ECON 575: Advanced Topics in Econometrics
Exercises for Bayesian Econometrics

A solution sheet is available on request. (This gives you the opportunity to attempt these questions without seeing the answers.)

Question 1

Let x be a (continuous) random variable that follows a (standard) Gamma Distribution. Then, its p.d.f. is:

$$p(x) = x^{\gamma-1}e^{-x} / \Gamma(\gamma) \quad ; \qquad x > 0\ ; \quad \gamma > 0$$

where $\Gamma(.)$ is the usual Gamma function:

$$\Gamma(z) = \int_0^{\infty} t^{z-1}e^{-t}dt\ ,$$

which satisfies the recursion relationship:

$$\Gamma(z+1) = z\Gamma(z)\ .$$

The parameter γ is the shape parameter for this distribution. It can be shown that $E(x) = \gamma$. Some examples of this density function appear below.

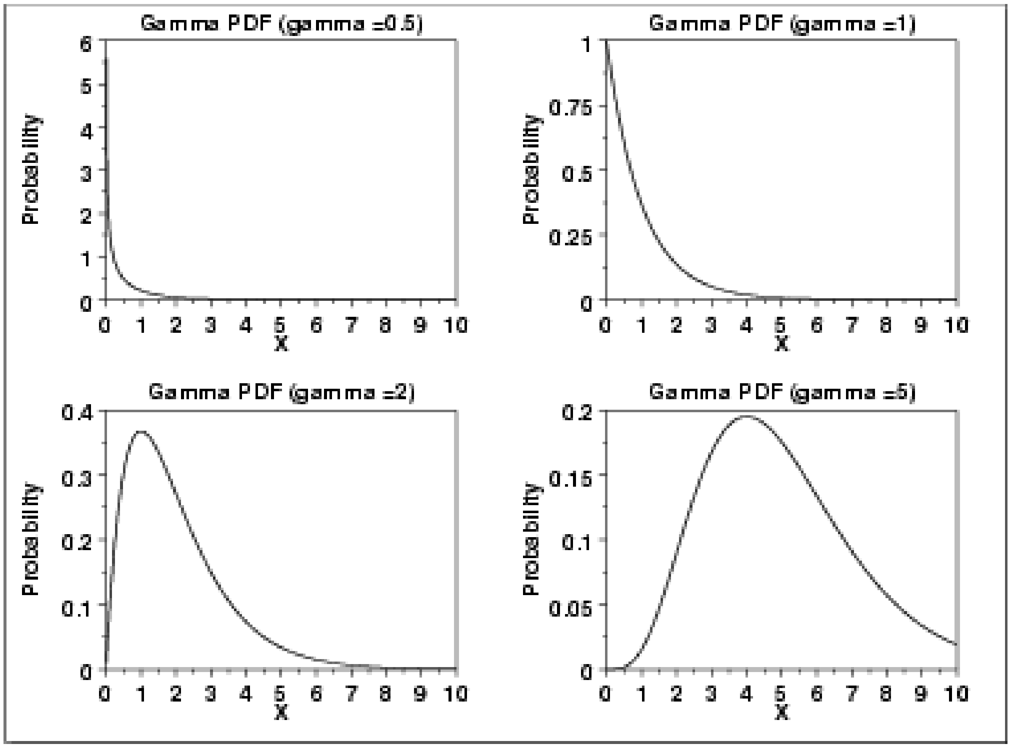
Let y be a (discrete) random variable that follows a Poisson Distribution. Then, its p.m.f. is:

$$p(y) = e^{-\lambda}\lambda^{x} / x! \qquad ; \qquad x = 0, 1, 2, \ldots\ldots; \qquad \lambda > 0.$$

(The parameter λ is both the mean and the variance of this distribution.)

(a) Now, suppose we have a random sample of n independent observations from a Poisson Distribution, and that we wish to estimate λ. Taking a Bayesian approach, we need to specify a prior p.d.f. for this parameter. Prove that the Gamma prior is the natural conjugate prior in this case.

(b) What is the Bayes estimator for λ, first under a quadratic loss function; and second under a ‘zero-one’ loss function? Are either of these Bayes estimators unbiased?



