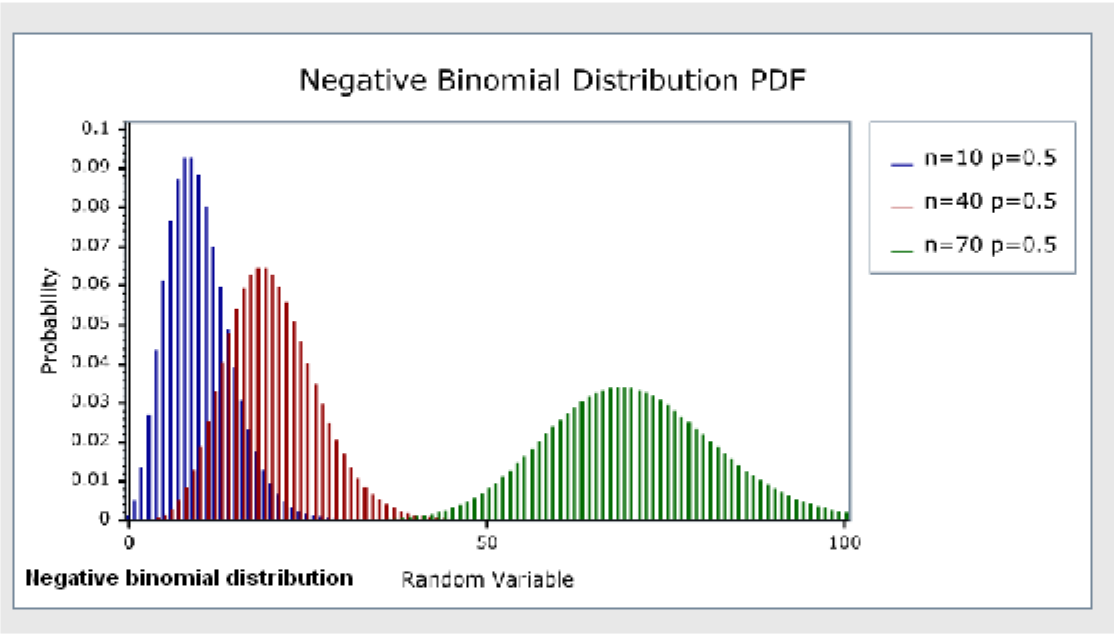
While it is often useful to determine probabilities for a specific discrete random variable or combined random variables, there are many situations in statistical inference and decision-making that involve the **s___** type of probability function.

I.e. Certain probability distributions _____ – have general characteristics which can be exploited.

Recognize the similarities between certain types of experiments and match a given case to the general formulas for mean, variance, independence and other characteristics of the random variables.

The examples we have seen so far have been different, specific, probability distributions. We are given information in each case.

Let's consider one specific, common, discrete random variable. In Topic 6 we will consider one specific continuous random variable.



The Binomial Distribution:

Many experiments share the common trait that their _______ can be classified into **one of two** events.

For Example:
1. Select a person: male or ______
2. Apply for a job: Success or ____
3. Your credit card bill: paid or due

It is often possible to describe the outcomes of events as either a "success" or "failure."

Use general terminology of **"success" or "failure"** to distinguish.

Experiments involving repeated independent trials, each with just two possible outcomes, form the basis of the ________ distribution (________ probability distribution).

Definition: **<u>Bernoulli Trial:</u>**

Each repetition of an experiment involving only ___ outcomes is called a Bernoulli trial.

- Interested in a series of independent, repeated Bernoulli trials.

- Independent results mean any one trial cannot influence the results of any other trial.

Note: Name the trials: t_1 t_2 t_3 ..., then,: P(t_2=success |t_1 = success)=P(t_2=success); etc.

By “repeated” trials we mean that each time the experiment is conducted, the conditions are __________.

The Binomial Distribution is **characterized** by:

n = number of _____

Π= probability of success in one independent trial.

The values of n and **Π** are referred to as the **parameters** of this distribution.

They help determine the ____________ of “success” and “failure.”

In a binomial distribution, the probabilities of interest are those of receiving a certain number **(x)** of __________ in '**n**' independent _______, each trial having the same probability **(Π)** of success.

Note: P(Failure) = 1- P(Success) as there are only 2 possible outcomes.

They are "**complementary**" outcomes.

The Binomial Formula:

To determine the probability of exactly x successes in n repeated Bernoulli trials, each with a _________ probability of success equal to Π, it is necessary to find the probability of any one _________ of outcomes where there are x successes.

If there are x successes in n trials, there must be (n-x) failures.

The probabilities of interest are those of getting x “successes” in n independent trials, each of which has probability Π of success.

P(Event)=P(Outcome #1) + P(Outcome #2)+ ...
= P(Outcomes) * (# of Outcomes)

if each outcome has the same probability.

