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Interpreting Dummy Variables in Semi-logarithmic Regression Models: Exact Distributional Results

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Abstract

Care must be taken when interpreting the coefficients of dummy variables in semi-logarithmic regression models. Existing results in the literature provide the best unbiased estimator of the percentage change in the dependent variable, implied by the coefficient of a dummy variable, and of the variance of this estimator. We extend these results by establishing the exact sampling distribution of an unbiased estimator of the implied percentage change. This distribution is non-normal, and is positively skewed in small samples. We discuss the construction of bootstrap confidence intervals for the implied percentage change, and illustrate our various results with two applications: one involving a wage equation, and one involving the construction of an hedonic price index for computer disk drives.

Keywords Semi-logarithmic regression, dummy variable, percentage change, confidence interval

JEL Classifications C13, C20, C52

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1. Introduction

Semi-logarithmic regressions, in which the dependent variable is the natural logarithm of the variable of interest, are widely used in empirical economics and other fields. It is quite common for such models to include, as regressors, "dummy" (zero-one indicator) variables which signal the possession (or absence) of qualitative attributes. Specifically, consider the following model:

$$ln(Y) = a + \sum_{i=1}^{l} b_i X_i + \sum_{j=1}^{m} c_j D_j + \varepsilon \quad , \tag{1}$$

where the X_i's are continuous regressors and the D_j's are dummy variables.

The interpretation of the estimated regression coefficients is straightforward in the case of the continuous regressors in (1): $100\hat{b}_i$ is the estimated percentage change in *Y* for a small change in X_i. However, as was pointed out initially by Halvorsen and Palmquist (1980), this interpretation does not hold in the case of the estimated coefficients of the dummy variables. The proper representation of the proportional impact, p_j, of a zero-one dummy variable, D_j , on the dependent variable, *Y*, is $p_j = [\exp(c_j) - 1]$, and there is a well-established literature on the appropriate estimation of this impact. More specifically, and assuming normal errors in (1), Kennedy (1981) proposes the consistent (and almost unbiased) estimator, $\hat{p}_j = [\exp(\hat{c}_j)/\exp(0.5\hat{V}(\hat{c}_j))] - 1$, where $\hat{c}_j$ is the OLS estimator of c_j, and $\hat{V}(\hat{c}_j)$ is its estimated variance. Giles (1982) provides the formula for the exact minimum variance unbiased estimator of p_j, and Van Garderen and Shah (2002) provide the formulae for the variance of the latter estimator, and the minimum variance unbiased estimator of this variance. Derrick (1984) and Bryant and Wilhite (1989) also investigate this problem.

Surprisingly, this literature is often overlooked by some practitioners who interpret $\hat{c}_j$ as if it were the coefficient of a continuous regressor. However, there is a diverse group of empirical applications that are more enlightened in this respect . Examples include the studies of Thornton and Innes (1989), Rummery (1992), Levy and Miller (1996), MacDonald and Cavalluzzo (1996), Lassibille (1998), Malpezzi *et al*. (1998) and Fedderson and Maennig (2009). There is general agreement on the usefulness of $\hat{p}_j$ (although see Krautmann and Ciecka, 2006 for an alternative viewpoint). However, the literature is silent on the issue of the precise form of the finite-sample distribution of this statistic. Such information is needed in order to conduct formal inferences about p_j. Asymptotically, of course, $\hat{p}_j$ is the maximum likelihood estimator of p_j, by invariance, and so its limit distribution is normal, in general. As we will show, however, appealing to this limit distribution can be extremely misleading even for quite large sample sizes. In addition, Hendry and Santos (2005) show that $\hat{p}_j$ will be inconsistent and asymptotically non-normal for certain specific formulations of the dummy variable, so particular case must be taken in such cases.

In the next section we provide more details about the underlying assumptions for the problem under discussion and introduce some simplifying notation. Our main result, the density function for $\hat{p}_j$, is derived in section 3, and in section 4 we present some numerical evaluations and simulations that explore the characteristics of this density. Section 5 discusses the construction of confidence intervals for p_j based on $\hat{p}_j$, and two empirical applications are discussed in section 6. Section 7 concludes.

