

The Solution!

(a)

$$y_i = \beta x_i + \varepsilon_i$$

Let $\hat{\beta} = \sum a_i y_i$ be any linear estimator.
So, $E(\hat{\beta}) = \sum a_i E(y_i) = \beta \sum a_i x_i$, and

$$\text{Bias}(\hat{\beta}) = E(\hat{\beta}) - \beta = \beta [\sum a_i x_i - 1].$$

Similarly,

$$\text{var.}(\hat{\beta}) = \sum a_i^2 \text{var}(y_i) = \sigma^2 \sum a_i^2.$$

$$\text{So, } \text{MSE}(\hat{\beta}) = M = \sigma^2 \sum a_i^2 + \beta^2 [\sum a_i x_i - 1]^2.$$

$$\frac{\partial M}{\partial a_j} = 2\sigma^2 a_j + 2\beta^2 [\sum a_i x_i - 1] x_j = 0, \quad \forall j. \quad (1)$$

Multiply by y_j & add over all j :

$$2\sigma^2 \sum a_j y_j + 2\beta^2 [\sum a_i x_i - 1] \sum x_j y_j = 0$$

$$\text{or: } \sigma^2 \hat{\beta} + \beta^2 [\sum a_i x_i - 1] \sum x_j y_j = 0 \quad (2)$$

Also, multiply (1) by x_j & sum over all j :

$$2\sigma^2 \sum a_j x_j + 2\beta^2 [\sum a_i x_i - 1] \sum x_j^2 = 0$$

$$\Rightarrow \sum a_i x_i = \left[\frac{\beta^2 \sum x_i^2}{\sigma^2 + \beta^2 \sum x_i^2} \right] \quad (\sum \varepsilon_i = 0)$$

Substituting into (2):

$$\sigma^2 \hat{\beta} + \beta^2 \left[\frac{\beta^2 \sum x_i^2}{\sigma^2 + \beta^2 \sum x_i^2} - 1 \right] \sum x_i y_i = 0$$

$$\text{or, } \sigma^2 \hat{\beta} + \beta^2 \left[\frac{\beta^2 \sum x_i^2 - \sigma^2 - \beta^2 \sum x_i^2}{\sigma^2 + \beta^2 \sum x_i^2} \right] \sum x_i y_i = 0$$

$$\text{or, } \hat{\beta} = \left[\frac{\beta^2 \sigma^2}{\sigma^2 + \beta^2 \sum x_i^2} \right] \frac{\sum x_i y_i}{\sigma^2}$$

$$= \left(\frac{\beta^2}{\sigma^2 + \beta^2 \sum x_i^2} \right) \sum x_i y_i$$

$$= \left(\frac{\beta^2 \sum x_i^2}{\sigma^2 + \beta^2 \sum x_i^2} \right) b \quad \#$$

where $b = (\sum x_i y_i / \sum x_i^2) = \text{OLS estimator}$.

As $\hat{\beta}$ depends on β & σ^2 , it can't be applied in practice.

(b) Now consider minimizing

$$H = h \left[\frac{\text{var}(\hat{\beta})}{\sigma^2} \right] + (1-h) \left[\frac{\text{Bias}(\hat{\beta})}{\beta} \right]^2; \text{ } 0 < h < 1$$

where again, $\hat{\beta} = \sum a_i y_i$. So,

$$\partial H / \partial a_j = 2 h a_j + 2 (1-h) x_j [\sum a_i x_i - 1] = 0.$$

Multiply by y_j & sum :

$$h \sum a_j y_j + (1-h) \sum x_j y_j [\sum a_i x_i - 1] = 0.$$

$$\text{or } h \hat{\beta} + (1-h) \sum x_i y_i [\sum a_i x_i - 1] = 0.$$

Sum over j after multiplying by x_j —

$$h \sum a_j x_j + (1-h) \sum x_j^2 [\sum a_i x_i - 1] = 0$$

$$\Rightarrow \sum a_i x_i [h + (1-h) \sum x_i^2] = (1-h) \sum x_i^2$$

$$\Rightarrow \sum a_i x_i = \frac{(1-h) \sum x_i^2}{h + (1-h) \sum x_i^2}$$

$$\text{So, } h \hat{\beta} + (1-h) \sum x_i y_i \left[\frac{(1-h) \sum x_i^2}{h + (1-h) \sum x_i^2} - 1 \right] = 0$$

$$\Rightarrow h \hat{\beta} + (1-h) \sum x_i^2 b \left[\frac{(1-h) \sum x_i^2 - h - (1-h) \sum x_i^2}{h + (1-h) \sum x_i^2} \right] = 0$$

$$\Rightarrow \hat{\beta} = \left[\frac{1}{h + (1-h) \sum x_i^2} \right] (1-h) \sum x_i^2 b$$

$$= \left[\frac{(1-h) \sum x_i^2}{h + (1-h) \sum x_i^2} \right] b.$$

This estimator is operational for any $h \in (0, 1)$, and if $h=0$, or $h=1$, $\hat{\beta} = b$, or $\hat{\beta} = 0$, respectively.