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THE ALMON ESTIMATOR:

METHODOLOGY AND USERS' GUIDE

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INTRODUCTION

The purposes of this paper are to survey and index recent contributions relating to the methodology of the "Almon Polynomial Approximation Estimator", and to provide guidance to users of computer programs related to this estimator. Giles [9] covers developments in the field of distributed lags up to October 1973. This paper incorporates his section on "Finite Lag Models" as well as subsequent developments.

PART I - METHODOLOGY

THE ALMON ESTIMATOR

Consider a simple distributed lag of (finite) length $n + 1$,

$$y_t = \sum_{i=o}^{n} \beta_i x_{t-i} + u_t , \quad t = 1, 2, \ldots, T \qquad (1)$$

where it is desired to estimate the coefficients (or lag weights) β_i; $i = 0,1,\ldots,n$.

When n is relatively small and successive observations are not collinear, these coefficients can be estimated directly by least squares. However, when n is large and/or successive observations too collinear it becomes necessary to impose certain restrictions on the coefficients, for otherwise the individual coefficient estimates may be very imprecise.

Almon [1] suggested that the β_i be restricted to lie on a polynomial of degree $P \leq n$.

Thus,

$$\beta_i = \sum_{j=o}^{P} a_j i^j = f(i) \; ; \quad i = 0,1,\ldots,n. \qquad (2)$$

where the polynomial degree, P, is given. Then by estimating the values taken by the polynomial at (P + 1) arbitrary points in the interval

An alternative form of the Almon technique (the "Direct Method") avoids the use of Lagrangian interpolation and is described by Fair and Jaffee [7], Robinson [19[and Cooper [6], for example.

The "Direct Method" combines (1) and (2) to give

$$y_t = \sum_{j=o}^{P} a_j \left(\sum_{i=o}^{n} i^j x_{t-i}\right) + u_t \qquad (3)$$

or

$$y_t = \sum_{j=o}^{P} a_j z_{jt} + u_t \qquad (4)$$

where

$$z_{jt} = \sum_{i=o}^{n} i^j x_{t-i}$$

Then, given n and P, the z_{jt} are constructed, and the $\hat{a}_j$ are obtained by applying O.L.S. to (4). The $\hat{\beta}_i$ are obtained as

$$\hat{\beta}_i = \sum_{j=o}^{P} \hat{a}_j i^j \; ; \quad i = 0,1,\ldots,n.$$

if we let,

$$A = (a_1, a_2, \ldots, a_p)$$

$$Z = [z_{jt}]$$

and $k_i = [1, i, i^2, \ldots, i^P]$

then, $\text{Var}(A) = \sigma^2 (Z'Z)^{-1}$

and $\text{Var}(\beta_i) = k_i \text{Var}(A) k'_i = \sigma^2 k_i (Z'Z)^{-1} k'_i$

Cooper demonstrates that the two approaches are algebraically identical, but he points out that the direct method is more likely to be hampered by multicollinearity in the Artificial Variables (z_{jt}) than is the Lagrangian interpolation variant. A small value of P reduces the imprecision arising from multicollinearity, since it reduces the number of regressors in the least squares regression, (4).

In both forms of the Almon technique, further linear restrictions may be placed on the value and/or slope of the polynomial at any point in [0, n]. However, it is common practice that these constraints be placed at the end points f(0) and/or f(n). This point is discussed in detail by Fair and Jaffee. Almon suggested that the restrictions f(-1) = f(n + 1) = 0, always be imposed (since $\beta_{-1} = \beta_{n+1} = 0$, from the specification of equation (1)), but these restrictions have been condemned as irrelevant (e.g. Schmidt and Waud [21]) since only the behaviour of the polynomial in the range [0, n] is of interest.

SPECIFICATION OF THE DISTRIBUTED LAG MODEL

Specification of the complete model entails

(1) choosing the correct lag length, n

(2) finding the appropriate polynomial order, P

(3) deciding which end-point restrictions (if any) are to be incorporated into the model.

Since *a priori* information is generally weak, the choice of n, P and end-point restrictions may not be immediately obvious. For this reason model selection is often based on the "Residual Variance Criterion" ($\bar{R}^2$ rule) applied to a small set of possible specifications. Giles and Smith [13] have shown that this criterion is valid for the Almon estimator and that it holds asymptotically when the errors are autocorrelated.

As is noted by Giles [11], one danger in using this rule is that it is very tempting to test a large number of alternative specifications and choose the one which looks "most sensible" *a posteriori*. He points out that such knowledge is *a priori* information and should be treated as such. The literature emphasises that caution must be exercised when applying classical methods to specify a lag model, especially if end-point restrictions are involved. See

Cohen et al. [5], Schmidt and Waud, and Trivedi [24].

Cohen et al provide evidence to show that the $\bar{R}^2$ criterion and the t-statistics[1] may be of little use for discrimination among alternative specifications due to pre-testing biases. Frost [8] suggests that, if one uses the $\bar{R}^2$ criterion, the estimated coefficients in the resulting specification will be biased. Furthermore, their distribution will deviate significantly from the normal distribution and their estimated variances will be biased downwards.

Godfrey and Poskitt [14] present a different approach for testing the restrictions imposed by the Almon estimator. Their method, which they show is equivalent to the likelihood ratio test, requires only that the unrestricted form be estimated. They assume that n is known which allows them to use their technique to select the correct degree of the polynomial. The "optimal" order of the polynomial is also considered by Amemiya and Morimune [3] and Schmidt and Sickles [20]. Following the restrictive assumption that the independent variables follow a first order autoregressive process with a varying correlation coefficient, Amemiya and Morimune attempt to find the order of polynomial which minimises some loss function (they assume n is known). The loss function they consider is the trace of the product of the mean square error matrix and the autocovariance matrix of the independent variable. An empirical study suggests that the "optimal" polynomial degree will be lower as: the lag distribution is smoother, collinearity greater, sample size smaller, and the ratio of the error variance to the variance of the dependent variable is greater.

