# Best linear estimator - The solution

Dave Giles

September 23, 2015

**(a)** We have:

$$y_i = \beta x_i + \epsilon_i$$

Let $\hat{\beta} = \sum_i a_i y_i$ be any linear estimator.

So,

$$\mathbb{E}(\hat{\beta}) = \sum_i a_i \mathbb{E}(y_i) = \beta \sum_i a_i x_i$$

and,

$$\text{Bias}(\hat{\beta}) = \mathbb{E}(\hat{\beta}) - \beta = \beta \left[ \sum_i a_i x_i - 1 \right].$$

Similarly,

$$\text{var}(\hat{\beta}) = \sum_i a_i^2 \text{var}(y_i) = \sigma^2 \sum_i a_i^2.$$

So,

$$\text{MSE}(\hat{\beta}) = M = \sigma^2 \sum_i a_i^2 + \beta^2 \left[ \sum_i a_i x_i - 1 \right]^2.$$

$$\frac{\partial M}{\partial a_j} = 2\sigma^2 a_j + 2\beta^2 \left[ \sum_i a_i x_i - 1 \right] x_j = 0 \text{ , } \forall j \tag{1}$$

Multiply by $y_j$ and add over all $j$:

$$2\sigma^2 \hat{\beta} + 2\beta^2 \left[ \sum_i a_i x_i - 1 \right] \sum_j x_j y_j = 0$$

or:

$$\sigma^2\hat{\beta} + \beta^2\left[\sum_i a_i x_i - 1\right]\sum_j x_j y_j = 0 \tag{2}$$

Also, multiply (1) by $x_j$ and sum over all $j$:

$$2\sigma^2\sum_j a_j x_j + 2\beta^2\left[\sum_i a_i x_i - 1\right]\sum_j x_j^2 = 0$$

$$\Rightarrow \sum_i a_i x_i = \left(\frac{\beta^2\sum_i x_i^2}{\sigma^2 + \beta^2\sum_i x_i^2}\right)$$

Substituting into (2):

$$\sigma^2\hat{\beta} + \beta^2\left[\frac{\beta^2\sum_i x_i^2}{\sigma^2 + \beta^2\sum_i x_i^2} - 1\right]\sum_i x_i y_i = 0$$

or,

$$\sigma^2\hat{\beta} + \beta^2\left[\frac{\beta^2\sum_i x_i^2 - \sigma^2 - \beta^2\sum_i x_i^2}{\sigma^2 + \beta^2\sum_i x_i^2}\right]\sum_i x_i y_i = 0$$

or,

$$\begin{aligned}\hat{\beta} &= \left[\frac{\beta^2\sigma^2}{\sigma^2 + \beta^2\sum_i x_i^2}\right]\frac{\sum_i x_i y_i}{\sigma^2} \\ &= \left(\frac{\beta^2}{\sigma^2 + \beta^2\sum_i x_i^2}\right)\sum_i x_i y_i \\ &= \left(\frac{\beta^2\sum_i x_i^2}{\sigma^2 + \beta^2\sum_i x_i^2}\right) b\end{aligned}$$

where $b = \left(\sum_i x_i y_i / \sum_i x_i^2\right)$ is the OLS estimator.
As $\hat{\beta}$ depends on $\beta$ and $\sigma^2$, it can't be applied in practice.

**(b)** Now consider minimizing:

$$H = h\left[\frac{\text{var}(\hat{\beta})}{\sigma^2}\right] + (1-h)\left[\frac{\text{Bias}(\hat{\beta})}{\beta}\right]^2 ; 0 < h < 1$$

where again, $\hat{\beta} = \sum_i a_i y_i$. So,

$$\partial H/\partial a_j = 2ha_j + 2(1-h)x_j \left[\sum_i a_i x_i - 1\right] = 0.$$

Multiply by $y_j$ and sum:

$$h \sum_j a_j y_j + (1-h) \sum_j x_j y_j \left[\sum_i a_i x_i - 1\right] = 0.$$

or

$$h\hat{\beta} + (1-h) \sum_i x_i y_i \left[\sum_i a_i x_i - 1\right] = 0.$$

Sum over $j$ after multiplying by $x_j$:

$$h \sum_j a_j x_j + (1-h) \sum_j x_j^2 \left[\sum_i a_i x_i - 1\right] = 0$$

$$\Rightarrow \sum_i a_i x_i \left[h + (1-h) \sum_i x_i^2\right] = (1-h) \sum_i x_i^2$$
$$\Rightarrow \sum_i a_i x_i = \frac{(1-h)\sum_i x_i^2}{h + (1-h)\sum_i x_i^2}$$

So,

$$h\hat{\beta} + (1-h) \sum_i x_i y_i \left[\frac{(1-h)\sum_i x_i^2}{h + (1-h)\sum_i x_i^2} - 1\right] = 0$$
$$\Rightarrow h\hat{\beta} + (1-h) \sum_i x_i^2 b \left[\frac{(1-h)\sum_i x_i^2 - h - (1-h)\sum_i x_i^2}{h + (1-h)\sum_i x_i^2}\right] = 0$$

$$\Rightarrow \hat{\beta} = \left[\frac{1}{h + (1-h)\sum_i x_i^2}\right] (1-h) \sum_i x_i^2 b$$
$$= \left[\frac{(1-h)\sum_i x_i^2}{h + (1-h)\sum_i x_i^2}\right] b.$$

This estimator is operational in any $h \in (0, 1)$, and if $h = 0$, or $h = 1$, $\hat{\beta} = b$, or $\hat{\beta} = 0$, respectively.