On the Bias of the Maximum Likelihood Estimator for the Two-Parameter Lomax Distribution

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Revised, June, 2011
Forthcoming in *Communications & Statistics – Theory & Methods*

Abstract

The Lomax (Pareto II) distribution has found wide application in a variety of fields. We analyze the second-order bias of the maximum likelihood estimators of its parameters for finite sample sizes, and show that this bias is positive. We derive an analytic bias correction which reduces the percentage bias of these estimators by one or two orders of magnitude, while simultaneously reducing relative mean squared error. Our simulations show that this performance is very similar to that of a parametric bootstrap correction based on a linear bias function. Three examples with actual data illustrate the application of our bias correction.

Keywords Maximum likelihood estimator; bias reduction; Lomax distribution; Pareto II distribution; bootstrap; linear bias function

MSC 60E05; 62E15; 62F10; 62F40

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1. Introduction

The objectives of this paper are to explore the bias of the maximum likelihood estimators (MLEs) of the parameters of the Lomax distribution, and to compare alternative methods of reducing this bias when the sample size is relatively small. We consider both analytical and simulation-based techniques for bias-correcting the MLEs, and find that a standard second-order analytic bias correction out-performs bias correction using a parametric bootstrap, in terms of both remaining bias and mean squared error.

The Lomax (1954), or Pareto II, distribution has been quite widely applied in a variety of contexts. Although introduced originally for modelling business failure data, the Lomax distribution has been used for reliability modelling and life testing (*e.g.*, Hassan and Al-Ghamdi, 2009), and applied to income and wealth distribution data (Harris, 1968; Atkinson and Harrison, 1978), firm size (Corbelini *et al.*, 2007) and queuing problems. It has also found application in the biological sciences and even for modelling the distribution of the sizes of computer files on servers (Holland *et al*., 2006). Some authors, such as Bryson (1974), have suggested the use of this distribution as an alternative to the exponential distribution when the data are heavy-tailed.

The two-parameter Lomax distribution has the following c.d.f. and p.d.f.:

$$F(y) = 1 - [1 + y/\lambda]^{-\alpha}$$

$$f(y) = (\alpha/\lambda)[1 + y/\lambda]^{-(1+\alpha)} \qquad ; \qquad y > 0$$

where $\alpha (> 0)$ is the shape parameter, and $\lambda (> 0)$ is the scale parameter. In some applications it is useful to incorporate a location parameter, but we do not pursue that here.

The Lomax distribution can be motivated in a number of ways. For example, Balkema and de Haan (1974) show that it arises as the limit distribution of residual lifetime at great age; Dubey (1970) shows that it can be derived as a special case of a particular compound gamma distribution; and Tadikamalla (1980) relates the Lomax distribution to the Burr family of distributions. On the other hand, the Lomax distribution has itself been used as the basis for several generalizations. For example, Ghitany *et al.* (2007) extend it by introducing an additional parameter using the Marshal and Olkin

(1997) approach; Al-Awadhi and Ghitany (2001) use the Lomax distribution as a mixing distribution for the Poisson parameter and derive a discrete Poisson-Lomax distribution; and Punathumparambath (2011) introduced the double-Lomax distribution and applied it to IQ data. The record statistics of the Lomax distribution have been studied by Ahsanullah (1991) and by Balakrishnan and Ahsanullah (1994); the implications of various forms of right-truncation and right-censoring are discussed by Myhre and Saunders (1982), Childs *et al*. (2001), Cramer and Schmiedt (2011) and others; and sample size estimation has been discussed by Abd-Elfattah *et al.* (2006).

The rest of the paper is structured as follows. In the next section we briefly discuss maximum likelihood estimation for this distribution. A standard methodology for analytic bias approximation and bias correction is introduced in section 3; and this is then applied to the MLEs of the Lomax distribution's parameters in section 4. Section 5 describes the results of a Monte Carlo simulation experiment that we have undertaken to examine the effectiveness of the analytic bias correction, and to compare its performance with bootstrap bias correction for this problem. Some illustrative empirical applications are provided in section 6; and section 7 concludes.

2. Maximum likelihood estimation

As noted already the p.d.f. for the Lomax distribution is:

$$f(y) = (\alpha / \lambda)[1 + y / \lambda]^{-(1+\alpha)} \quad ; \qquad y > 0 \qquad (1)$$

where $\alpha (> 0)$ is the shape parameter, and $\lambda (> 0)$ is the scale parameter. Using integral 3.241, no. 4, from Gradshteyn and Ryzhik (1965), the r^{th} central moment of the Lomax distribution is

$$E[Y^r] = \alpha\lambda^r \Gamma(r+1)\Gamma(\alpha - r) / \Gamma(\alpha + 1) \ , \quad ; \quad \alpha > r \quad ; r = 1, 2, 3, \ldots.. \qquad (2)$$

So, $E[Y] = \lambda / (\alpha - 1)$, for $\alpha > 1$; and $V[Y] = \alpha\lambda^2 / [(\alpha - 1)^2 (\alpha - 2)]$, for $\alpha > 2$.

We will focus on maximum likelihood estimation of this distribution's parameters, but it should be noted that other estimation procedures have been considered by Lingappaiah (1986/87) and Howlader and Hossain (2002), for example. Based on n independent observations, the log-likelihood function is:

$$l = n\log(\alpha) - n\log(\lambda) - (1+\alpha)\sum_{i=1}^{n}\log(1 + y_i / \lambda) \ , \tag{3}$$

and

$$\partial l / \partial\lambda = -(n/\lambda) + [(1+\alpha)/\lambda]\sum_{i=1}^{n}[y_i /(\lambda + y_i)] \tag{4}$$

$$\partial l / \partial\alpha = (n/\alpha) - \sum_{i=1}^{n}\log(1 + y_i / \lambda) \quad . \tag{5}$$

Note that there is no closed-form solution to the likelihood equations based on (4) and (5), and a suitable numerical algorithm must be used to obtain the MLEs of λ and α. In practice, this poses little difficulty as the log-likelihood in (3) is strictly concave. This can be verified from the second-order derivatives that are derived in section 4. Although we cannot obtain exact closed-form expressions for the MLEs, $\hat{\lambda}$ and $\hat{\alpha}$, analytic expressions for the $O(n^{-1})$ biases of these MLEs *can* be derived and then used to reduce the finite-sample biases of these estimators, as we now discuss.

