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INSTRUMENTAL VARIABLES REGRESSIONS INVOLVING SEASONAL DATA

David E.A. GILES *

Monash University, Clayton, Victoria 3168, Australia

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Various approaches to handling seasonal data are known to be equivalent in the context of least squares estimation of a fixed-regressor linear model. This note extends these results to models which have stochastic regressors and are estimated by the method of Instrumental Variables.

1. Introduction

In ordinary least squares regression it is well known that the following, and various other, strategies result in identical estimates of the coefficients of interest: [1] (i) include a set of seasonal dummy variables in the regression; (ii) regress the dependent and independent variables against these dummy variables, then use the residual vectors as 'seasonally adjusted' series and fit the regression of interest.

Frequently, least squares estimation is inappropriate, such as when the relationship is part of a simultaneous equations model. In this paper we show that a range of results of the above type hold for a very general class of estimation problems, namely for any regression model (typically involving stochastic regressors) which is estimated by the method of

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[1] These, and some related results, are due to Lovell (1963) and extend earlier work by Frisch and Waugh (1933). Also, see Johnston (1972, pp. 189–192) and Maddala (1977, pp. 338–342 and 462).

Instrumental Variables (IV). The least squares results are special cases of ours.

2. The model and estimators

Consider the basic model

$$y = X\beta + u_1, \tag{1}$$

where y and u_1 are $(T \times 1)$; X is $(T \times k)$, stochastic and of rank k; and the parameters of interest are the k elements of β. To allow for seasonality we may extent (1) to

$$y = X\beta + D\gamma + u_2, \tag{2}$$

where D is $(T \times s)$, non-stochastic and of rank s; and $T > (k+s)$.

The columns of D will be interpreted as seasonal dummy variables, but the results which follow apply for any fixed D. As D is non-stochastic, its columns act as their own instruments. Let Z be a $(T \times k)$ matrix whose columns are instruments for the columns of X. In the following discussion the necessary inverse matrices are assumed to exist, and we consider only the case of equal numbers of regressors and instruments as this includes such common estimators as two-stage least squares and other members of the k-class.

The IV estimators for (1) and (2) are $b_1 = (Z'X)^{-1}Z'y$ and

$$\begin{pmatrix} b_2 \\ c_2 \end{pmatrix} = \begin{pmatrix} Z'X & Z'D \\ D'X & D'D \end{pmatrix}^{-1} \begin{pmatrix} Z'y \\ D'y \end{pmatrix}.$$

As an alternative allowance for data seasonality, regress y and each column of X on D. Least squares regression suffices as D is exogenous, and the residuals are

$$y_s = Ay, \qquad X_s = AX,$$

where $A = (I - D(D'D)^{-1}D')$ is idempotent. These residuals may be viewed as 'seasonally adjusted' series. [2] Then, estimate

$$y_s = X_s\beta + u_3 \tag{3}$$

[2] For a critique of this interpretation, see Johnston (1972). However, if D includes trend and cyclical variables as well as seasonal dummies and if the data are measured as deviations about sample means, then this interpretation may be sensible.

by IV, yielding $b_3 = (Z'X_s)^{-1}Z'y_s$. We shall show that $b_2 = b_3$, *inter alia*.

We may consider sixteen variations of the IV estimation of (1) by choosing y or y_s, and X or X_s; including or excluding D; and using Z or its 'seasonally adjusted' counterpart, $Z_s = AZ$, as the instrument matrix. So, in addition to (1)–(3) we have the following specifications:

$$y_s = X\beta + u_4, \qquad y_s = X\beta + D\gamma + u_5, \tag{4), (5}$$

$$y = X_s\beta + u_6, \tag{6}$$

$$y = X_s\beta + D\gamma + u_7, \tag{7}$$

$$y_s = X_s\beta + D\gamma + u_8. \tag{8}$$

Let b_i and b_i^* be the IV estimators of β in eq. (i) when Z and Z_s, respectively, is the instrument matrix. We show below that thirteen of these estimators are identical to each other.

