

***Proof that the Median Minimizes
the Mean Absolute Deviation***

In our Bayesian analysis we meet the result that if we have an absolute-error loss function, then the MEL estimator is the median of the posterior p.d.f.. This result arises as a consequence of the more general result, that the median is the value, M , of a random variable, X , that minimizes the quantity $E | X - M |$. In the Bayes case, the expectation is simply being taken with respect to the posterior distribution, and the loss function is $L[\theta, \hat{\theta}(x)] = | \theta - \hat{\theta}(x) |$. (Note that the problem, and its solution, would not be altered if we used the more general loss function, $L[\theta, \hat{\theta}(x)] = c | \theta - \hat{\theta}(x) |$, for some $c > 0$.)

To prove the result in question, we need to recall how to differentiate an integral when the range of integration depends on the variable with respect to which we are differentiating. This result is:

Lemma: (“Leibniz’s Rule”)

Let
$$\phi(\alpha) = \int_{u_1}^{u_2} f(x, \alpha) dx \quad ; \quad a \leq \alpha \leq b$$

where u_1 and u_2 may depend on α . Then

$$\frac{d\phi}{d\alpha} = \int_{u_1}^{u_2} \frac{\partial f}{\partial \alpha} dx + f(u_2, \alpha) \frac{du_2}{d\alpha} - f(u_1, \alpha) \frac{du_1}{d\alpha}$$

for $a \leq \alpha \leq b$.

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Our main result now follows:

Theorem:

Let X be random, with distribution function F , and let $E(\cdot)$ denote expectation with respect to F . Then the quantity $E | X - M |$ is minimized when M is the median of F .

Proof:

Assume that F is zero to the left of some constant, A , and is unity to the right of some constant, B .

$$\text{Let } \phi = E | X - M | = \int_A^M (M - x)dF + \int_M^B (x - M)dF .$$

Applying the above Lemma twice, we have:

$$\begin{aligned} \frac{d\phi}{dM} &= \int_A^M (1)dF + (M - M)(1) - (M - A)(0) + \int_M^B (-1)dF + (B - M)(0) - (M - M)(1) \\ &= \int_A^M dF - \int_M^B dF = 0 , \quad \text{for an extremum.} \end{aligned}$$

Solving, we obtain:

$$\int_A^M dF = \int_M^B dF .$$

So, M must be the value of X such that $\text{Pr.}[X < M] = \text{Pr.}[X > M]$. That is, M must be the **median** of the distribution.

To verify that we have *minimized* $E | X - M |$, note that:

$$\begin{aligned} \frac{d^2\phi}{dM^2} &= \frac{\partial}{\partial M} \int_A^M (1)dF - \frac{\partial}{\partial M} \int_M^B (1)dF \\ &= \left[\int_A^M (0)dF + (1)\left(\frac{dM}{dM}\right) - (1)\left(\frac{dA}{dM}\right) \right] - \left[\int_M^B (0)dF + (1)\left(\frac{dB}{dM}\right) - (1)\left(\frac{dM}{dM}\right) \right] \\ &= [0 + 1 - 0] - [0 + 0 - 1] = 2 \quad (> 0). \quad \# \end{aligned}$$

Also, note that if we let $A \rightarrow -\infty$ and $B \rightarrow \infty$, the support of F can be taken to be the whole real line, so there is no loss of generality in establishing the above result with a support of $[A, B]$.

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