3. Bias approximation to $O(n^{-1})$

If $l(\theta)$ is the (total) log-likelihood based on a sample of n observations, with p-dimensional parameter vector, θ, and $l(\theta)$ is regular with respect to all derivatives up to and including the third order, then the joint cumulants of the derivatives of $l(\theta)$ are defined as

$$k_{ij} = E(\partial^2 l / \partial\theta_i \partial\theta_j) \quad ; \quad i, j = 1, 2, \ldots, p \tag{6}$$

$$k_{ijl} = E(\partial^3 l / \partial\theta_i \partial\theta_j \partial\theta_l) \quad ; \quad i, j, l = 1, 2, \ldots, p \tag{7}$$

$$k_{ij,l} = E[(\partial^2 l / \partial\theta_i \partial\theta_j)(\partial l / \partial\theta_l)] \quad ; \quad i, j, l = 1, 2, \ldots, p \tag{8}$$

The derivatives of these cumulants are:

$$k_{ij}^{(l)} = \partial k_{ij} / \partial\theta_l \quad ; \quad i, j, l = 1, 2, \ldots, p \tag{9}$$

and all of the expressions in (6) to (9) are assumed to be $O(n)$.

Cox and Snell (1968) proved that for sample data that are independent, but not necessarily identically, distributed the bias of the s^{th} element of the MLE of θ is:

$$Bias(\hat{\theta}_s) = \sum_{i=1}^{p}\sum_{j=1}^{p}\sum_{l=1}^{p} k^{si}k^{jl}[0.5k_{ijl} + k_{ij,l}] + O(n^{-2})\,;\;\; s = 1, 2, \ldots, p \qquad (10)$$

where k^{ij} is the $(i,j)^{\text{th}}$ element of the inverse of the information matrix, $K = \{-k_{ij}\}$. Cordeiro and Klein (1994) showed that (10) also holds for non-independent data, and that this expression can be written as:

$$Bias(\hat{\theta}_s) = \sum_{i=1}^{p} k^{si} \sum_{j=1}^{p}\sum_{l=1}^{p} [k_{ij}^{(l)} - 0.5k_{ijl}]k^{jl} + O(n^{-2})\,;\; s = 1, 2, \ldots, p. \qquad (11)$$

The bias formula in (11) is generally easier to evaluate than (10), as it does not require the expectations of products, as in (8).

Defining $a_{ij}^{(l)} = k_{ij}^{(l)} - (k_{ijl}/2)$, for $i, j, l = 1, 2, \ldots, p$; and gathering up terms into matrices:

$$A^{(l)} = \{a_{ij}^{(l)}\}\,; \quad i, j, l = 1, 2, \ldots, p$$

$$A = [A^{(1)} \,|\, A^{(2)} \,|\, \ldots\ldots. \,|\, A^{(p)}],$$

the $O(n^{-1})$ bias of the MLE of θ in (11) can be re-written in the compact matrix form:

$$Bias(\hat{\theta}) = K^{-1}A\,vec(K^{-1}) + O(n^{-2})\,.$$

Then, a "bias-corrected" MLE for θ can then be constructed as:

$$\theta^{*} = \hat{\theta} - \hat{K}^{-1}\hat{A}\;vec(\hat{K}^{-1}),$$

where $\hat{K} = (K)|_{\hat{\theta}}$ and $\hat{A} = (A)|_{\hat{\theta}}$, and values for the elements of $\hat{\theta}$ can be obtained by solving the likelihood equations based on (4) and (5) by numerical methods. Importantly, it is not necessary to be able to solve these equations analytically in order to determine the bias expression. It can be shown that the bias of θ^{*} is $O(n^{-2})$.

This approach to analytically bias-adjusting MLEs has been widely and successfully used, and some recent applications are reported by Cordeiro and McCullagh (1991), Cordeiro *et al*. (1996), Cribari-Neto and Vasconcellos (2002), Giles and Feng (2009), Giles (2011) and Giles *et al*. (2011).

4. Bias reduction and the Lomax distribution

In what follows, we will require the following higher-order derivatives of the log-likelihood function:

$$\partial^2 l / \partial\alpha^2 = -(n/\alpha^2)$$

$$\partial^3 l / \partial\alpha^3 = (2n/\alpha^3)$$

$$\partial^2 l / \partial\lambda^2 = (n/\lambda^2) - [(1+\alpha)/\lambda^2]\sum_{i=1}^{n} y_i/(\lambda + y_i) - [(1+\alpha)/\lambda]\sum_{i=1}^{n}[y_i/(\lambda + y_i)^2]$$

$$\partial^3 l / \partial\lambda^3 = -(2n/\lambda^3) + 2[(1+\alpha)/\lambda^3]\sum_{i=1}^{n} y_i/(\lambda + y_i) + 2[(1+\alpha)/\lambda^2]\sum_{i=1}^{n}[y_i/(\lambda + y_i)^2]$$
$$+ 2[(1+\alpha)/\lambda]\sum_{i=1}^{n}[y_i/(\lambda + y_i)^3]$$

$$\partial^2 l / \partial\lambda\partial\alpha = (1/\lambda)\sum_{i=1}^{n}[y_i/(\lambda + y_i)]$$

$$\partial^3 l / \partial\lambda\partial\alpha^2 = 0$$

$$\partial^3 l / \partial\lambda^2\partial\alpha = -(1/\lambda^2)\sum_{i=1}^{n}[y_i/(\lambda + y_i)] - (1/\lambda)\sum_{i=1}^{n}[y_i/(\lambda + y_i)^2]$$

To evaluate the expectations of these derivatives we will use the following result:

<u>*Theorem*</u>

Let Y follow a Lomax distribution with parameters α and λ. Then

$$E[Y(\lambda + Y)^{-r}] = \alpha\lambda^{1-r}/[(\alpha + r)(\alpha + r - 1)] \quad ; \quad r = 1, 2, 3, \ldots\ldots \quad \alpha + r > 1 \qquad (12)$$

<u>*Proof*</u>

$$E[Y(\lambda + Y)^{-r}] = \int_0^\infty y(\lambda + y)^{-r}(\alpha/\lambda)(1 + y/\lambda)^{-(\alpha+1)}dy$$
$$= \alpha\lambda^\alpha \int_0^\infty y(\lambda + y)^{-(r+\alpha+1)}dy$$

Applying the change of variable, $x = \lambda + y$:

$$E[Y(\lambda+Y)^{-r}] = \alpha\lambda^{\alpha}\int_{\lambda}^{\infty}(x-\lambda)x^{-(r+\alpha+1)}dx$$
$$= \alpha\lambda^{\alpha}[\int_{\lambda}^{\infty}x^{-(r+\alpha)}dx - \lambda\int_{\lambda}^{\infty}x^{-(r+\alpha+1)}dx]$$
$$= \alpha\lambda^{\alpha}\{[x^{1-(r+\alpha)}(1-r-\alpha)^{-1}]_{\lambda}^{\infty} - \lambda[x^{-(r+\alpha)}(r+\alpha)^{-1}]_{\lambda}^{\infty}\}$$
$$= \alpha\lambda^{\alpha}\{\lambda^{1-r-\alpha}(r+\alpha-1)^{-1} - \lambda^{1-r-\alpha}(r+\alpha)^{-1}\}$$
$$= \alpha\lambda^{1-r}/[(\alpha+r)(\alpha+r-1)]$$

and this expression is positive only if $(\alpha+r)>1$. □

We now have the following results for the joint cumulants of the derivatives of the log-likelihood function:

$$k_{11} = E[\partial^2 l/\partial\lambda^2] = -n\alpha/[\lambda^2(\alpha+2)]$$
$$k_{12} = E[\partial^2 l/\partial\lambda\partial\alpha] = n/[\lambda(\alpha+1)]$$
$$k_{22} = E[\partial^2 l/\partial\alpha^2] = -n/\alpha^2$$
$$k_{111} = E[\partial^3 l/\partial\lambda^3] = 4n\alpha/[\lambda^3(\alpha+3)]$$
$$k_{112} = E[\partial^3 l/\partial\lambda^2\partial\alpha] = -2n/[\lambda^2(\alpha+2)]$$
$$k_{222} = E[\partial^3 l/\partial\alpha^3] = 2n/\alpha^3$$
$$k_{122} = E[\partial^3 l/\partial\lambda\partial\alpha^2] = 0 \quad .$$

In addition,

$$k_{11}^{(1)} = \partial k_{11}/\partial\lambda = 2n\alpha/[\lambda^3(\alpha+2)]$$
$$k_{12}^{(1)} = \partial k_{12}/\partial\lambda = -n/[\lambda^2(\alpha+1)]$$
$$k_{22}^{(1)} = \partial k_{22}/\partial\lambda = 0$$
$$k_{11}^{(2)} = \partial k_{11}/\partial\alpha = -2n/[\lambda^2(\alpha+2)^2]$$
$$k_{12}^{(2)} = \partial k_{12}/\partial\alpha = -n/[\lambda(\alpha+1)^2]$$
$$k_{22}^{(2)} = \partial k_{22}/\partial\alpha = 2n/\alpha^3 \quad .$$

So,

$$a_{11}^{(1)} = k_{11}^{(1)} - 0.5k_{111} = 2n\alpha/[\lambda^3(\alpha+2)(\alpha+3)]$$
$$a_{12}^{(1)} = k_{12}^{(1)} - 0.5k_{121} = k_{12}^{(1)} - 0.5k_{112} = -n/[\lambda^2(\alpha+1)(\alpha+2)]$$
$$a_{22}^{(1)} = k_{22}^{(1)} - 0.5k_{221} = k_{22}^{(1)} - 0.5k_{122} = 0$$

and,

$$a_{11}^{(2)} = k_{11}^{(2)} - 0.5k_{112} = n\alpha /[\lambda^2(\alpha+2)^2]$$

$$a_{12}^{(2)} = k_{12}^{(2)} - 0.5k_{122} = -n /[\lambda(\alpha+1)^2]$$

$$a_{22}^{(2)} = k_{22}^{(2)} - 0.5k_{222} = n / \alpha^3 \quad .$$

The information matrix is (see, also, Brazouskas, 2003, p. 321; Arnold, 1983, p. 210):

$$K = \{-k_{ij}\} = n\begin{bmatrix} \alpha /[\lambda^2(\alpha+2)] & -1/[\lambda(\alpha+1)] \\ -1/[\lambda(\alpha+1)] & 1/\alpha^2 \end{bmatrix} \quad , \qquad (13)$$

which is clearly positive-definite (so –K is negative-definite and the log-likelihood is strictly concave).

Defining $A^{(q)} = \{a_{ij}^{(q)}\}$, q = 1, 2; and $A = [A^{(1)} \mid A^{(2)}]$, we have:

$$A = n\begin{bmatrix} \dfrac{2\alpha}{\lambda^3(\alpha+2)(\alpha+3)} & \dfrac{-1}{\lambda^2(\alpha+1)(\alpha+2)} & \dfrac{\alpha}{\lambda^2(\alpha+2)^2} & \dfrac{-1}{\lambda(\alpha+1)^2} \\ \dfrac{-1}{\lambda^2(\alpha+1)(\alpha+2)} & 0 & \dfrac{-1}{\lambda(\alpha+1)^2} & \dfrac{1}{\alpha^3} \end{bmatrix}$$