We can choose x from n in : $\boxed{\_\_\_ = \dfrac{n!}{x!(n-x)}}$ ways.

So, there are nCx outcomes consistent with the event of x "s_________" and (n-x) failures.

Each of these (nCx) outcomes occurs with probability $\left[\Pi^{x}(1-\Pi)^{n-x}\right]$.

So:

$$P(\text{X successes in n trials})$$

$$= \Pi^{x}(1-\Pi)^{n-x}(nCx)$$

$$= \left[\frac{n!}{x!(n-x)!}\right]\Pi^{x}(1-\Pi)^{n-x}$$

$$for \quad \begin{cases} \text{x} = 0,\ 1,\ 2,\ 3,\ \ldots,\text{n} \\ \text{n} = 1,\ 2,\ 3,\ldots etc. \end{cases}$$

This set of probabilities gives the probability distribution for a _______ Random Variable.

<u>Note</u>: If x is B(n, **Π**), it is a _________ random variable:

$$P(X) = nCx(\Pi)^{x}(1-\Pi)^{n-x}$$

$$\left\{\begin{matrix} \text{x} = 0,\ 1,\ 2,\ 3, \ldots, \text{n} \\ \text{n} = 1,\ 2,\ 3,\ \ldots. \end{matrix}\right\}$$ is its probability ____ function:

So:

$$(i) \quad P(X) \geq 0; \quad \text{all x.}$$

$$(ii) \quad \sum_{x=0}^{n} P(X) = 1$$

<u>Example:</u> $\Pi = 0.2;\ n = 4;\ (1 - \Pi) = 0.8$

$$P(0) = {}_4C_0 (0.2)^0 (0.8)^{4-0}$$

$$= \left[\frac{4!}{0!4!}\right](1)(0.8)^4 = 0.\_\_\_\_$$

$$P(1) = {}_4C_1 (0.2)^1 (0.8)^{4-1}$$

$$= \left[\frac{4!}{1!3!}\right](0.2)^1(0.8)^3 = 0.\_\_\_\_$$

$$P(2) = {}_4C_2 (0.2)^2 (0.8)^{4-2}$$

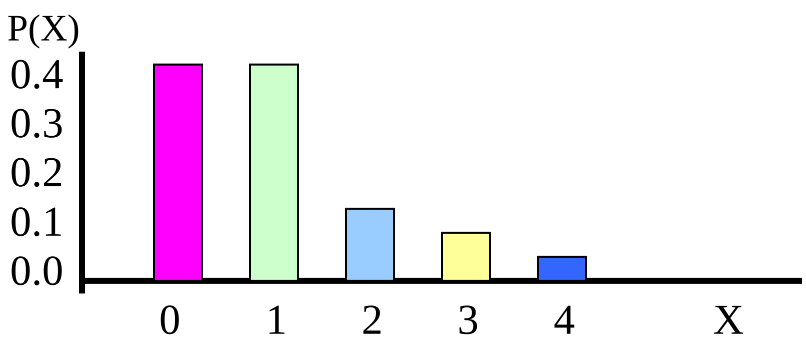
$$= \left[\frac{4!}{2!2!}\right](0.2)^2(0.8)^2 = 0.1536$$

$$P(3) = {}_4C_3(0.2)^3(0.8)^{4-3}$$

$$= \left[\frac{4!}{3!1!}\right](0.2)^3(0.8)^1 = 0.0256$$

$$P(4) = {}_4C_4(0.2)^4(0.8)^{4-4}$$

$$= \left[\frac{4!}{4!0!}\right](0.2)^4(0.8)^0 = 0.\_\_\_\_$$



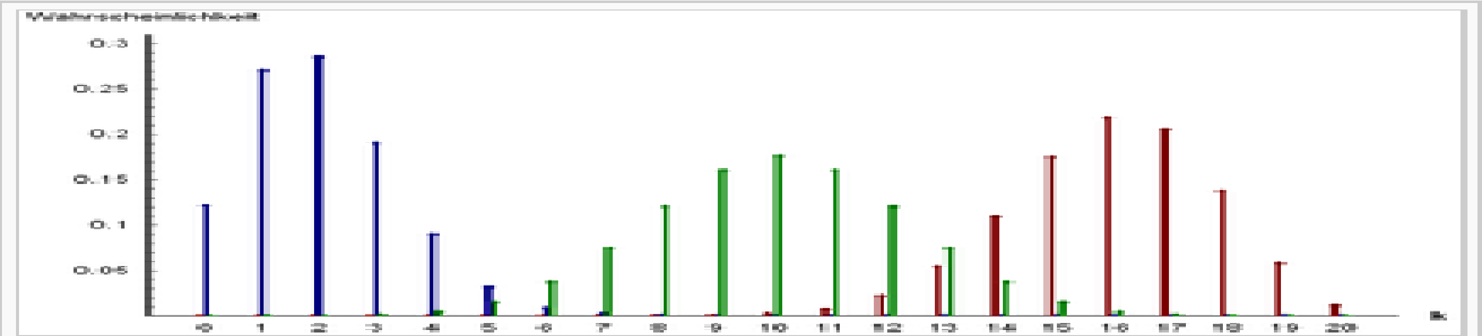
Binomial tables assist in the calculations.