Schmidt and Sickles apply the "weak" mean square error criterion that Amemiya and Morimune use to investigate the efficiency of the Almon estimator. Since the Almon restricted estimator has a

1. The choice of lag length can not be made on the basis of t-tests, since these tests are invalid unless the correct lag length is chosen. See Schmidt and Waud.

smaller variance than the O.L.S. estimator they are interested in comparin the unbiased O.L.S. estimator with the possibly biased Almon estimator (i.e. when the coefficients do not lie exactly on the polynomial or if other imposed restrictions are false). To do this they compare the "weak" mean square errors of the two estimators and find that the difference depends on the closeness of fit of the polynomial (as well as on P). They also attempt to find the "optimal" degree of the polynomial in a way similar to that of Amemiya and Morimune.

As an alternative to classical statistical inference, Bayesian techniques appear to offer scope for dealing with prior information (Zellner and Williams [25] and Giles [12]) and model selection. The fact that meaningful probability statements may be attached to alternative specifications makes the Bayesian methodology especially appealing. Further discussion on related Bayesian methods is contained in Chetty [4], Leamer [15], [16] and Maddala [17].

FURTHER ISSUES

Two adverse features of the polynomial estimation procedure are that it has the tendency to "smear" or "spread" the effect of the lag back over preceding time periods, and that the presence of a lag is not a testable hypothesis in this context.

Giles [10] shows that in the special case of exactly P independent restrictions on the coefficients, the t-statistics for each $\hat{\beta}_i$ are identical and the same as that for $\hat{a}_j$, where $\hat{a}_j$ is the estimate o the only Almon coefficient not eliminated by the restrictions. He also points out that the "Average Lag" is "superfluous" in this case, since it is independent of the estimates and is determined solely by the lag length, n.

The possibility of a lag structure which varies not only over time but also with other variables is discussed by Tinsley [23], Almon [2], Pesando [18] and Tanner [22]. Pesando demonstrates how the "variable

CONCLUSION

Although the Almon technique appears very useful for estimating distributed lags, users must not overlook special problems which may arise when it is used. Because the optimal lag length can not be found using t-tests, many of the methods mentioned above for selecting the optimal order of polynomial are of little practical use (since they require that the correct lag length be known). Bearing this in mind it seems that the most sensible approach for specifying an Almon model is to use the $\bar{R}^2$ rule. However, problems may arise due to pre-testing biases which suggests that extreme care must be taken if this rule is to be used.

PART II - USERS' GUIDE

This section discusses Regression Strategy and provides some mathematical analysis of equations and test statistics generated by the Almon method. An appendix contains illustrations and equations of second, third and fourth degree polynomials. Specifications for the computer program will follow in a separate note.

MODEL SPECIFICATION

The Almon technique reduces the number of coefficients to be estimated from n + 1 lag weights to P + 1 "Almon coefficients"; $P \leqslant n$. Placing end-point restrictions on the polynomial further reduces the number of Almon coefficients - one coefficient for each independent linear restriction. Hence model specification involves:

1. finding the correct lag length, n
2. choosing the optimal degree of polynomial, P.
3. imposing appropriate end-point restrictions, R.

n is rarely known *a priori* and cannot be determined using t-tests, hence many of the techniques discussed in the literature for choosing the "optimal" degree of polynomial (see Amemiya and Morimune)

For this reason, a sensible strategy to adopt is to consider a small set of possible combinations of n, P and R based on economic theory and then to choose the "best" by applying the residual variance criterion ($\bar{R}^2$ rule). (Note that pre-testing biases may result when this rule is used - see Part I.)

THE CHOICE OF LAG LENGTH

As pointed out above, the lag length cannot be chosen on the basis of t-tests, since these tests are invalid when the lag length has not been correctly chosen. For this reason the choice of n must be made on prior economic information. By carefully analysing the lag in question and applying sound economic reasoning it should be possible to arrive at a few possible values for n, one of which is hopefully correct.

POLYNOMIAL DEGREE AND END-POINT RESTRICTIONS

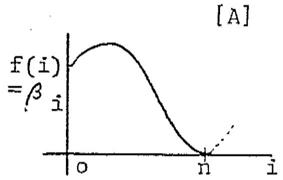
The choice of a small set of eligible shapes must again be subject to sound economic reasoning. It is stressed that it is not appropriate to randomly test many different polynomial degrees and restrictions then pick the combination with the highest $\bar{R}^2$. If a polynomial shape is to be included in the eligible set there must exist strong economic justification for its inclusion.

The normal procedure for choosing P and R would entail analysing the lag to isolate various characteristics then choosing a few shapes from the appendix to this paper (or elsewhere) which best represent the lag.

Some of the common characteristics to look for are listed below:

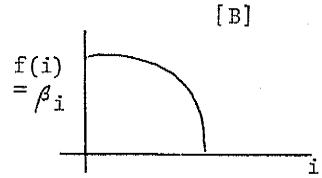
(1) Does the lag "tail off" slowly [A] or quickly [B]?