Using Cordeiro and Klein's (1994) modification of the Cox and Snell (1968) result,

$$Bias\begin{pmatrix} \hat{\lambda} \\ \hat{\alpha} \end{pmatrix} = K^{-1}Avec(K^{-1}),$$

and bias-adjusted estimators can be obtained as $\begin{pmatrix} \lambda^* \\ \alpha^* \end{pmatrix} = \hat{K}^{-1}\hat{A}vec(\hat{K}^{-1})$, where

$$\hat{K} = K\big|_{\lambda=\hat{\lambda};\alpha=\hat{\alpha}} \text{ ; and } \hat{A} = A\big|_{\lambda=\hat{\lambda};\alpha=\hat{\alpha}}.$$

5. A simulation experiment

The above bias expressions are valid to $O(n^{-1})$. We have compared the actual bias and mean squared error (MSE) of the MLEs and bias-corrected MLEs in a small Monte Carlo experiment. The MLEs were obtained using the Newton-Raphson algorithm in the *maxLik* package (Toomet and Henningsen, 2008) for the *R* statistical software environment (R, 2008). In addition to $\hat{\alpha}$, α^*, $\hat{\lambda}$ and λ^*, we have also investigated bias-correcting the estimators using a simple parametric bootstrap ($\tilde{\alpha}$ and $\tilde{\lambda}$), as well as the linear bootstrap correction ($\breve{\alpha}$ and $\breve{\lambda}$) proposed by MacKinnon and Smith (1998). These bootstrap-bias-corrected estimators are also unbiased to $O(n^{-2})$, but it is known that this may come at the expense of increased variance, and perhaps MSE.

In the case of α, for example, the simple (constant) bootstrap bias-corrected estimator is $\tilde{\alpha} = 2\hat{\alpha} - (1/N_B)[\sum_{j=1}^{N_B} \hat{\alpha}_{(j)}]$, where $\hat{\alpha}_{(j)}$ is the MLE of α obtained from the j^{th} of the N_B bootstrap samples. The linear bootstrap bias-corrected estimator assumes that the bias is a linear function of the parameter (for a given sample size), rather than being constant. In applying the MacKinnon-Smith procedure we have used the original MLE and the simple bootstrap bias-corrected MLE as the two points at which to evaluate the bias function in order to solve for the latter, as suggested by those authors. The computational cost of using the linear bootstrap-corrected estimator is double that associated with the standard (constant) bootstrap bias correction.

Each part of the experiment uses 20,000 Monte Carlo replications. In the case of the bootstrap-corrected estimators we use 500 bootstrap samples *per* replication when $n > 250$, and 1,000 samples when $n \leq 250$. Some preliminary experimentation showed that these numbers were sufficient to ensure stable results. The values reported in Table 1 are *percentage* biases and MSEs. The latter are defined as $[100 \times \text{MSE}(\hat{\alpha}) / \alpha^2]$, *etc*., and are reported in square brackets below the corresponding percentage biases.

The scale parameter, λ, is set to unity in this experiment. Some additional experimentation indicated that the orders of magnitude of the percentage biases and MSEs are relatively invariant to the value of this parameter. Various values of the shape parameter, α, have been considered, including ones that are consistent with some of the empirical studies discussed in section 1. For example, Corbellini *et al*. (2007) obtained estimates ranging from 0.96 to 1.06; and Holland *et al.* (2006) reported estimates of α in the range 0.5 to 3.3. We report results based on $\alpha = 1.1$ and 2.1. The latter value ensures that the first two moments of the Lomax distribution exist, while the former value ensures that the first moment is finite. The minimum sample sizes that are considered are determined by factors that are discussed at the end of this section.

The simulation results in Table 1 are strikingly clear, and the following comments apply to the estimators for both of the parameters. We see that the MLEs, $\hat{\alpha}$ and $\hat{\lambda}$ are positively biased, and the percentage bias is sizeable in moderately sized samples. The biases and mean squared errors of these MLEs fall quickly as the sample size increases, reflecting the consistency of the estimators. The analytic bias adjustment, leading to α^* and λ^*, is extremely successful, reducing the (absolute) percentage bias by one or even two orders of magnitude, except for the smallest of the sample sizes considered for $\alpha = 2.1$. In addition, this success comes with an improvement in precision – the percentage MSEs of both α^* and λ^* are less than those of $\hat{\alpha}$ and $\hat{\lambda}$, regardless of sample size. For very small sample sizes, this reduction in %MSE can be more than 50%. Of course, for very large samples the bias adjustment becomes irrelevant, and the biases and MSEs of the unadjusted and adjusted MLEs converge. Another important feature of these results is that in many cases, application of the analytic bias adjustment reverses the sign of the bias, especially for the smaller sample size considered. This "over-correction" has been reported in similar studies involving the Cox-Snell methodology, including those of Cribari-Neto and Vasconcellos (2002) for the beta distribution, Giles (2011) for the half-logistic distribution, and Giles *et al*. (2011) for the generalized Pareto distribution. A practical implication of our results is that researchers will know the direction of the biases in the estimators in small samples.

The two bootstrap bias corrections also perform very well in terms of their primary task, with the linear bootstrap bias-correction out-performing the constant bootstrap bias correction when $n < 500$. The superior performance of the latter bootstrap correction is easily explained. Some simulations of the bias function itself revealed it to be upward sloping and approximately linear over the part of the parameter space being considered. However, the small bias associated with the linear bootstrap correction comes at the expense of increased %MSE (relative to the constant bootstrap and analytic corrections). In some cases, the linear bootstrap bias correction increases the %MSE of the estimator above that of the original MLE. These features of the MSE are not surprising in view of the observation of MacKinnon and Smith (1998, p.211) that increased variance, of the linear corrected estimator over the constant corrected estimator, is likely to occur "…in almost all cases of interest."