2. Assumptions and Notation

Consider the linear regression model (1) based on n observations on the data:

$$\ln(Y) = a + \sum_{i=1}^{l} b_i X_i + \sum_{j=1}^{m} c_j D_j + \varepsilon \quad ,$$

(where the continuous regressors may also have been log-transformed, without affecting any of the following discussion or results), and the random error term satisfies $\varepsilon \sim N(0, \sigma^2 I)$. Let d_{jj} be the j^{th} diagonal element of $(X'X)^{-1}$, where $X = (X_1, X_2, \ldots.., X_l, D_1, D_2, \ldots. D_m)$. In addition, let $\hat{c}_j$ be the OLS estimator of c_j, so that $\hat{c}_j \sim N(c_j, \sigma^2 d_{jj})$. The usual unbiased estimator of the variance of $\hat{c}_j$ is

$$\hat{V}(\hat{c}_j) = \hat{\sigma}^2 d_{jj} = (d_{jj}\sigma^2 / \nu) u ,$$

where $\nu = (n - l - m)$, $\hat{\sigma}^2 = (e'e)/\nu$, e is the OLS residual vector, and $u \sim \chi^2_\nu$.

Giles (1982) shows that the exact minimum variance estimator of p_j is

$$\tilde{p}_j = \exp(\hat{c}_j)\sum_{i=0}^{\infty}\left(\frac{(\nu/2)^i\,\Gamma(\nu/2)}{\Gamma(i+\nu/2)}\frac{(-0.5\hat{V}(\hat{c}_j))^i}{i!}\right)-1. \qquad (2)$$

He also shows that the approximation, $\hat{p}_j$, provided by Kennedy (1981) is extremely accurate even in quite small samples. Van Garderen and Shah (2002) offer some further insights into the accuracy of this approximation, and provide strong evidence that favours its use. They show that $\tilde{p}_j$ may be expressed more compactly as

$$\tilde{p}_j = \exp(\hat{c}_j)\ {}_0F_1\big((\nu/2)\,;-\nu\hat{V}(\hat{c}_j)/4\big)-1 \quad , \qquad (3)$$

where ${}_0F_1(.;.)$ is the confluent hypergeometric limit function (*e.g.*, Abramowitz and Segun, 1965, Ch.15 ; Abadir, 1999, p.291). In addition, they derive the variance of $\tilde{p}_j$, the exact unbiased estimator of this variance, and a convenient approximation to this variance estimator as is discussed in section 5 below.

Hereafter, and without loss of generality, we suppress the "*j*" subscripts to simplify the notation. Our primary objective is to derive the density function of the following statistic, which estimates the proportional impact of a dummy variable on the variable Y itself, in (1):

$$\hat{p} = [\exp(\hat{c})/\exp(0.5\,\hat{V}(\hat{c}))]-1 \quad .$$

Note that $\hat{p} > -1$.

3. Main Result

First, consider the two random components of $\hat{p}$, and their joint probability distribution.

Lemma 1: Let $x = \exp(\hat{c})$, and $y = \exp(0.5\hat{V}(\hat{c}))$. The joint probability p.d.f. of x and y is:

$$f(x,y) = k''x^{-1}y^{-(1+\nu/(\sigma^2 d))}(\ln y)^{\nu/2-1}\exp\{-[\ln x - c]^2/(2\sigma^2 d)\}\ ,$$

where

$$k'' = \frac{\nu^{\nu/2}}{(2\pi)^{1/2}(\sigma^2 d)^{(\nu+1)/2}\Gamma(\nu/2)} .$$

Proof: Under our assumptions, the random variable $x = \exp(\hat{c})$ is log-normally distributed, with density function:

$$f(x) = \frac{1}{x\sigma\sqrt{2\pi d}} \exp\{-[\ln x - c]^2 / (2\sigma^2 d)\} \qquad ; \qquad x > 0 . \tag{4}$$

Let $y = \exp(0.5\hat{V}(\hat{c})) = \exp(ku)$, where $k = (\sigma^2 d / 2\nu)$. As $\hat{c}$ and $\hat{\sigma}^2$ are independent, so are x and y. Note that the density of a χ^2_ν variate, u, is

$$f(u) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)} u^{\nu/2-1} e^{-u/2} \qquad ; \qquad u > 0 . \tag{5}$$

It follows immediately that the p.d.f. of y is

$$\begin{aligned} f(y) &= \frac{1}{k\, y 2^{\nu/2}\Gamma(\nu/2)} (\frac{1}{k}\ln y)^{\nu/2-1} e^{-(\ln y/2k)} \qquad ; \qquad y > 1 \\ &= k'(\ln y)^{\nu/2-1} y^{-(1+\nu/(\sigma^2 d))} \end{aligned} \tag{6}$$

where

$$k' = \frac{1}{(\sigma^2 d/\nu)^{\nu/2}\Gamma(\nu/2)} . \tag{7}$$

Using the independence of x and y,

$$f(x, y) = k'' x^{-1} y^{-(1+\nu/(\sigma^2 d))} (\ln y)^{\nu/2-1} \exp\{-[\ln x - c]^2 / (2\sigma^2 d)\} , \tag{8}$$

where

$$k'' = \frac{\nu^{\nu/2}}{(2\pi)^{1/2}(\sigma^2 d)^{(\nu+1)/2}\Gamma(\nu/2)} . \tag{9}$$

■

We now have the joint p.d.f. of the two random components of $\hat{p}$, and this can now be used to derive the pd.f. of $\hat{p}$ itself.

