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Coefficient Sign Changes when Restricting Regression Models Under Instrumental Variables Estimation*

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It is now well known that deleting a variable from (or otherwise restricting the coefficients of) a least squares regression model has implications for the signs of the remaining variables' coefficients which are predictable under certain conditions. This topic was discussed by Leamer (1975) who showed, among other things, that if the absolute t-value associated with the deleted variable is less than that associated with another variable in the model, then the sign of the latter's estimated coefficient cannot change as a result of the deletion. Leamer's necessary condition was extended to include a sufficient condition by Visco (1978); was presented in a simple alternative form by Oksanen (1987); and was generalized by McAleer *et al.* (1986) to apply to situations where the deleted variables are combined in arbitrary linear combinations (such as in certain distributed lag models). See, also, Visco (1988).

The results obtained by these authors are extremely useful to the applied econometrician — certain possibilities may be ruled out, *a priori*, simply by inspecting the values of appropriate t-values or F-statistics. However, all of these results are derived only in the context of Ordinary Least Squares (O.L.S.) estimation. Given the widespread use of estimators in the Instrumental Variables (I.V.) family, it is interesting to ask whether similar results hold in this more general context. If so, then prescriptions could be made to cover the estimation of structural equations from simultaneous systems, as well as dynamic models with autocorrelated errors, for example.

This question is readily answered. As Oksanen (1987, p. 229) notes, the existing results relate to a problem in the *algebra* (not statistics) of least squares. The algebra of I.V. estimation can be written in a form analogous to that of O.L.S., and so the results of interest hold in the I.V. case too. To see this, let the model be

$$y = X\beta + u; \quad E(u) = 0; \quad V(u) = \sigma^2 I$$

where X is (Txk) with at least some columns stochastic, and $E(u|X) \neq 0$. Let the (Txg) instrument matrix be Z, with $g \geq k$. Assume that X and Z each have full column rank and that

$$\text{plim}(T^{-1}X'Z) = \Sigma_{xz}; \text{plim}(T^{-1}Z'Z) = \Sigma_{zz}$$

*Darren Gibbs asked some interesting questions which led the author to prepare this note.

where Σ_{xz} and Σ_{zz} are both finite and of rank k and g respectively. By the Mann-Wald Theorem, $(T^{-1/2}Z'u)^d \to N(0, \sigma^2\Sigma_{zz})$.

The general I.V. estimator of β is

$$\hat{\beta} = (X'MX)^{-1}X'My,$$

where $M = Z(Z'Z)^{-1}Z'$, and is idempotent. The usual consistent estimator of the asymptotic covariance matrix of $\hat{\beta}$ is $V = s^2(X'MX)^{-1}$, where $s^2 = (y - X\hat{\beta})'(y - X\hat{\beta})/T = \hat{u}'\hat{u}/T$.

If q exact linear independent restrictions are placed on the elements of β, we have $R\beta = r$, where R is (qxk) and of rank q. The restricted I.V. estimator is (see, e.g., Giles, 1982)

$$\hat{\beta}^* = \hat{\beta} - (X'MX)^{-1}R'[R(X'MX)^{-1}R']^{-1}(R\hat{\beta} - r),$$

and the validity of the restrictions may be tested via the Wald (and Lagrange Multiplier) test statistic

$$W = (R\hat{\beta} - r)'[R(X'MX)^{-1}R']^{-1}(R\hat{\beta} - r)/s^2.$$

Under the null, W is asymptotically χ^2_q (see, e.g., Giles, 1982).

Now, because M is idempotent, defining $X^* = MX$, we have:

$$\begin{aligned}
\hat{\beta} &= (X^{*\prime}X^*)^{-1}X^{*\prime}y \\
\hat{\beta}^* &= \hat{\beta} - (X^{*\prime}X^*)^{-1}R'[R(X^{*\prime}X^*)^{-1}R']^{-1}(R\hat{\beta} - r) \\
V &= s^2(X^{*\prime}X^*)^{-1} \\
W &= (R\hat{\beta} - r)'[R(X^{*\prime}X^*)^{-1}R']^{-1}(R\hat{\beta} - r)/s^2 \\
&= (\hat{u}^{*\prime}\hat{u}^* - \hat{u}'\hat{u})/s^2,
\end{aligned}$$

where $\hat{u}^* = y - X\hat{\beta}^*$.

So, in this transformed space the algebra of I.V. estimation is the same as that of O.L.S. estimation in the original space. Accordingly, the results of all of the authors referred to at the beginning of this note hold precisely if I.V., rather than O.L.S. estimation is used. For example, the 'asymptotic-t-values' reported with I.V. estimates are of the form $t_i = (\hat{\beta}_i / V_{ii}^{1/2})$ for the i'th element of β. Although t_i is asymptotically standard normal, rather than t-distributed, its algebraic structure is identical to that of its least squares counterpart. Accordingly, the proof of the Lemma given by Leamer (1975, p. 38) follows exactly in this case: there cannot be a sign change for a regression coefficient estimated by I.V. when the absolute 'asymptotic t-value' for that coefficient exceeds that of the coefficient for the deleted variable.

The algebra of I.V. estimation, rather than any statistical assumptions, facilitate this generalization of the results. Given the widespread application of I.V. estimators in econometrics, this observation should be of assistance to applied researchers.

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