Overall, the Cox-Snell / Cordeiro-Klein analytic adjustment and the linear bootstrap bias correction perform extremely well, and in many of the situations considered they provide similar results. The analytic adjustment is somewhat superior to the bootstrap correction when the sample size exceeds approximately 100. The converse is true if the sample size is smaller, provided that the focus is primarily on bias itself. In very small samples, the Cox-Snell correction results in a reduction in %MSE and an increase in (absolute) percentage bias; while the opposite is true for the bootstrap corrections. Unless the sample size is relatively large (in which case the relative biases of $\hat{\alpha}$ and $\hat{\lambda}$ are very small, in any case), the constant bootstrap bias correction is not recommended. The analytical bias adjustment can be recommended on the basis of its overall performance, and on the grounds of its computational simplicity. Given the formulae that we have provided, it is almost costless to implement this bias correction in practice.

It should be noted that for very small sample sizes, the decision regarding bias correction must be made carefully. For small n, all of the bias correction methods were found to be capable of producing outlier results, and this is more likely to occur with larger values of α. This was a particular problem with the linear bootstrap bias correction in the simulation experiment, and was found to be related to the bias function itself becoming downward sloping for values of α above some level. We dealt with this by excluding the results for all of the estimators in such cases and substituting additional replications. A practical solution to this is proposed below.

When bias-adjusting $\hat{\alpha}$ using the analytical formula when n is relatively small and α is relatively large, it is possible for α^*, $\tilde{\alpha}$ or $\breve{\alpha}$ to be negative, which is clearly nonsensical. This may be a consequence of using a bias correction that is valid only to $O(n^{-1})$. In practice, negative estimates would not be accepted, and such rare cases have been discarded in our simulation experiment. Figure 1 shows a representative relationship between our recommended α^*, and $\hat{\alpha}$. This relationship depends on n, but is invariant to the value of $\hat{\lambda}$. It is a trivial arithmetic matter to avoid a negative estimate of α. However, recalling from Table 1 that the bias of $\hat{\alpha}$ is always positive, while that of α^* is always negative (for small n), we can see from Figure 1 that if an estimate based on $\hat{\alpha}$ exceeds $\hat{\alpha}^*$, then there is an increased likelihood of the *absolute* bias of α^* exceeding that of $\hat{\alpha}$.

So, to assist practitioners, Figure 1 suggests that a *conservative* rule of thumb would be to analytically bias-adjust $\hat{\alpha}$ to α^* only if $\hat{\alpha} < \hat{\alpha}^*$. This also ensures a one-to-one relationship between $\hat{\alpha}$ and α^*. Simulations of the bias function, as used in the case of the linear bootstrap bias correction, show that this *same* rule of thumb will also avoid situations where the bias function is downward sloping. This is recommended by MacKinnon and Smith (1998, p.229) to ensure that the bootstrap bias corrected estimators do not exhibit excessive variance (and perhaps MSE). It should be noted that negative values of λ^* can also arise, but it can be shown that this cannot occur if our rule of thumb is applied, so we don't need to consider this estimator further in the present discussion.

A table of the exact values of $\hat{\alpha}^*$, for n = 20(1)500, can be downloaded from http://web.uvic.ca/~dgiles/downloads/data. The value of $\hat{\alpha}^*$ increases monotonically with n, and by way of illustration, its values are 1.49 and 4.43 for n = 20, 100 respectively. The following OLS regression model, can be used to determine $\hat{\alpha}^*$ accurately for various values of n:

$$\begin{aligned}\hat{\alpha}^* = &-0.991404 + 0.027773n - 0.0000328n^2 + 0.64541 ln(n) - 3.617656D \\ &\quad (0.024038)\ (0.000267)\ (0.0000009)\ (0.009075)\ (0.074549) \\ &- 0.014898(D \times n) + 0.0000293(D \times n^2) + 1.036946(D \times ln(n)) \\ &\quad (0.000291)\ (0.0000009)\ (0.019257)\end{aligned} \tag{14}$$

(Sample size = 481; $\bar{R}^2 = 0.999999$)

Here, D is a dummy variable, defined as D = 0 if $20 \leq n \leq 150$; D = 1 if $151 \leq n \leq 500$; and White's (1980) heteroskedasticity-consistent standard errors appear in parentheses. The latter are presented as both the White (1980) and Breusch-Pagan (1979) tests led to a rejection of the null hypothesis of homoskedastic errors, at the 5% significance level,

7. Illustrative applications

We have considered three simple applications, the first two of which involve data that have been used previously with the Lomax distribution. The third application uses new data that is of a type found previously to be modelled better by a Lomax distribution than by various competitors. In each case, the sample size is quite modest. We have fitted the Lomax distribution to each data-set using MLE, using the Marquardt algorithm, and then bias-adjusted the parameter estimates using the analytic bias formulae derived in section 4. Given the relative inferiority of the bootstrap bias correction noted already, we do not report those results here.

The first application involves American Insurance Association data relating to insurance losses in excess of $5million (in 1981 dollars) due to major hurricanes between 1949 and 1980. The data are provided by Hogg and Klugman (1983, p.92), and we have subtracted 5million from each of their sample values, so allowing for the scale of the data reported by those authors, our first datum is

1766.0, *etc.* The second set of data consists of observations for precipitation in a Florida meteorological study by Simpson (1972), and further analyzed by Bryson (1974). Finally, we consider a sample of computer file sizes (in bytes) for all 269 files with the .ini extension on the first author's Windows-based personal computer. These data can be downloaded from http://web.uvic.ca/~dgiles/downloads/data. Previous work by Holland *et al.* (2006) has demonstrated the superiority of the Lomax distribution over several other competitors for modeling such file sizes. Those authors also provide technical information suggesting that in this context the distribution should have infinite variance (*i.e.*, $\alpha < 2$).