Theorem 1: The exact finite-sample density function of $\hat{p}$ is

$$f(\hat{p}) = \left(\frac{\nu^{\nu/2} e^{-\alpha^2 \beta/2}}{(\hat{p}+1) 2^{(\nu+2)/4} (\sigma^2 d)^{(\nu+1)/2}} \right) \times$$

$$\left(\frac{1}{\Gamma((\nu+2)/4)} {}_1F_1((\nu/4), 0.5; (\alpha^2\beta)) - \frac{(\alpha+\nu)\sqrt{2\beta}}{\Gamma(\nu/4)}) \; {}_1F_1((2-\nu)/4, 1.5; (\alpha^2\beta)) \right) \quad ; \quad \hat{p} > -1$$

where ν is the regression degrees of freedom, σ^2 is the regression error variance, d is the diagonal element of the $(X'X)^{-1}$ matrix associated with the dummy variable in question, c is the true value of the coefficient of that dummy variable, $\beta = 1/(2\sigma^2 d)$, $\alpha = \ln(\hat{p}+1) - c$, and ${}_1F_1(.,.;.)$ is the confluent hypergeometric function (e.g., Gradzshteyn and Ryzhik, 1965, p.1058).

Proof: Consider the change of variables from x and y to $w = (\ln x - c)$ and $\hat{p} = (x-y)/y$. The Jacobian of the transformation is $[\exp(w+c)/(\hat{p}+1)]^2$, so

$$f(w, \hat{p}) = k''(\hat{p}+1)^{\nu/(\sigma^2 d)-1} \exp\{-[(w^2/2) + \nu(w+c)]/(\sigma^2 d)\}[w + c - \ln(\hat{p}+1)]^{\nu/2-1} \; ;$$

for $\hat{p} > -1$; $w > \ln(\hat{p}+1) - c$. (10)

The marginal density of $\hat{p}$ can then be obtained as

$$f(\hat{p}) = k''(\hat{p}+1)^{\nu/(\sigma^2 d)-1} \int_{\alpha}^{\infty} \exp\{-[(w^2/2) + \nu(w+c)]/(\sigma^2 d)\}[w + c - \ln(\hat{p}+1)]^{\nu/2-1} dw, \quad (11)$$

where $\alpha = \ln(\hat{p}+1) - c$.

Making the change of variable, $z = w + c - \ln(\hat{p}+1)$, we have

$$f(\hat{p}) = k''(\hat{p}+1)^{\nu/(\sigma^2 d)-1}\int_0^\infty z^{\nu/2-1}\exp\{-[(z+\alpha)^2/2+\nu(z+\ln(\hat{p}+1))]/(\sigma^2 d)\}dz\,. \tag{12}$$

Then, defining $\beta = 1/(\sigma^2 d)$, (12) can be written as:

$$f(\hat{p}) = k''(\hat{p}+1)^{\nu\beta-1}e^{-\beta[\nu\ln(\hat{p}+1)+\alpha^2/2]}\int_0^\infty z^{\nu/2-1}e^{-(\beta/2)z^2-\beta(\alpha+\nu)z}dz\,. \tag{13}$$

Then, using the integral result 3.462 (1) of Gradzshteyn and Ryzhik (1965, p.337),

$$f(\hat{p}) = k''(\hat{p}+1)^{\nu\beta-1}e^{-\beta[\nu\ln(\hat{p}+1)+\alpha^2/2]}e^{\beta(\alpha+\nu)^2/4}\beta^{-\nu/4}\Gamma(\nu/2)D_{-\nu/2}((\alpha+\nu)\sqrt{\beta}) \tag{14}$$

for $\hat{p} > -1$, where $D_\gamma(.)$ is the parabolic cylinder function (Gradzshteyn and Ryzhik, 1965, p.1064). Using the relationship between the parabolic cylinder function and (Kummer's) confluent hypergeometric function, we have:

$$D_{-\nu/2}((\alpha+\nu)\sqrt{\beta}) = 2^{-\nu/4}\sqrt{\pi}e^{-[\beta(\alpha+\nu)^2]/4}\times$$
$$\left(\frac{1}{\Gamma((\nu+2)/4)}\,{}_1F_1((\nu/4)\,,0.5\,;\,\beta(\alpha+\nu)^2/2) - \frac{(\alpha+\nu)\sqrt{2\beta}}{\Gamma(\nu/4)})\,{}_1F_1((2+\nu)/4\,,1.5\,;\,\beta(\alpha+\nu)^2/2)\right) \tag{15}$$

where the confluent hypergeometric function is defined as (Gradzshteyn and Ryzhik, 1965, p.1058):

$$\begin{aligned}{}_1F_1(a\,,c\,;z) &= \sum_{j=0}^{\infty}\frac{(a)_j}{(c)_j}\frac{z^j}{j!}\\ &= 1+\frac{a}{c}z+\frac{a(a+1)}{c(c+1)}\frac{z^2}{2!}+\frac{a(a+1)(a+2)}{c(c+1)(c+2)}\frac{z^3}{3!}+\ldots\ldots\ldots\ldots\end{aligned} \tag{16}$$

Parenthetically, Pochhammer's symbol is

$$(\xi)_j = \prod_{k=0}^{j-1}(\xi+k) = \frac{\Gamma(\xi+j)}{\Gamma(\xi)}\,, \tag{17}$$

where it is understood that empty products in its construction are assigned the value unity. So, recalling the definition of k'' in (9), the density function of $\hat{p}$ can be written as:

$$f(\hat{p}) = \left(\frac{\nu^{\nu/2} e^{-\alpha^2 \beta/2}}{(\hat{p}+1) 2^{(\nu+2)/4} (\sigma^2 d)^{(\nu+1)/2}} \right) \times \tag{18}$$

$$\left(\frac{1}{\Gamma((\nu+2)/4)} \, {}_1F_1((\nu/4), 0.5 \,;\, (\alpha^2 \beta)) - \frac{(\alpha+\nu)\sqrt{2\beta}}{\Gamma(\nu/4)}) \, {}_1F_1((2-\nu)/4, 1.5 \,;\, (\alpha^2 \beta)) \right) \quad ; \quad \hat{p} > -1$$

■

4. Numerical Evaluations

Given its functional form, the numerical evaluation of the density function in (18) is non-trivial. A helpful discussion of confluent hypergeometric (and related) functions is provided by Abadir (1999), for example, and some associated computational issues are discussed by Nardin *et al*. (1992) and Abad and Sesma (1996). In particular, it is well known that great care has to be taken over the computation of the confluent hypergeometric functions, and the leading term in (18) also poses challenges for even modest values of the degrees of freedom parameter, ν . Our evaluations were undertaken using a FORTRAN 77 program, written by the author. This program incorporates the double-precision complex code supplied by Nardin *et al.* (1989), to implement the methods described by Nardin *et al*. (1992), for the confluent hypergeometric function; and the GAMMLN routine from Press *et al*. (1992) for the (logarithm of the) gamma function. Monte Carlo simulations were used to verify the exact numerical evaluations, and hence the validity of (18) itself.

Figures 1 and 2 illustrate $f(\hat{p})$ for small degrees of freedom and various choices of the other parameters in the p.d.f.. The true value of p is 6.39 in Figure 1, and its values in Figure 2 are 2980.0 (c = 8) and 4446.1 (c = 8.4).

The quality of a normal asymptotic approximation to $f(\hat{p})$ has been explored in a small Monte Carlo simulation experiment, involving 5,000 replications, with code written for the SHAZAM econometrics package (Whistler *et al*., 2004). The data-generating process used is

$$\ln(Y_i) = a + b_1 X_{1i} + b_2 X_{2i} + cD_i + \varepsilon_i \quad ;$$

$$\varepsilon_i \sim i.i.d.\, N[0, \sigma^2] \qquad ; \quad i = 1, 2, 3, \ldots, n. \tag{19}$$

Figure 1: p.d.f.'s of p-hat
(*v* = 5, c = 2, d = 0.022)