[A]



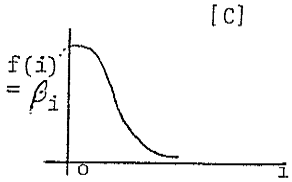
Restriction $f'(n) = 0$

[B]



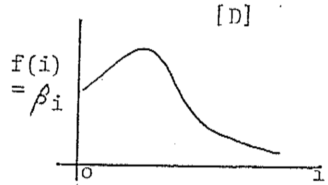
(2) In what position is the largest lag weight expected to lie, i.e. at i = 0 [C] or elsewhere [D]?

[C]



Restriction $f'(0) = 0$

[D]



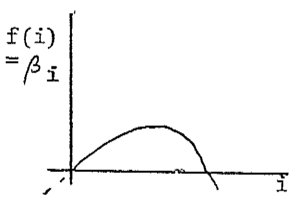
(3) What is the expected value of the polynomial (β_i) at i = 0, i = n, i.e. if $\beta_o = 0$ then $f(0) = 0$

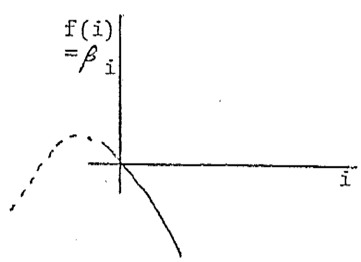
$\beta_n = 0$ $\quad f(n) = 0$

Each restricted polynomial (a given combination of P and R) may assume many different shapes in the interval [0, n] depending upon which particular lag is being estimated. When estimating a given specification users should be careful to observe which shape has been generated to avoid using one which doesn't make "economic sense". Different shapes arise depending on which part(s) of the polynomial obeys the restriction(s) [E], and because each shape has a "mirror image" with respect to the horizontal axis [F].

[E] P = 2

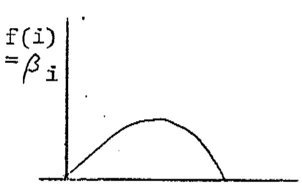
$\beta_o = 0$

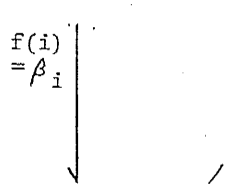




[F] P = 2

$\beta_o = 0$





MATHEMATICAL ANALYSIS AND TEST STATISTICS[2].

Consider,

$$y_t = \sum_{i=o}^{n} \beta_i x_{t-i} + u_t$$

which has T - n - 1 degrees of freedom (T observations). Restricting the coefficients to lie on a polynomial of degree P increases the number of degrees of freedom by [(n + 1) - (P + 1)] to (T - P - 1).

If,

$$\beta_i = \sum_{j=o}^{P} a_j i^j, \text{ where } 0^o = 1$$

then,

$$y_t = \sum_{j=o}^{P} a_j z_{jt} + u_t \qquad (5)$$

At this stage (P + 1) coefficients require estimation. When end-point restrictions are imposed on the polynomial they are incorporated directly into equation (5) by eliminating the appropriate Almon coefficients (a_j) and transforming the z_{jt} matrix. Imposing R end-point restrictions yields the final Almon specification with T + R - P - 1 degrees of freedom.

Example $$y_t = \sum_{i=o}^{n} \beta_i x_{t-i} + u_t$$

subject to $\beta_i = a_o + a_1 i + a_2 i^2$

and, $\beta_n = 0$

Thus,

$$a_o + a_1 n + a_2 n^2 = 0$$

or, $$a_o = - n(a_1 + a_2 n)$$

Hence, $\beta_i = -n(a_1 + a_2 n) + a_1 i + a_2 i^2$

$$= a_1(i - n) + a_2(i^2 - n^2)$$

$$= A^* k_i^{*\prime}$$

where A^* is a (P + 1 - R) vector consisting of the Almon coefficients <u>not</u> eliminated by the R end-point restrictions and k_i^* is a (P + 1 - R) vector of transformed polynomial variables.

The $\hat{\beta}_i$ are obtained from

$$\hat{\beta}_i = \hat{a}_1(i - n) + \hat{a}_2(i^2 - n^2)$$

Now,

$$y_t = a_o z_{ot} + a_1 z_{1t} + a_2 z_{2t} + u_t$$

$$= -(a_1 + a_2 n)n z_{ot} + a_1 z_{1t} + a_2 z_{2t} + u_t$$

$$= a_1 (z_{1t} - n z_{ot}) + a_2 (z_{2t} - n^2 z_{ot}) + u_t$$

$$= a_1 z_{1t}^* + a_2 z_{2t}^* + u_t$$

and,

$$Y = Z^* A^* + U$$

where Z^* is a Tx(P + 1 - R) matrix of observations on the (P + 1 - R) transformed z_{jt} variables.

Further,

$$VAR(A^*) = \sigma^2 (Z^{*\prime} Z^*)^{-1}$$

and,

$$VAR(\beta_i) = k_i^* \, VAR(A^*) \, k_i^{*\prime}$$

$$= \sigma^2 k_i^* (Z^{*\prime} Z^*)^{-1} k_i^{*\prime} \qquad (6)$$

To estimate var($\hat{\beta}_i$) replace σ^2 by $\hat{\sigma}^2$ in (6) where,

$$\hat{\sigma}^2 = (\hat{U}'\hat{U})/(T + R - P - 1)$$

TESTING THE RESTRICTIONS

(1) Given that the true lag length is used, the following statistic may be used to test the hypothesis that the restrictions imposed by the Almon estimator are correct.

$$F = [(\hat{U}'\hat{U} - \tilde{U}'\tilde{U})(T - n - 1)/(\tilde{U}'\tilde{U})(n - P + R)]$$

is F-distributed with (n - P + R) and (T - n - 1) degrees of freedom, where $\tilde{U}$ is the vector of O.L.S. residuals. Note that this statistic assumes that n is correct, so when n has been mis-specified the statistic is invalid.