General Characteristics of Binomial Distribution

(A) If n is small and $\pi \leq 0.50$: distribution is skewed to the ______.

(B) If n is small and $\pi > 0.50$: distribution is skewed to the _____.

(C) If n is large and/ or $\pi = 0.50$: distribution is ___________.



Binomial distribution for $n = 20$
$p = 0.1$ (blue), $p = 0.5$ (green) and $p = 0.8$ (red)

Example: 30% of job applicants are from a minority group. If we take __applicants at random, what is the probability of including __from the minority group?

Let X = “minority person”
n=5; X=2; Π=0.3

$$P(X) = nCx(\Pi)^{x}(1-\Pi)^{n-x} \Leftarrow Binomial$$

$$P(2) = \left(\frac{5!}{2!\,3!}\right)(0.3)^{2}(0.7)^{3} = 0.\_\,\_\,\_\,\_$$

<u>Example</u>: Customers in a bank. One in four customers require service for more than 4 minutes. ___ people are in a queue. What is the probability that exactly 3 customers will take more than 4 minutes each?

Π =0.25;n=6; x=3
X=”customer takes more than 4 minutes.”

$$P(X) = nCx(\Pi)^x(1-\Pi)^{n-x}$$

$$P(3) = \left(\frac{6!}{3!3!}\right)(0.25)^3(0.75)^3 = 0.1318$$

What is the probability that at least 3 customers need more than 4 minutes each?

I.e. P(3)+ P(4) + P(5) + P(6):

$$P(4) = \left(\frac{6!}{4!2!}\right)(0.25)^4(0.75)^2 = 0.0330$$

$$P(5) = \left(\frac{6!}{5!1!}\right)(0.25)^5(0.75)^1 = 0.0044$$

$$P(6) = \left(\frac{6!}{6!0!}\right)(0.25)^6(0.75)^0 = 0.0002$$

So the probability is:

(0.1318+ 0.0330 + 0.0044 + 0.0002) = 0.____

& P(< 3 customers take more than 4 minutes each)=1- 0.1694 = 0.____

An example from sports

[edit]

A soccer player makes multiple attempts to score goals. If she has a shooting success probability of 0.25 and takes 4 shots in a match, then the number of goals she scores can be modeled as B(4, 0.25). Note that *p* represents the probability of any given shot becoming a goal, and 1 − *p* represents the probability of failure. The probability of the player scoring 0, 1, 2, 3, or 4 goals on 4 shots is:

$$\Pr(0 \text{ goals}) = f(0) = \Pr(K=0) = \binom{4}{0}0.25^0(1-0.25)^{4-0} \approx 0.32$$

$$\Pr(1 \text{ goal}) = f(1) = \Pr(K=1) = \binom{4}{1}0.25^1(1-0.25)^{4-1} \approx 0.42$$

$$\Pr(2 \text{ goals}) = f(2) = \Pr(K=2) = \binom{4}{2}0.25^2(1-0.25)^{4-2} \approx 0.21$$

$$\Pr(3 \text{ goals}) = f(3) = \Pr(K=3) = \binom{4}{3}0.25^3(1-0.25)^{4-3} \approx 0.05$$

$$\Pr(4 \text{ goals}) = f(4) = \Pr(K=4) = \binom{4}{4}0.25^4(1-0.25)^{4-4} \approx 0.004$$

The Expected Value of a Binomial Random Variable:

The ____ number of successes in any given experiment must equal the number of trials (n) times the probability of a success on each trial (π).

For example, the probability that a process produces a defective item is π = 0.___;

Then the mean number of defectives in ___trials is:

$$(10)(0.25) = 2.5.$$

Let X be ~ B(n, Π). Then E(X)=nΠ :

$\Rightarrow$ Probability of success = Π

$\Rightarrow$ number of successes expected from a single trial is:

$$\{1(\Pi) + 0(1 - \Pi)\} = \Pi$$

So, in "n" trials, expected $n\Pi$ "successes."