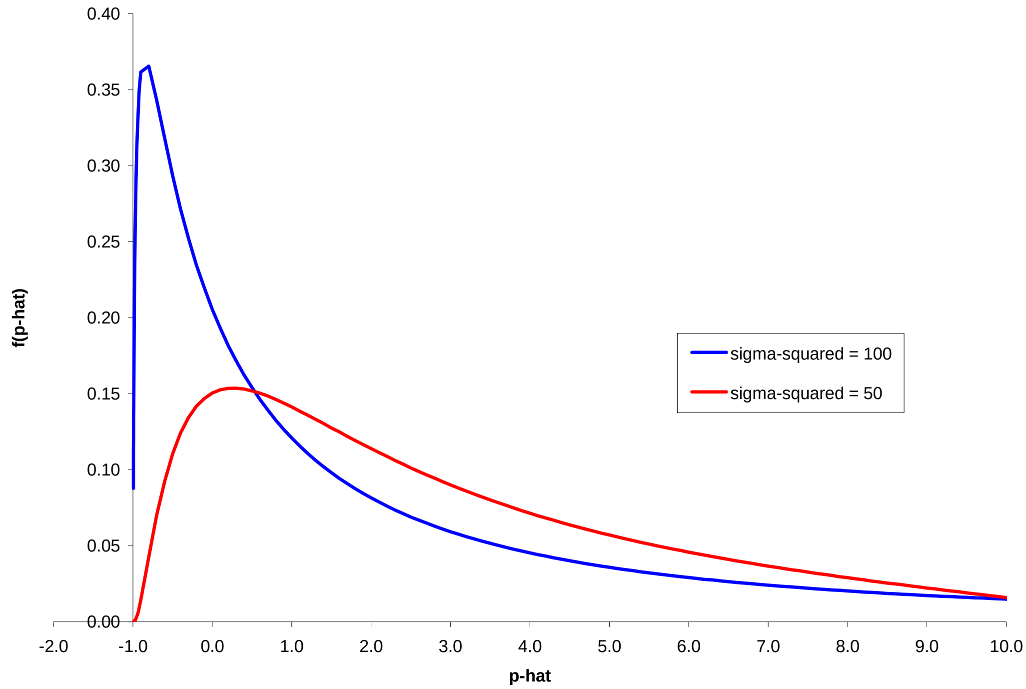
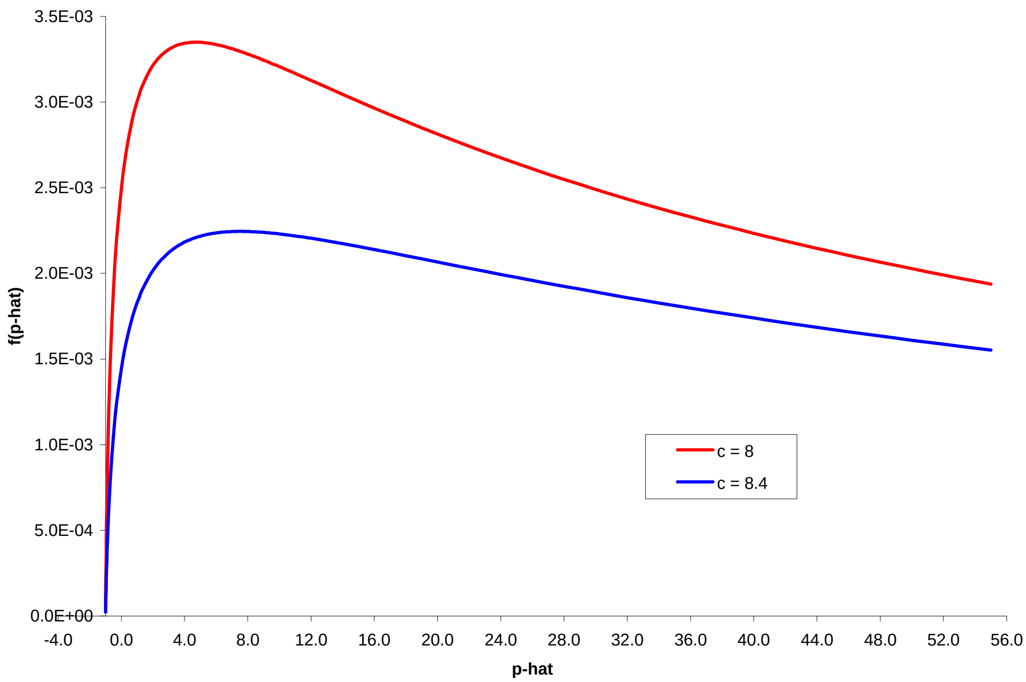


Figure 2: p.d.f.'s of p-hat
(*v* = 10, d = 1.5, sigma-squared = 2.4)