Table 2 reports the MLEs and the bias-adjusted MLEs when the Lomax distribution is fitted to each of these three data-sets. Each of the estimates based on $\hat{\alpha}$ are less than the values of $\hat{\alpha}^*$ for the respective sample sizes, so our rule of thumb from section 5 suggests that the bias-adjusted estimates based on α^* and λ^* should be used. In each case the estimated values of α imply that the distributions have infinite variance, and for the third data-set the distribution has an infinite mean. For the other two data-sets the MLEs of α exceed unity before bias-adjustment (although the excess is not statistically significant), implying a finite mean. However, this is reversed by bias-adjusting the estimates. So, focusing on the point estimates of α in these first two cases, bias-adjusting the MLEs induces an important change to the characteristics of the Lomax distribution, even though the numerical differences may not seem large. Interestingly, the values of $\hat{\alpha}$ and $\hat{\lambda}$ associated with computer file data are very similar to those obtained by Holland *et al.* (2006) using a sample of 410 *.ini files. Their estimates were $\hat{\alpha} = 0.556$ and $\hat{\lambda} = 119.091$. They did not report standard errors, and of course, they did not bias-adjust their estimates. We can see that for sample size associated with the third data-set, bias-adjusting the MLEs has negligible impact on the point estimates.

The fitted c.d.f.'s based on the first two sets of MLEs are shown in Figures 1 and 2, over the ranges of the samples. (The fitted c.d.f.'s for the third data-set are indistinguishable.) For the hurricane loss data (Figure 1), the effect of the bias adjustment is to increase the intercept of the Lomax *density* at the origin and slightly "thicken" the extreme right tail. The effect on the fitted Lomax distribution for the precipitation data (Figure 2) is just the opposite, so that the probability of extreme values is somewhat reduced. However, it should be noted that for the parameter estimates in Table 2, neither the skewness nor kurtosis coefficients are defined.

7. Conclusions

In this paper we have evaluated some of the small-sample properties of the maximum likelihood estimator for the two-parameter Lomax (Pareto II) distribution. This distribution has been applied in a broad range of settings, and information about the bias and mean squared errors of the parameter estimators is of practical importance. Although the maximum likelihood estimators of the shape and scale parameters of the Lomax distribution cannot be expressed in closed form, we have used the methods of Cox and Snell (1968) to derive analytic second-order bias expressions which can then be used to bias-adjust the numerical point estimates.

Our simulation results show that this procedure is very effective. The maximum likelihood estimators of both parameters are positively biased in small samples. Applying the analytic bias correction reduces the absolute value of this bias by one or two orders of magnitude relative to that of the original estimators. As an added bonus, the bias correction also has the effective of sharply reducing the mean squared errors of the estimators in small samples.

We have also considered two parametric bootstrap bias corrections for this problem. A simple bootstrap correction, based on the assumption of a constant bias function, is found to be inferior to our analytic bias correction in the situations studied. However, a bootstrap bias correction based on the assumption of a linear bias function is found to be extremely effective. The analytic adjustment is somewhat superior to the latter bootstrap correction for sample sizes in excess of 100, and the converse is true if the sample size is smaller. In very small samples, the Cox-Snell correction results in a reduction in %MSE and an increase in (absolute) percentage bias; while the opposite is true for the linear bootstrap correction.

The analytic bias correction that is derived in this paper is very simple to apply in practice, and is recommended over the bootstrap alternative. Except in quite extreme cases, when the sample size is very small, this bias correction is strongly recommended over the choice of no correction at all. We provide precise guidelines to enable a practitioner to determine when the correction should not be applied.

Acknowledgments

We are grateful to an anonymous referee and Jacob Schwartz for their very helpful comments, and to Lief Bluck and Kees van Kooten for providing access to the additional computing facilities needed to complete the simulation experiments. We also thank Matthew Giles for writing the Python script that was used to harvest the data on computer file sizes. The second author acknowledges financial support from King's University College at the University of Western Ontario.

Table 1. Percentage biases and MSE's

n	$\%Bias(\hat{\alpha})$ [$\%MSE(\hat{\alpha})$]	$\%Bias(\alpha^{*})$ [$\%MSE(\alpha^{*})$]	$\%Bias(\tilde{\alpha})$ [$\%MSE(\tilde{\alpha})$]	$\%Bias(\breve{\alpha})$ [$\%MSE(\breve{\alpha})$]	$\%Bias(\hat{\lambda})$ [$\%MSE(\hat{\lambda})$]	$\%Bias(\lambda^{*})$ [$\%MSE(\lambda^{*})$]	$\%Bias(\tilde{\lambda})$ [$\%MSE(\tilde{\lambda})$]	$\%Bias(\breve{\lambda})$ [$\%MSE(\breve{\lambda})$]
				α = 1.1 ; λ = 1.0				
35	17.486	-15.229	-4.251	-2.079	31.341	-27.500	-5.199	-0.892
	[25.325]	[5.584]	[17.788]	[21.030]	[78.191]	[17.354]	[51.782]	[59.835]
50	11.556	-7.464	-1.858	-0.199	19.775	-13.808	-2.557	0.789
	[14.286]	[5.272]	[14.011]	[15.489]	[38.706]	[14.250]	[40.159]	[43.626]
100	8.158	-0.353	-2.624	-0.700	14.225	-0.942	-4.998	-1.234
	[8.493]	[4.925]	[4.389]	[5.117]	[24.413]	[13.515]	[12.109]	[14.110]
150	4.949	-0.169	-0.970	-0.443	8.847	-0.255	-1.693	-0.575
	[4.459]	[3.311]	[3.078]	[3.298]	[12.961]	[9.310]	[8.647]	[9.256]
250	2.943	0.063	-0.150	-0.135	5.142	0.047	-0.331	-0.196
	[2.209]	[1.871]	[1.848]	[1.972]	[6.291]	[5.233]	[5.170]	[5.724]
500	1.339	-0.037	-0.085	-0.114	2.338	-0.087	-0.172	-0.290
	[1.010]	[0.937]	[0.934]	[1.152]	[2.834]	[2.603]	[2.594]	[2.914]
750	0.841	-0.062	-0.080	0.031	1.568	-0.025	-0.056	0.116
	[0.631]	[0.601]	[0.601]	[0.938]	[1.789]	[1.691]	[1.693]	[2.243]
1000	0.704	0.030	0.018	0.072	1.216	0.028	0.009	0.178
	[0.469]	[0.451]	[0.451]	[0.518]	[1.325]	[1.269]	[1.270]	[1.441]