(2) The method for testing the restrictions put forward by Godfrey and Poskitt and discussed in the methodology only requires that the unrestricted form be estimated.

TESTING THE COEFFICIENTS

Given that the Almon restrictions comply with the lag and that n is correct, the following statistics may be used to test the significance of the lag weights.

(1) The t-test

Let a^*_{ii} be the ith diagonal element of $k^*_i (Z^{*'}Z^*)^{-1} k^{*'}_i$

then,

$$\text{s.e.}(\hat{\beta}_i) = \hat{\sigma} a^*_{ii}$$

Now,

$t_i = (\hat{\beta}_i - \bar{\beta}_i)/\text{s.e.}(\hat{\beta}_i)$ is t-distributed with (T + R - P - 1) degrees of freedom and may be used to test the hypothesis that $\beta_i = \bar{\beta}_i$, where $\bar{\beta}_i$ is known (e.g. $\bar{\beta}_i = 0$).

(When the residual variance criterion is used for model selection, it is possible that one or more lag weights are insignificant. This does not mean that those lag weights should be eliminated because the test assumes that the optimal lag length has been used.)

(2) The F-test

Given that n is chosen correctly and the Almon restrictions are appropriate, the hypothesis that $\beta = \bar{\beta}$, ($\bar{\beta}$ known), may be tested using the F-test. It can be shown that

$$F^* = [(\hat{\beta} - \bar{\beta})' Z^{*'} Z^* (\hat{\beta} - \bar{\beta})](T + R - P - 1)/[\hat{U}'\hat{U}(P + 1 - R)]$$

is F-distributed with $(P + 1 - R)$ and $(T + R - P - 1)$ degrees of freedom.

SERIAL CORRELATION

The Durbin-Watson and Wallis statistics may be applied to the residuals resulting from the Almon estimator to test for autocorrelated error terms.

$$D.W. = \sum_{t=2}^{n} (u_t - u_{t-1})^2 / \sum_{t=1}^{n} u_t^2$$

$$d_4 = \sum_{t=5}^{n} (u_t - u_{t-4})^2 / \sum_{t=1}^{n} u_t^2$$

When consulting the tables note that $k' = (P - R)$ instead of $(k - 1)$ as is usually the case, and that the tabulated critical values are invalid when the regression does not contain an intercept, if there is a lagged dependent variable as a regressor or if any of the restrictions, etc. are incorrect.

GOODNESS OF FIT

The multiple correlation coefficient associated with the Almon estimator is given by

$$R_A^2 = 1 - [(\hat{U}'\hat{U})/(Y'Y - (\tfrac{1}{T})(\sum_i y_i)^2)]$$

and the "adjusted" R_A^2 ($\bar{R}_A^2$) is given by

$$(1 - \bar{R}_A^2) = (T - 1)(1 - R_A^2)/(T + R - P - 1).$$

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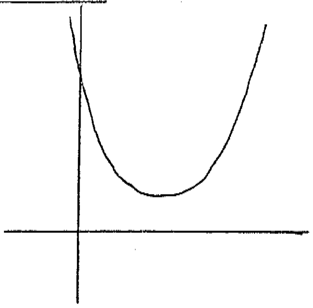
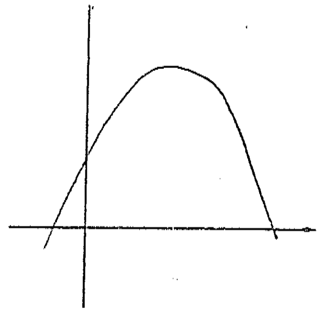
APPENDIX

Second, third and fourth order polynomials are illustrated by the following diagrams (mirror images are given only for second degree polynomials). The sets of transformed equations which follow the diagrams are given in the same order as the illustrations to which they correspond.