The regressors X_1 and X_2 were (pre-) generated as χ_1^2 and standard normal variables respectively, and held fixed in repeated samples. We considered a range of sample sizes, *n*, to explore both the finite-sample and asymptotic features of $f(\hat{p})$; and we set a = 1, $b_1 = b_2$ = 0.1 c = 0.5, and $\sigma^2 = 2$. The implied true value of *p* is 0.65, and the value of *d* is determined by the data for *X*, the construction of the dummy variable, *D*, and the sample size, *n*, and two cases can be considered. First, the number of non-zero values in *D* is allowed to grow at the same rate of *n*, so the usual asymptotics apply. In this case we set *D* = 1 for *i* = 1, 2, …, $(n/2)$, and zero otherwise. The sample R^2 values for the fitted regressions are typical for cross-section data. Averaged over the 5,000 replications, they are in the range 0.423 (*n* = 10) to 0.041 (*n* = 15,000). Second, the number of the non-zero values in *D* is *fixed* at some value, n_D, in which case the usual asymptotics do *not* apply. More specifically, in this second case the OLS estimator of *c* is *inconsistent*, and its limit distribution is non-normal. This arises as a natural generalization of the results in Hendry and Santos (2005), for the case where n_D = 1. In this second case we set n_D = 5, and assign only the last five values of *D* to unity, without loss of generality.

Table 1 reports summary statistics from this experiment, namely the %Bias of $\hat{p}$, and the standard deviation and skewness and kurtosis coefficients for its empirical sampling distribution. All of the p-values associated with the Jarque-Bera (J-B) normality test are essentially zero, except the one indicated. As we can see, for Case 1 (where standard asymptotics apply) the consistency of $\hat{p}$ is reflected in the decline in the % biases and standard deviations as *n* increases. For small samples, the distribution of $\hat{p}$ has positive skewness and excess kurtosis, as expected from Figures 1 and 2. In Case 2 the usual asymptotics do not apply. The inconsistency of $\hat{p}$ is obvious, as is the non-normality of its limit distribution. The latter is positively skewed with large positive excess kurtosis. Figures 3 and 4 illustrate the sampling distributions of $\hat{p}$ when n = 1,000, for Case 1 and Case 2 respectively in Table 1.

Table 1: Characteristics of Sampling Distribution for $\hat{p}$

Case 1: $n_D = (n/2)$

n	*d*	**%Bias(** $\hat{p}$ **)**	**S.D.(** $\hat{p}$ **)**	**Skew**	**Excess Kurtosis**
10	0.528	13.057	2.470	5.395	47.425
20	0.202	4.335	1.187	2.338	9.431
50	0.081	4.280	0.722	1.535	4.457
100	0.040	1.987	0.484	0.979	1.878
1000	0.004	0.696	0.149	0.165	-0.029
5000	0.001	0.177	0.065	0.156	0.197
15000	3×10^{-4}	0.045	0.037	0.070	0.034*

Case 2: $n_D = 5$

n	*d*	**%Bias(** $\hat{p}$ **)**	**S.D.(** $\hat{p}$ **)**	**Skew**	**Excess Kurtosis**
10	0.462	9.100	2.091	4.225	29.437
20	0.282	5.587	1.490	3.636	29.490
50	0.192	2.811	1.171	2.508	12.452
100	0.190	4.461	1.147	2.220	8.459
1000	0.168	-1.152	1.013	1.839	6.302
5000	0.167	-0.991	1.034	2.119	8.618
15000	0.167	-1.794	1.013	2.075	7.596

* J-B p-value = 0.115. J-B p-values for all other tabulated cases are zero, to *at least* 3 decimal places.