3. Results

Theorem 1

$$b_i = b_j^* = (Z'AX)^{-1}Z'Ay, \qquad i = 2, 3, 5, 7, 8, \quad j = 1,\ldots,8.$$

Proof. The equivalence of b_1^*, b_3, b_3^*, b_4^* and b_6^* follows immediately from the definitions of y_s, X_s and Z_s and the idempotency of A. The remaining eight estimators are easily shown to be equal to these five by applying the usual partitioned inverse formula and noting that $Z_s'D = D'y_s = 0$. For example,

$$\begin{aligned} b_2 &= \left(Z'X - Z'D(D'D)^{-1}D'X\right)^{-1}Z'y \\ &\quad -\left(Z'X - Z'D(D'D)^{-1}D'X\right)^{-1}D'y \\ &= (Z'AX)^{-1}Z'Ay. \end{aligned}$$

Q.E.D.

Theorem 2. The estimators shown to be equivalent in Theorem 1 are weakly consistent under the usual assumptions. [3]

[3] Clearly, b_1 is also consistent under these assumptions. Further, b_4 (b_6) is consistent if (4) [(6)] is the correct specification and $\text{plim}(T^{-1}Z'u_4) = 0$ [$\text{plim}(T^{-1}Z'u_6) = 0$]. Otherwise b_4 (b_6) is inconsistent in general.

Proof. Let limit $(T^{-1}D'D)=\Sigma_{dd}$ be finite and non-singular and let $\text{plim}(T^{-1}E'F)=\Sigma_{ef}$ be finite, for any E and F. Assume that (1) is the correct model specification and that $\text{plim}(T^{-1}Z'u_1)=0$. Then,

$$\begin{aligned}\text{plim}(Z'AX)^{-1}(Z'Ay) &= \beta+\left(\Sigma_{zx}-\Sigma_{zd}\Sigma_{dd}^{-1}\Sigma_{dx}\right)^{-1}\\ &\quad\times\left(\text{plim}\ T^{-1}Z'u_1-\Sigma_{zd}\Sigma_{dd}^{-1}\ \text{plim}\ T^{-1}D'u_1\right)\\ &= \beta \quad (D \text{ is exogenous}).\end{aligned}$$

Q.E.D.

Theorem 3. The residual vectors from the IV estimation of (2), (3), (5), (7) and (8) are identical whether Z or Z_s is the instrument matrix.

Proof. Under IV estimation the residual vector is orthogonal to the instrument matrix, so $D'e_i=0$ and $Ae_i=e_i$, where e_i is the IV residual vector for eq. (i); i = 2, 5, 7, 8. As $b_i=b_i^*$ (i = 2, 3, 5, 7, 8) we need not distinguish between the use of Z and Z_s in defining the residual vectors.[4]

Recalling that $AD=0$ and $b_2=b_3$,

$$e_2=y-Xb_2-Dc_2=Ae_2=y_s-X_sb_3=e_3.$$

Similarly, $e_5=e_7=e_8=e_3$. Q.E.D.

Theorem 4. The (consistently) estimated asymptotic covariance matrices for b_i and b_i^* (i = 2, 3, 5, 7, 8) are each equal to $(T^{-1}e_2'e_2)(Z'AX)^{-1}(Z'AZ)(X'AZ)^{-1}$.

Proof. The conventional formula[5] for the estimated asymptotic covariance matrix under IV estimation is $s^2(Z^{*\prime}X^*)^{-1}(Z^{*\prime}Z^*)(X^{*\prime}Z^*)^{-1}$, where X^* is the regressor matrix, Z^* is the instrument matrix, and s^2 is T^{-1} times the sum of the squared IV residuals. Theorem 3 ensures that s^2 is invariant to the choice of residuals. For b_3^* we have $X^*=X_s=AX$ and $Z^*=AZ$, so the result follows immediately by substitution and Theorem 1. Consistency is assured by Theorem 2. Q.E.D.

[4] We have not shown that $c_i=c_i^*$, but this result is not needed below.

[5] For example, see Johnston (1972, p. 280).

4. Discussion

We have extended some well known least squares regression results to the important case of stochastic regressors and Instrumental Variables estimation. The IV estimator is invariant to the following ways of handling data seasonality: 'seasonally adjusting' the instrument matrix for X, 'seasonally adjusting' *both* y and X, and including seasonal dummies in the regression. These same options result in a consistent IV estimator, and the geometrical motivation for these results is that the IV estimator is invariant to the order in which y and X are projected onto the subspace of the space spanned by Z that is orthogonal to D. Finally, the IV residuals and the estimated asymptotic covariance matrix for the IV estimator of the regression coefficients are invariant to 'seasonally adjusting' *both* y and X, or including seasonal dummies in the regression (whether or not the instruments are 'seasonally adjusted'). These results should be helpful to applied researchers estimating simultaneous equations models from time-series data.

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