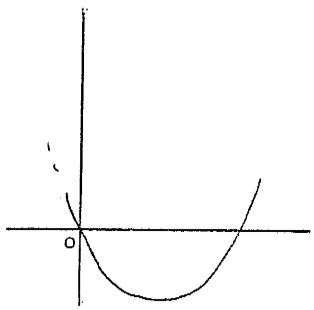
SECOND DEGREE POLYNOMIALS

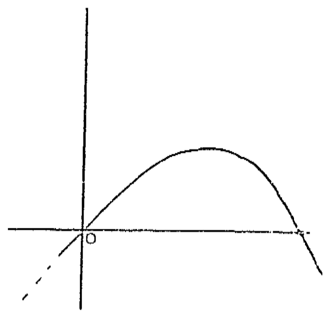
.. Unrestricted



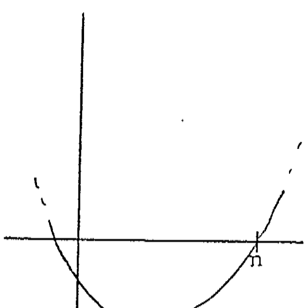


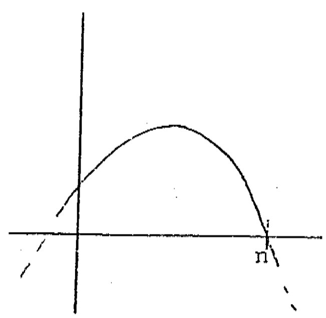
. f(0) = 0



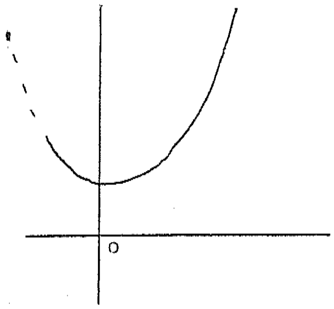


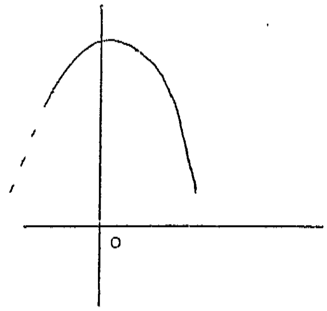
f(n) = 0



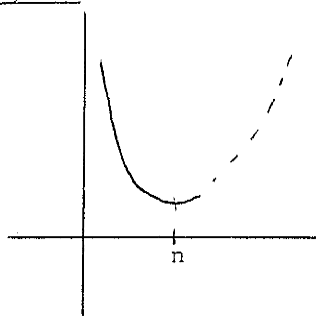


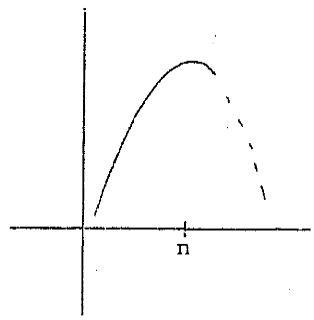
. $f'(0) = 0$



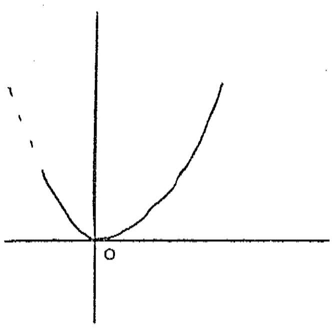


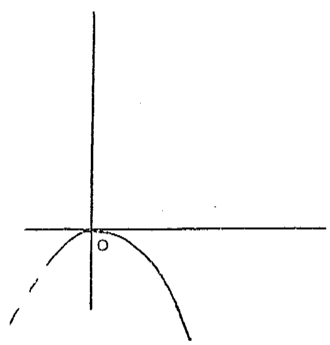
. $f'(n) = 0$



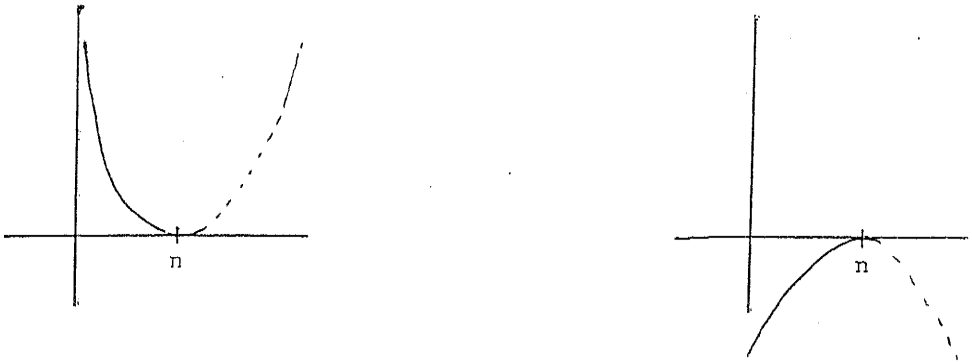


$f(0) = f'(0) = 0$





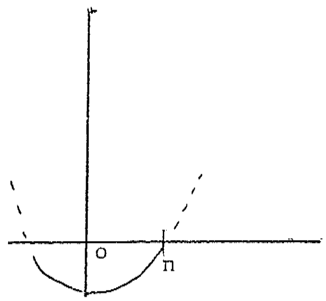
$\underline{f(n) = f'(n) = 0}$

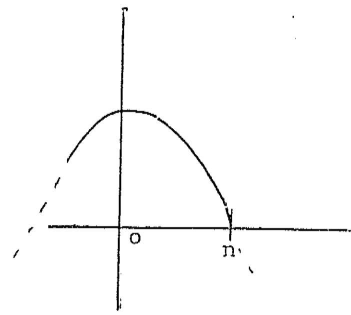


$\underline{f(0) = f(n) = 0}$

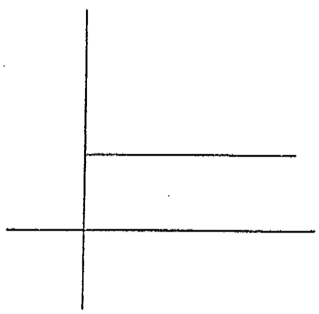
$\underline{f(0) = f'(n) = 0}$

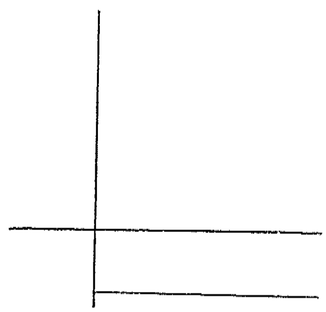
10. f(0) = f (n) = 0



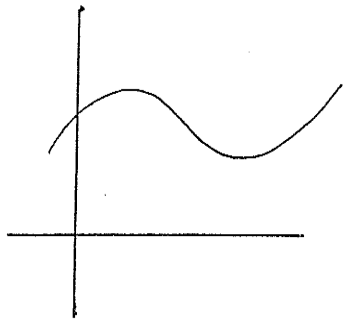


11. f'(0) = f'(n) = 0

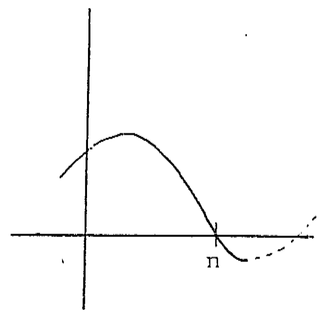




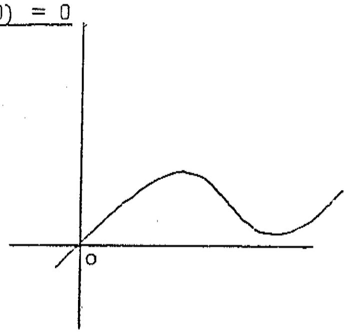
12. Unrestricted



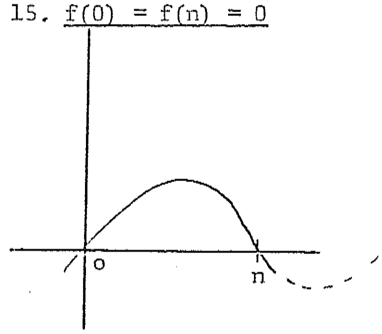
13. f(n) = 0



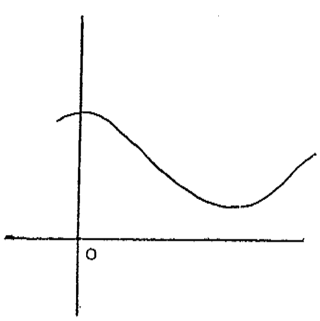
4. f(0) = 0



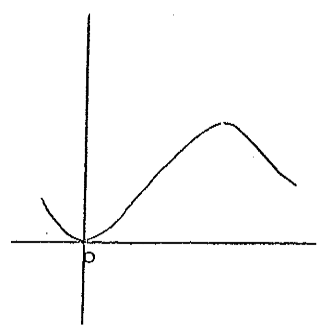
15. f(0) = f(n) = 0



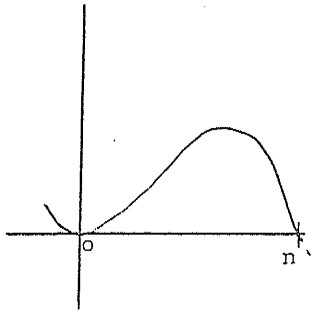