Figure 3: Case 1; n = 1000

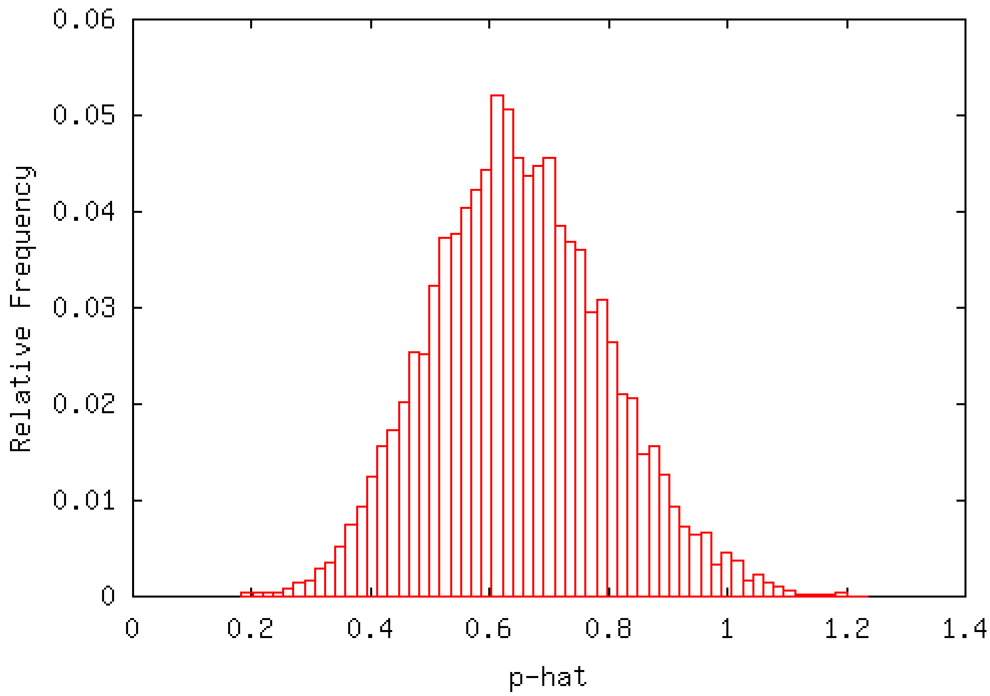
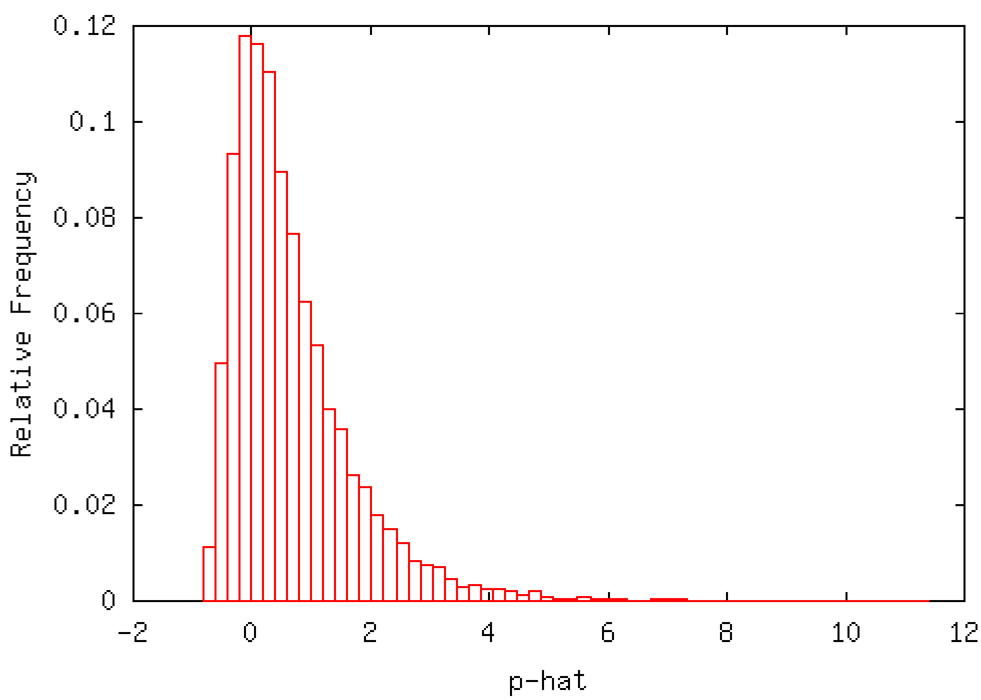


Figure 4: Case 2; n = 1000



5. Confidence Intervals

For very large samples, $\hat{p}$ converges to the MLE of *p*, and the usual asymptotics apply. So, inferences about *p* can be drawn by constructing standard (asymptotic) confidence intervals by using the approximation $\sqrt{n}(\hat{p}-p)\overset{d}{\rightarrow}N[0\,,V(\hat{p})]$, where

$$V(\hat{p})=\exp(2\hat{c})\left\{\exp(V(\hat{c}))\ {}_0F_1\ \left((\nu/2);[V(\hat{c})]^2\right)-1\right\} \tag{20}$$

is derived by van Garderen and Shah (2002, p.151). They also show that the minimum variance unbiased estimator of $V(\hat{p})$ is

$$\hat{V}(\hat{p})=\exp(2\hat{c})\left\{[{}_0F_1\left((\nu/2)\,;-(\nu/4)\hat{V}(\hat{c})\right)^2-{}_0F_1\ \left((\nu/2)\,;-\nu\hat{V}(\hat{c})\right)\right\} \quad . \tag{21}$$

Here, $\hat{V}(\hat{c})$ is just the square of the standard error for $\hat{c}$ from the OLS regression results, and ${}_0F_1(.;.)$ is the confluent hypergeometric limit function defined in section 2. Van Garderen and Shah (2002, p.152) suggest using the approximately unbiased estimator of $V(\hat{p})$, given by

$$\tilde{V}(\hat{p})=\exp(2\hat{c})\left\{\exp(-\hat{V}(\hat{c}))-\exp(-2\hat{V}(\hat{c}))\right\} \quad , \tag{22}$$

and they note that in this context it is superior to the approximation based on the delta method. So, using (22) and the asymptotic normality of $\hat{p}$, large-sample confidence intervals are readily constructed.