Question 2

Suppose that we have a random sample of n non-negative observations from an exponential distribution with mean $(1 / \theta)$. So, the p.d.f. for the i^{th} observation is:

$$p(y_i) = \theta \exp\{-\theta\, y_i\} \quad .$$

Suppose also that the prior for θ is a gamma density, with parameters α and β, and mean equal to $(\alpha\beta)$, and mode at $[(\alpha - 1)\beta]$, if $\alpha > 1$:

$$p(\theta) = \theta^{\alpha-1} e^{-(\theta/\beta)} / [\beta^{\alpha}\Gamma(\alpha)] \quad .$$

(a) Prove that the posterior for θ is a gamma density, with parameters $(\alpha + n)$ and $(\beta^{-1} + \Sigma y_i)^{-1}$.

(b) What is the Bayes estimator of θ under a quadratic loss function?

(c) What is the Bayes estimator of θ under a "zero – one" loss function?

Question 3

Consider the Natural Conjugate Bayes estimator of β in the standard Normal multiple linear regression model, under a quadratic loss function.

(a) Show that this estimator is biased.

(b) Why does this really not matter to a pure Bayesian econometrician?

(c) If the conditional prior covariance matrix for β were chosen to be equal to the covariance matrix for the Maximum Likelihood estimator of β in this model, show that the expected value of the Bayes estimator of β is a simple average of the conditional prior mean for β, and β itself.

(d) Why would the choice of prior covariance matrix in (c) really be a "non-Bayesian" choice?

Question 4

Suppose that we have a sample of '*n*' random observations from a normal population whose mean is ***known***, but whose variance is unknown. We wish to estimate the latter parameter using Bayesian inference.

(a) Show that the natural-conjugate prior for the "precision", $\tau = \sigma^{-2}$, is a Gamma density.

(b) Give the formulae for the Bayes estimator of this precision parameter under both quadratic and zero-one loss functions.

(c) What would be a consistent estimator for the variance itself under each of these loss functions?

Question 5

(a) Suppose that we have n = 10 observations drawn randomly from a population with an unknown mean, μ, and known variance of unity. If the sample mean is 1.5, calculate the posterior odds relating to the hypotheses H_1: $\mu = 1.0$ and H_2: $\mu = 2.0$ when the prior probability for each hypothesis is 0.5.

(b) Consider the following loss structure:

		State of the World	
		H_1 True	*H_2 True*
Action	*Accept H_1*	0	4
	Accept H_2	2	0

Which hypothesis would you choose?

Question 6

You need to know the following result. If a random variable, *X*, follows a **Gamma Distribution**, then its p.d.f. is:

$$p(x) \propto x^{\alpha-1}e^{-x/\gamma} \quad ; \quad \alpha, \gamma > 0 \,; \quad x > 0$$

This density is skewed to the right, it has a single mode at $x = \gamma(\alpha - 1)$, provided $\alpha \geq 1$, and $E(X) = \alpha\gamma$.

Now, consider the following estimation problem. We have '*n*' independent observations from a **Pareto distribution**, whose p.d.f. is:

$$p(y_i) = (\theta A^{\theta})/(y_i^{\theta+1}) \quad ; \quad 0 < A < y_i < \infty \quad ; \quad 0 < \theta < \infty \quad ; \quad i = 1, 2, \ldots, n.$$

(a) Suppose we are ignorant, *a priori*, about θ. Show that the posterior p.d.f. is $p(\theta \mid y) \propto \theta^{n-1}(A/G)^{n\theta}$, where *G* is the geometric mean of the sample data.

(b) What is the Bayes estimator for θ under quadratic loss? What about under zero-one loss?

Question 7

Suppose that we have '*n*' independent drawings from a distribution that is uniform on [0 , θ], where θ is unknown and is to be estimated. That is:

$$p(y_i \mid \theta) = 1/\theta \quad ; \quad 0 \leq y_i \leq \theta$$

We are totally 'ignorant' about the value of θ, except for its sign.

(a) What is the MLE for θ ?

(b) Assume we are ignorant, *a priori*. Show that the posterior p.d.f. for θ is

$$p(\theta \mid y) \propto 1/\theta^{n+1} \quad ; \quad \theta \geq y_{\max}$$

(c) Prove that the normalizing constant for this posterior is $ny_{\max}^n$.

(d) Obtain the Bayes (MEL) estimator for θ when the loss function is quadratic.

(e) Interpret the difference between the MLE and Bayes estimators here, for finite '*n*'.