6. f'(0) = 0



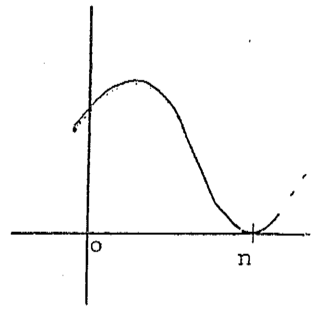
17. f(0) = f'(0) = 0



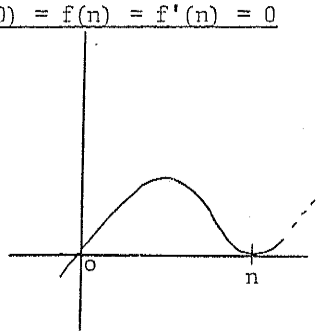
18. f(0) = f'(0) = f(n) = 0



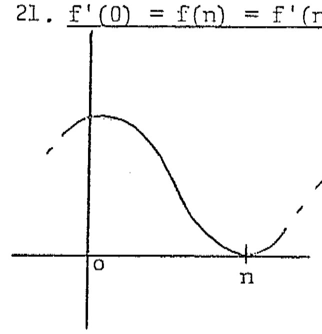
19. f(n) = f'(n) = 0



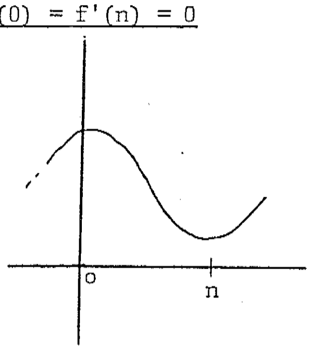
'0. f(0) = f(n) = f'(n) = 0



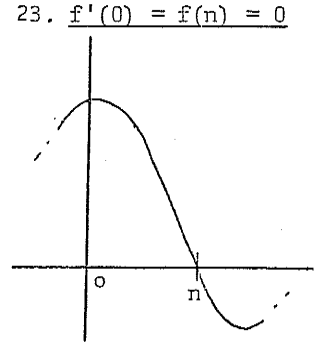
21. f'(0) = f(n) = f'(n) = 0



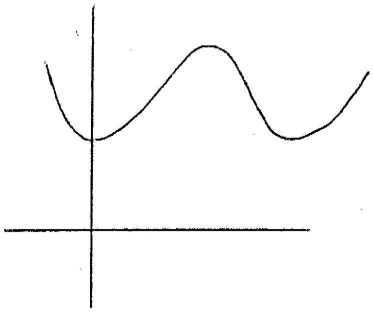
2. f'(0) = f'(n) = 0



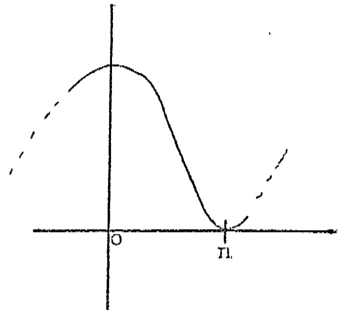
23. f'(0) = f(n) = 0



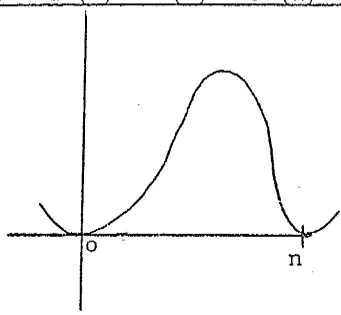
4. Unrestricted



25. $f'(0) = f(n) = f'(n) = 0$



. $f(0) = f'(0) = f(n) = f'(n) = 0$



FORMULAE

1. $$y_t = a_o z_{ot} + a_1 z_{1t} + a_2 z_{2t}$$
$$\hat{\beta}_i = \hat{a}_o + \hat{a}_1 i + \hat{a}_2 i^2$$
$$k_i^* = (1, i, i^2)$$
$$Z_t^* = (z_{ot}, z_{1t}, z_{2t})$$

2. $$y_t = a_1 z_{1t} + a_2 z_{2t}$$
$$\hat{\beta}_i = \hat{a}_1 i + \hat{a}_2 i^2$$
$$k_i^* = (i, i^2)$$
$$Z_t^* = (z_{1t}, z_{2t})$$

3. $$y_t = a_1 (z_{1t} - n z_{ot}) + a_2 (z_{2t} - n^2 z_{ot})$$
$$\hat{\beta}_i = \hat{a}_1 (i-n) + \hat{a}_2 (i^2 - n^2)$$
$$k_i^* = (i-n, i^2-n^2)$$
$$Z_t^* = (z_{1t} - n z_{ot}, z_{2t} - n^2 z_{ot})$$

4. $$y_t = a_o z_{ot} + a_2 z_{2t}$$
$$\hat{\beta}_i = \hat{a}_o + \hat{a}_2 i^2$$
$$k_i^* = (1, i^2)$$
