Question 1:

Write brief notes (and/or diagrams) to explain what we mean by each of the following:

(a) The Lindeberg-Lévy Central Limit Theorem.

(b) A Mini-Max estimator.

(c) An Asymptotically Efficient estimator.

Total: 18 Marks

Question 2:

A Pearson Type III distribution has three parameters. When one of these parameters is set to 0.5, the density function for a random variable, \(Y\), that follows this distribution is:

\[
p(y) = \frac{1}{\beta \sqrt{\pi}} \left( \frac{y - \alpha}{\beta} \right)^{-1/2} \exp \left\{ -\frac{(y - \alpha)}{\beta} \right\} ; \quad y > \alpha
\]

and the associated characteristic function can be shown to be:

\[
\phi_y(t) = \exp \{i\alpha t\} \left(1 - i\beta t \right)^{-1/2}.
\]

(a) Suppose we construct a new random variable, \(X = Y_1 + Y_2\), where \(Y_1\) and \(Y_2\) are independent. What is the characteristic function for \(X\)?

3 marks

(b) Use this characteristic function to prove that \(E(X) = 2\alpha + \beta\).

8 marks

(c) Now, suppose that we know that \(\alpha = 0\). If we have \(n\) independent sample values, show that the MLE of \(\beta\) is \(\tilde{\beta} = 2\bar{Y} = \frac{2}{n} \sum_{i=1}^{n} y_i\). (Don’t forget the second-order condition.)

8 marks

(d) What is the MLE for \(E(X)\) in part (b), under the conditions stated in part (c)?

1 mark

Total: 20 Marks
Question 3:

The density for a random variable, \( Y \), that follows a Rayleigh distribution is:

\[
p(y \mid \theta) = \frac{(y / \theta^2)}{\exp \{-y^2 / (2\theta^2)\}} ; \quad y > 0 ; \quad \theta > 0 .
\]

and the \( k \)th moment about the origin is \( \mu_k = \theta^k 2^{k/2} \Gamma \left[ 1 + \frac{k}{2} \right] \). Here, the Gamma function satisfies the recurrence relationship, \( \Gamma(x + 1) = x\Gamma(x) \); \( \Gamma(1) = 1 \) and \( \Gamma(1/2) = \sqrt{\pi} \).

(a) If we have \( n \) independent sample values, show that the MLE of \( \theta \) is \( \hat{\theta} = \sqrt{(1/2n) \sum y_i^2} \).

(b) Show that the mean of \( Y \) is \( \theta\sqrt{\pi / 2} \) and the variance of \( Y \) is \( \theta^2 \left( 4 - \pi \right) / 2 \). What are the MLE’s for the mean, variance and standard deviation of \( Y \), and what desirable properties will these estimator have?

(c) Derive the Likelihood Ratio Test statistic for testing \( H_0: \theta = 1 \) against \( H_1: \theta \neq 1 \).

(d) Suppose that \( n = 100 \) and \( \sum_{i=1}^{n} y_i^2 = 180 \). Apply the LRT. What assumptions have you made? Is your conclusion sensitive to your choice of significance level?

Question 4:

This question relates to the estimation of a particular “Tobit” model, which explains hours worked in terms of several explanatory variables. The data for the dependent variable are truncated from below – we don’t observe the characteristics of people who work zero hours. In addition, the underlying distribution is non-Normal.

(a) What does OUTPUT suggest about the success (or otherwise) of this estimation?

(b) What do you conclude from OUTPUT 2?

(c) Use the results in OUTPUT 1 and OUTPUT 3 to test the same hypothesis as in OUTPUT 2, but using a different test.

(d) What do you conclude from OUTPUT 4?