Table 1. (continued) Percentage biases and MSE's

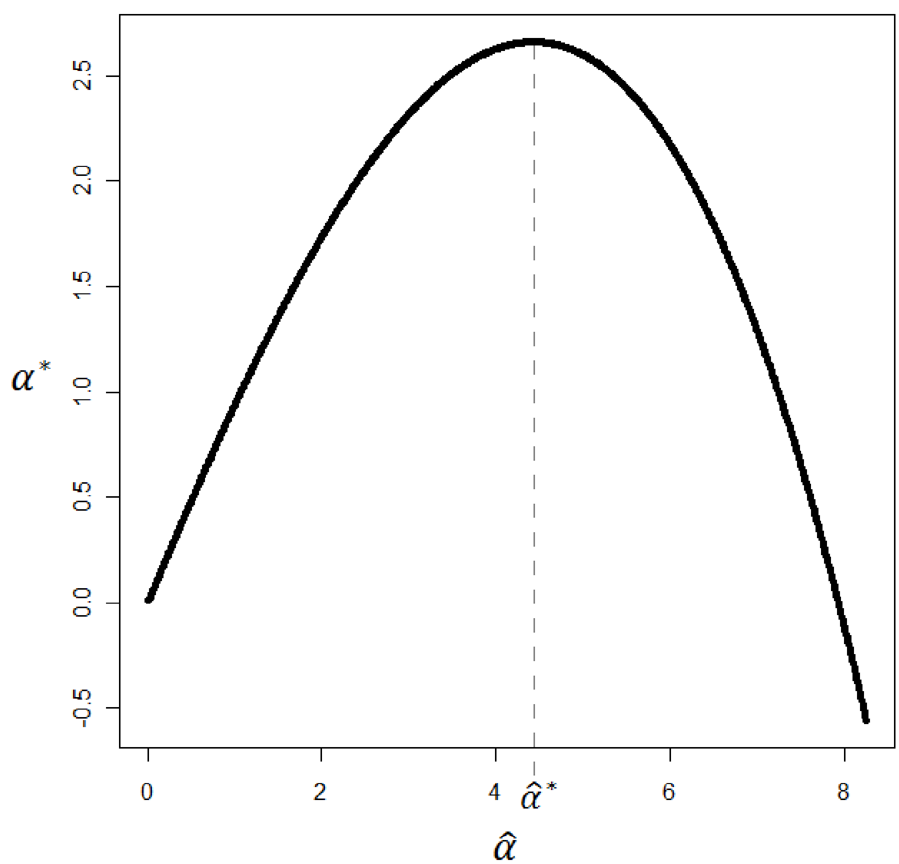
n	$\%Bias(\hat{\alpha})$ [$\%MSE(\hat{\alpha})$]	$\%Bias(\alpha^*)$ [$\%MSE(\alpha^*)$]	$\%Bias(\tilde{\alpha})$ [$\%MSE(\tilde{\alpha})$]	$\%Bias(\breve{\alpha})$ [$\%MSE(\breve{\alpha})$]	$\%Bias\ (\hat{\lambda})$ [$\%MSE\cdots(\hat{\lambda})$]	$\%Bias(\lambda^*)$ [$\%MSE(\lambda^*)$]	$\%Bias\ (\tilde{\lambda})$ [$\%MSE\ (\tilde{\lambda})$]	$\%Bias\ (\breve{\lambda})$ [$\%MSE\ (\breve{\lambda})$]
				***α* = 2.1 ; *λ* = 1.0**				
80	16.238	-16.182	-3.261	-0.460	23.274	-22.948	-3.957	-0.108
	[24.107]	[5.256]	[19.232]	[23.342]	[49.259]	[11.031]	[37.452]	[44.358]
100	11.592	-10.316	-2.787	-0.322	16.132	-14.750	-3.604	-0.139
	[15.601]	[4.630]	[15.920]	[18.568]	[29.970]	[9.192]	[31.256]	[36.013]
150	9.834	-3.685	-1.874	-0.221	13.873	-5.275	-2.443	-0.021
	[11.721]	[4.940]	[10.078]	[10.791]	[23.003]	[9.573]	[20.124]	[21.402]
250	6.699	-0.496	-2.150	-0.715	9.447	-0.727	-3.073	-0.955
	[6.899]	[4.164]	[3.819]	[4.335]	[13.452]	[8.013]	[7.329]	[8.378]
500	2.933	-0.181	-0.460	-0.432	4.138	-0.250	-0.644	-0.538
	[2.413]	[1.992]	[1.961]	[2.311]	[4.719]	[3.875]	[3.815]	[4.134]
750	1.939	-0.064	-0.174	-0.130	2.746	-0.076	-0.231	-0.076
	[1.518]	[1.344]	[1.336]	[1.867]	[2.971]	[2.620]	[2.603]	[3.533]
1000	1.512	0.033	-0.029	0.040	2.103	0.022	-0.065	0.044
	[1.099]	[1.003]	[0.999]	[1.229]	[2.146]	[1.955]	[1.947]	[2.353]

Table 2. MLEs for empirical applications

Data-set	***n***	$\hat{\lambda}$	$\hat{\alpha}$	λ^*	α^*
1	35	78670.267	1.157	51701.358	0.926
		(64882.53)	(1.78)		
2	26	137.972	1.097	84.616	0.856
		(100.06)	(0.58)		
3	269	128.307	0.498	124.032	0.492
		(33.70)	(0.06)		

Notes: Asymptotic standard errors appear in parentheses. The data-sets are 1: Hurricane losses; 2: Precipitation; 3: Computer file sizes.