$$Z_t^* = (z_{ot}, z_{2t})$$

5. $$y_t = a_o z_{ot} + a_2 (z_{2t} - 2n z_{1t})$$
$$\hat{\beta}_i = \hat{a}_o + \hat{a}_2 (i^2 - 2ni)$$
$$k_i^* = (1, i^2 - 2ni)$$
$$Z_t^* = (z_{ot}, z_{2t} - 2n z_{1t})$$

6. $y_t = a_2 z_{2t}$

$\hat{\beta}_i = \hat{a}_2 i^2$

$k_i^* = (i^2)$

$Z_t^* = (z_{2t})$

7. $y_t = a_2 (z_{2t} - 2nz_{1t} + n^2 z_{ot})$

$\hat{\beta}_i = \hat{a}_2 (i^2 - 2ni + n^2)$

$k_i^* = (i^2 - 2ni + n^2)$

$Z_t^* = (z_{2t} - 2nz_{1t} + n^2 z_{ot})$

8. $y_t = a_2 (z_{2t} - nz_{1t})$

$\hat{\beta}_i = \hat{a}_2 (i^2 - ni)$

$k_i^* = (i^2 - ni)$

$Z_t^* = (z_{2t} - nz_{1t})$

9. $y_t = a_2 (z_{2t} - 2nz_{1t})$

$\hat{\beta}_i = \hat{a}_2 (i^2 - 2ni)$

$k_i^* = (i^2 - 2ni)$

$Z_t^* = (z_{2t} - 2nz_{1t})$

10. $y_t = a_2 (z_{2t} - n^2 z_{ot})$

$\hat{\beta}_i = \hat{a}_2 (i^2 - n^2)$

$k_i^* = (i^2 - n^2)$

$Z_t^* = (z_{2t} - n^2 z_{ot})$

11. $y_t = a_o z_{ot}$

$\hat{\beta}_i = \hat{a}_o$

$k_i^* = (1)$

$Z_t^* = (z_{ot})$

12. $y_t = a_o z_{ot} + a_1 z_{1t} + a_2 z_{2t} + a_3 z_{3t}$

$\hat{\beta}_i = \hat{a}_o + \hat{a}_1 i + \hat{a}_2 i^2 + \hat{a}_3 i^3$

$k_i^* = (1, i, i^2, i^3)$

$Z_t^* = (z_{ot}, z_{1t}, z_{2t}, z_{3t})$

13. $y_t = a_1(z_{1t} - nz_{ot}) + a_2 (z_{2t} - n^2 z_{ot}) + a_3 (z_{3t} - n^3 z_{ot})$