Total: 24 Marks

Total: 13 Marks
OUTPUT 1

Dependent Variable: HOURS
Method: ML - Censored Extreme Value (Newton-Raphson / Marquardt steps)
Date: 02/17/16  Time: 13:45
Sample: 1753
Included observations: 753
Left censoring (value) at zero
Convergence achieved after 8 iterations
Coefficient covariance computed using the Huber-White method

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
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<td>486.2058</td>
<td>3.536720</td>
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<td>0.021072</td>
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<tr>
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<td>EDUC</td>
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<td>21.26279</td>
<td>-1.153327</td>
<td>0.2488</td>
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<tr>
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<td>12.04116</td>
<td>-3.085017</td>
<td>0.0020</td>
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<tr>
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Error Distribution

| SCALE:C(10)  | 1112.469 | 103.8358 | 10.71373 | 0.0000 |

Mean dependent var 740.5764  S.D. dependent var 871.3142
S.E. of regression 577.0212  Akaike info criterion 9.899979
Sum squared resid 2.47E+08  Schwarz criterion 9.961388
Log likelihood -371.7342  Hannan-Quinn criter. 9.932637
Avg. log likelihood -4.935709

Left censored obs 325  Right censored obs 0
Uncensored obs 428  Total obs 753

OUTPUT 2

Wald Test:
Equation: TOBIT

<table>
<thead>
<tr>
<th>Test Statistic</th>
<th>Value</th>
<th>df</th>
<th>Probability</th>
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</thead>
<tbody>
<tr>
<td>F-statistic</td>
<td>0.729663</td>
<td>(3, 743)</td>
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<tr>
<td>Chi-square</td>
<td>2.188988</td>
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</table>

Null Hypothesis: C(4)=C(6)=C(9)=0
OUTPUT 3

Dependent Variable: HOURS
Method: ML - Censored Extreme Value (Newton-Raphson / Marquardt steps)
Date: 02/17/16 Time: 13:36
Sample: 1753
Included observations: 753
Left censoring (value) at zero
Convergence achieved after 9 iterations
Coefficient covariance computed using the Huber-White method

<table>
<thead>
<tr>
<th>Variable</th>
<th>Coefficient</th>
<th>Std. Error</th>
<th>z-Statistic</th>
<th>Prob.</th>
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<tbody>
<tr>
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<tr>
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Error Distribution

<table>
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<tr>
<th>SCALE C(7)</th>
<th>1113.790</th>
<th>104.0929</th>
<th>10.69996</th>
<th>0.0000</th>
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</thead>
</table>

- Mean dependent var: 740.5764
- S.D. dependent var: 871.3142
- S.E. of regression: 565.3175
- Akaiki info criterion: 9.896655
- Sum squared resid: 2.58E+08
- Schwarz criterion: 9.939641
- Log likelihood: -3719.091
- Hannan-Quinn criterion: 9.913216
- Avg. log likelihood: -4.933032

Left censored obs: 325
Right censored obs: 0
Uncensored obs: 428
Total obs: 753

OUTPUT 4

Gradients of the Objective Function
Gradients evaluated at estimated parameters
Equation: TOBIT
Method: ML - Censored Extreme Value
Specification: HOURS C FAMINC NWIFEINC EXPER AGE KIDSLT6

<table>
<thead>
<tr>
<th>Variable</th>
<th>Sum</th>
<th>Mean</th>
<th>Weighted Grad</th>
</tr>
</thead>
<tbody>
<tr>
<td>C</td>
<td>1.47E-16</td>
<td>1.95E-19</td>
<td>4.55E-10</td>
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<tr>
<td>FAMINC</td>
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<td>6.29E-15</td>
<td>-1.74E-17</td>
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<tr>
<td>NWIFEINC</td>
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<td>1.14E-17</td>
<td>1.00E-12</td>
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<tr>
<td>EXPER</td>
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<td>1.53E-19</td>
<td>-1.06E-12</td>
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<tr>
<td>AGE</td>
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<td>1.37E-17</td>
<td>2.73E-13</td>
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<tr>
<td>KIDSLT6</td>
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<td>6.95E-20</td>
<td>-4.50E-10</td>
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<td>SCALE: C(7)</td>
<td>3.13E-16</td>
<td>4.16E-19</td>
<td>1.05E-10</td>
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END OF TEST