Figure 1. Relationship between $\hat{\alpha}$ and $\tilde{\alpha}$: $\boldsymbol{n}$ = 100



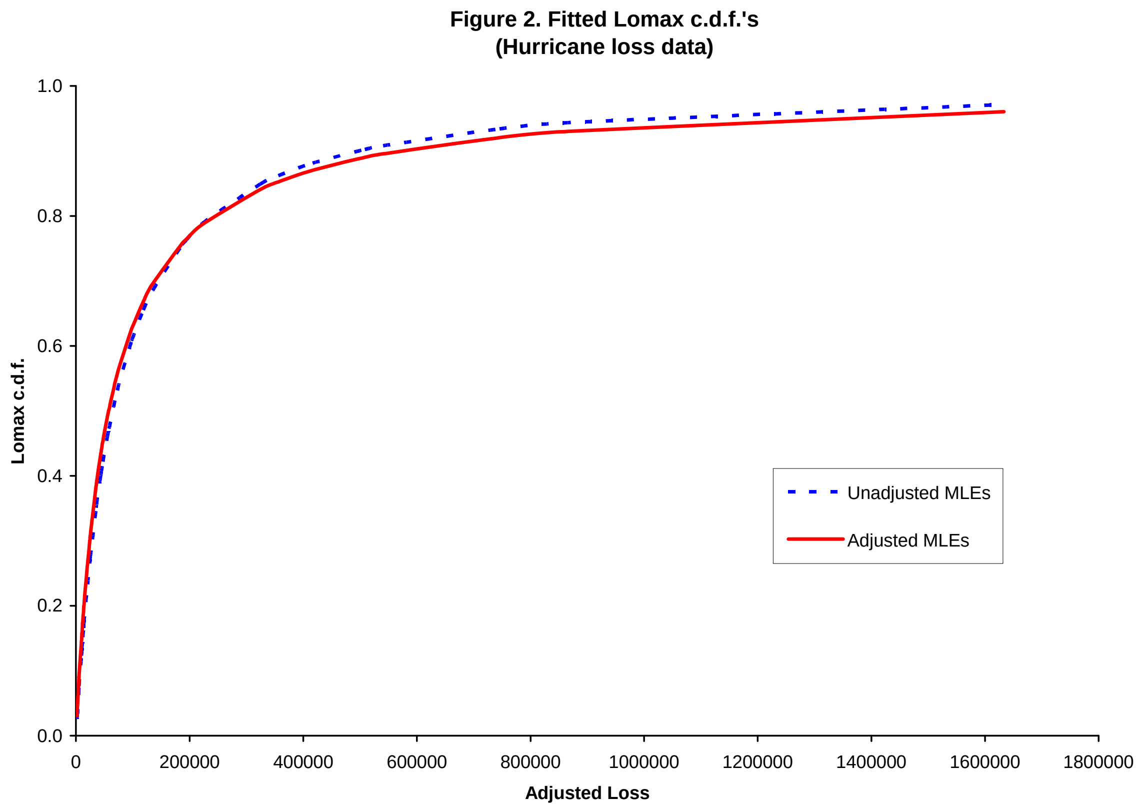


Figure 2. Fitted Lomax c.d.f.'s
(Hurricane loss data)
Lomax c.d.f.
1.0
0.8
0.6
0.4
0.2
0.0
0
200000
400000
600000
800000
1000000
1200000
1400000
1600000
1800000
Adjusted Loss
Unadjusted MLEs
Adjusted MLEs

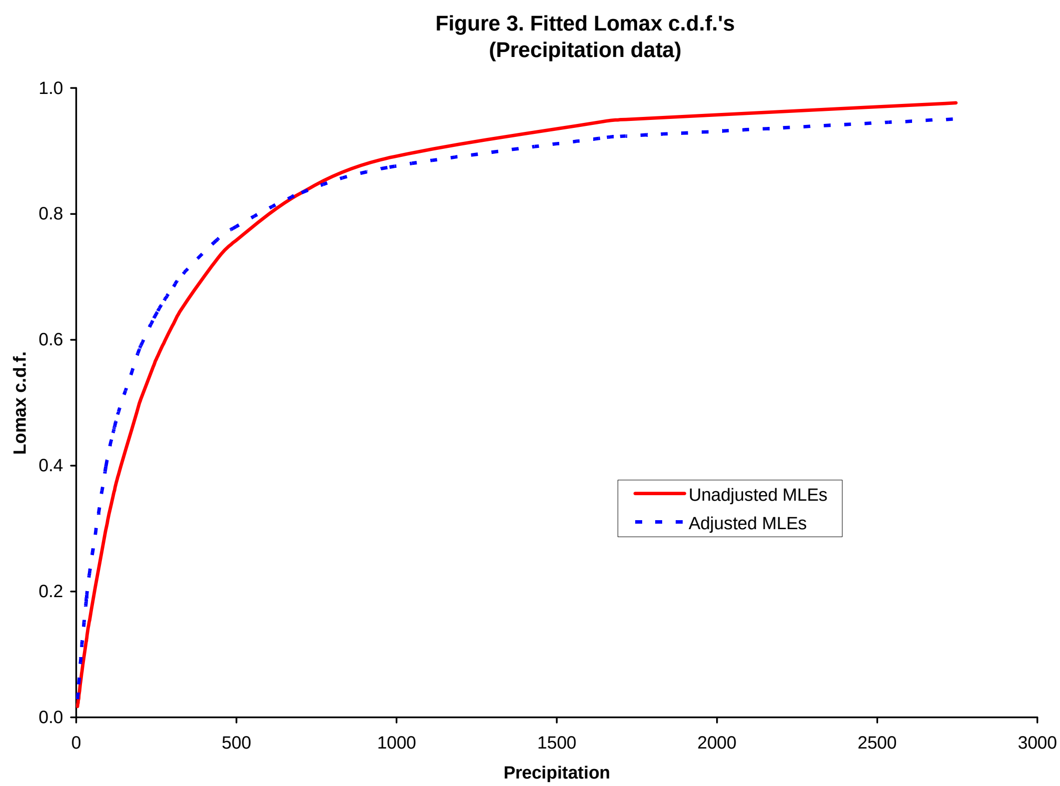


Figure 3. Fitted Lomax c.d.f.'s
(Precipitation data)
Lomax c.d.f.
1.0
0.8
0.6
0.4
0.2
0.0
0
500
1000
1500
2000
2500
3000
Precipitation
Unadjusted MLEs
Adjusted MLEs

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