$\hat{\beta}_i = \hat{a}_1 (i-n) + \hat{a}_2 (i^2-n^2) + \hat{a}_3 (i^3-n^3)$

$k_i^* = (i-n, i^2-n^2, i^3-n^3)$

$Z_t^* = (z_{1t} - nz_{ot}, z_{2t} - n^2 z_{ot}, z_{3t} - n^3 z_{ot})$

14. $y_t = a_1 z_{1t} + a_2 z_{2t} + a_3 z_{3t}$

$\hat{\beta}_i = \hat{a}_1 i + \hat{a}_2 i^2 + \hat{a}_3 i^3$

$k_i^* = (i, i^2, i^3)$

$Z_t^* = (z_{1t}, z_{2t}, z_{3t})$

15. $y_t = a_2 (z_{2t} - nz_{1t}) + a_3 (z_{3t} - n^2 z_{1t})$

$\hat{\beta}_i = \hat{a}_2 (i^2-ni) + \hat{a}_3 (i^3 - n^2 i)$

$k_i^* = (i^2-ni, i^3-n^2 i)$

$Z_t^* = (z_{2t} - nz_{1t}, z_{3t} - n^2 z_{1t})$

16. $y_t = a_o z_{ot} + a_2 z_{2t} + a_3 z_{3t}$

$\hat{\beta}_i = \hat{a}_o + \hat{a}_2 i^2 + \hat{a}_3 i^3$

$k_i^* = (1, i^2, i^3)$

$Z_t^* = (z_{ot}, z_{2t}, z_{3t})$

17. $y_t = a_2 z_{2t} + a_3 z_{3t}$

$\hat{\beta}_i = \hat{a}_2 i^2 + \hat{a}_3 i^3$

$k_i^* = (i^2, i^3)$

$Z_t^* = (z_{2t}, z_{3t})$

18. $y_t = a_3 (z_{3t} - nz_{2t})$

$\hat{\beta}_i = \hat{a}_3 (i^3 - ni^2)$

$k_i^* = (i^3 - ni^2)$

$Z_t^* = (z_{3t} - nz_{2t})$

19. $y_t = a_2 (z_{2t} - 2nz_{1t} + n^2 z_{ot}) + a_3 (z_{3t} - 3n^2 z_{1t} + 2n^3 z_{ot})$

$\hat{\beta}_i = \hat{a}_2 (i^2 - 2ni + n^2) + \hat{a}_3 (i^3 - 3n^2 i + 2n^3)$

$k_i^* = (i^2 - 2ni + n^2, i^3 - 3n^2 i + 2n^3)$

$Z_t^* = (z_{2t} - 2nz_{1t} + n^2 z_{ot}, z_{3t} - 3n^2 z_{1t} + 2n^3 z_{ot})$

20. $y_t = a_3 (z_{3t} - 2nz_{2t} + 3n^2 z_{1t})$

$\hat{\beta}_i = \hat{a}_3 (i^3 - 2ni^2 + 3n^2 i)$

$k_i^* = (i^3 - 2ni^2 + 3n^2 i)$

$Z_t^* = (z_{3t} - 2nz_{2t} + 3n^2 z_{1t})$

21. $$y_t = a_3 \left(z_{3t} - \frac{3}{2} n z_{2t} + \frac{1}{2} n^3 z_{ot}\right)$$
$$\hat{\beta}_i = \hat{a}_3 \left(i^3 - \frac{3}{2} n i^2 + \frac{1}{2} n^3\right)$$
$$k_i^* = \left(i^3 - \frac{3}{2} n i^2 + \frac{1}{2} n^3\right)$$
$$Z_t^* = \left(z_{3t} - \frac{3}{2} n z_{2t} + \frac{1}{2} n^3 z_{ot}\right)$$

22. $$y_t = a_o z_{ot} + a_3 \left(z_{3t} - \frac{3}{2} n z_{2t}\right)$$
$$\hat{\beta}_i = \hat{a}_o + \hat{a}_3 \left(i^3 - \frac{3}{2} n i^2\right)$$
$$k_i^* = \left(1,\ i^3 - \frac{3}{2} n i^2\right)$$
$$Z_t^* = \left(z_{ot},\ z_{3t} - \frac{3}{2} n z_{2t}\right)$$

23. $$y_t = a_3 \left(z_{3t} - n^3 z_{ot}\right) + a_2 \left(z_{2t} - n^2 z_{ot}\right)$$
$$\hat{\beta}_i = \hat{a}_3 \left(i^3 - n^3\right) + \hat{a}_2 \left(i^2 - n^2\right)$$
$$k_i^* = \left(i^3 - n^3,\ i^2 - n^2\right)$$
$$Z_t^* = \left(z_{3t} - n^3 z_{ot},\ z_{2t} - n^2 z_{ot}\right)$$

24. $$y_t = a_o z_{ot} + a_1 z_{1t} + a_2 z_{2t} + a_3 z_{3t} + a_4 z_{4t}$$
$$\hat{\beta}_i = \hat{a}_o + \hat{a}_1 i + \hat{a}_2 i^2 + \hat{a}_3 i^3 + \hat{a}_4 i^4$$
$$k_i^* = \left(1,\ i,\ i^2,\ i^3,\ i^4\right)$$
$$Z_t^* = \left(z_{ot},\ z_{1t},\ z_{2t},\ z_{3t},\ z_{4t}\right)$$

25. $$y_t = a_3 \left(z_{3t} - \frac{3}{2} n z_{2t} + \frac{1}{2} n^3 z_{ot}\right) + a_4 \left(z_{4t} - 2n^2 z_{2t} + n^4 z_{ot}\right)$$
$$\hat{\beta}_i = \hat{a}_3 \left(i^3 - \frac{3}{2} n i^2 + \frac{1}{2} n^3\right) + \hat{a}_4 \left(i^4 - 2n^2 i^2 + n^4\right)$$
$$k_i^* = \left(i^3 - \frac{3}{2} n i^2 + \frac{1}{2} n^3,\ i^4 - 2n^2 i^2 + n^4\right)$$
$$Z_t^* = \left(z_{3t} - \frac{3}{2} n z_{2t} + \frac{1}{2} n^3 z_{ot},\ z_{4t} - 2n^2 z_{2t} + n^4 z_{ot}\right)$$

26. $y_t = a_4 (z_{4t} - 2nz_{3t} + n^2 z_{2t})$

$\hat{\beta}_i = \hat{a}_4 (i^4 - 2ni^3 + n^2 i^2)$

$k_i^* = (i^4 - 2ni^3 + n^2 i^2)$

$Z_t^* = (z_{4t} - 2nz_{3t} + n^2 z_{2t})$