Binomial mean: E(X $_{\text{Binomial}}$)=_____

Proof:

$$E(X) = \sum_{x=0}^{n} xP(x)$$

$$= \sum_{x=0}^{n} x \cdot nCx(\Pi)^{x}(1-\Pi)^{n-x}$$

$$= \sum_{x=1}^{n} (x) \cdot \frac{n!}{x!(n-x)!} \Pi^{x}(1-\Pi)^{n-x}$$

$$= \sum_{x=1}^{n} \frac{n!}{(x-1)!(n-x)!} \Pi^{x}(1-\Pi)^{n-x}$$

$$= n\Pi \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!(n-1-(x-1))!} \Pi^{x-1}(1-\Pi)^{(n-1)-(x-1)}$$

$$= (n\Pi)\sum_{y=0}^{m} \underbrace{\left[\frac{m!}{y!(m-y)!}\right]\Pi^{y}(1-\Pi)^{m-y}}_{Y=B(m,\Pi)}$$

$[y = x-1; \quad m = n-1]$

So,

$E(x) = (n\Pi)(1) = n\Pi$

Similarly, we can show that V(X)=________

So the standard deviation of X: $\boxed{\sqrt{n\Pi(1-\Pi)}}$

when X~ B(n, Π).

Example: If a manufacturing process is working properly, 10% of the items produced will be defective. We take a random sample of 15 items. Calculate the _________ number of defectives:

X = defective ~ B(15, 0.1)

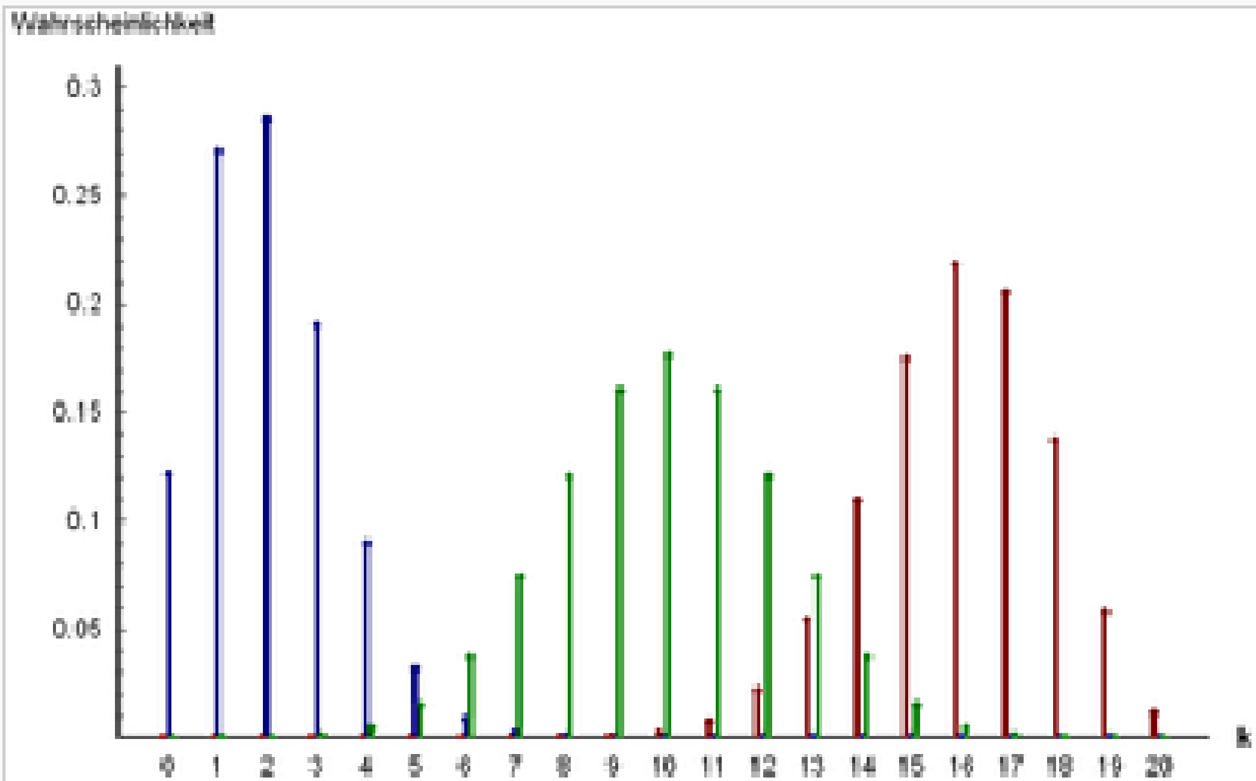
So, $E(x) = n\pi = (15)(0.1) =$ **1.5**

One or two defectives expected.

What is the _________ deviation of the number of defectives?

$V(X) = n\pi(1-\pi) = (15)(0.1)(0.9) =$ **1.35**

So, the standard deviation (X)= **1.16**



Binomial distribution for $n = 20$
$p = 0.1$ (blue), $p = 0.5$ (green) and $p = 0.8$ (red)