In small samples, however, the situation is considerably more complicated. Although $\hat{p}$ is essentially unbiased (case 1), and a suitable estimator of its variance is available, Figures 1 and 2 and the results in Table 1 indicate that the sampling distribution of $\hat{p}$ is far from normal, even for moderate sample sizes. The complexity of the density function for $\hat{p}$ in (18), and the associated c.d.f., strongly suggest the use of the bootstrap to construct confidence intervals for *p*.

We have adapted the Monte Carlo experiment described in section 4 to provide a comparison of the coverage properties of bootstrap percentile intervals and intervals based (wrongly) on the normality assumption together with the variance estimator $\tilde{V}(\hat{p})$. We use 1,000 Monte Carlo replications and 999 bootstrap samples – the latter number being justified by the results of Efron (1987, p.181). In applying the bootstrap to the OLS regressions we use the "normalized residuals" (Efron, 1979; Wu, 1986, p.1265). We limit our investigation to "Case 1" as far as the construction of the dummy variable in model (19) is concerned, so that the usual asymptotics apply to $\hat{c}$ (and hence $\hat{p}$).

The results appear in Table 2, where c_L and c_U are the lower and upper end-points of the 95% confidence intervals. It will be recalled from Table 1 that the density for $\hat{p}$ is positively skewed. So, in the case of the bootstrap confidence intervals the upper and lower end-points are taken as the 0.025 and 0.975 percentiles of the bootstrap samples for $\hat{p}$, averaged over the 999 such samples. In the case of the normal approximation the limits are $\hat{p} \pm 1.96\sqrt{\tilde{V}(\hat{p})}$. In each case, average values taken over the 1,000 Monte Carlo replications are reported in Table 2. A standard bootstrap confidence interval has second-order accuracy. That is, if the intended coverage probability is, say, α, then the coverage probability of the bootstrap confidence interval is $\alpha + O(n^{-1})$. We also report the actual coverage probabilities (CP), and their associated standard errors, for the intervals based on the normal approximation and $\tilde{V}(\hat{p})$. The confidence intervals based on the normal approximation are always "shifted" downwards, relative to the bootstrap intervals. The associated CP values are less than 0.95, but approach this nominal level as the sample size increases. For sample sizes $n \geq 100$ the coverage probabilities of the approximate intervals are within two standard errors of 0.95. Finally, we see that the constraint, $p > -1$, is violated by the approximate intervals for $n \leq 20$.

The simple bootstrap confidence intervals discussed here can, no doubt, be improved upon by considering a variety of refinements to their construction, including those suggested in DiCiccio and Efron (1996) and the associated published comments. However, we do not pursue this here.

Table 2: 95% Confidence Intervals for *p*

n	Bootstrap		Normal Approximation Using $\widetilde{V}(\hat{p})$			
	c_L	c_U	c_L	c_U	CP	(s.e.)
10	-0.800	21.436	-1.982	3.236	0.777	(0.059)
20	-0.515	4.769	-1.174	2.421	0.882	(0.021)
30	-0.436	3.622	-0.954	2.142	0.898	(0.015)
40	-0.306	3.059	-0.724	2.063	0.912	(0.012)
50	-0.270	2.719	-0.632	1.915	0.917	(0.011)
100	-0.055	1.883	-0.251	1.547	0.935	(0.008)
500	0.283	1.111	0.239	1.052	0.947	(0.007)
1000	0.384	0.968	0.362	0.939	0.948	(0.007)
5000	0.521	0.781	0.517	0.774	0.950	(0.007)
10000	0.559	0.743	0.557	0.740	0.950	(0.007)
15000	0.575	0.725	0.574	0.723	0.950	(0.007)

6. Applications

We consider two simple empirical applications to illustrate the various results discussed above. The first application compares both the point and interval estimates of a dummy variable's percentage impact when these estimates are calculated in two ways: first, by naïvely interpreting the coefficient in question as if it were associated with a continuous regressor; and second, using the appropriate (and widely recommended) $100\,\hat{p}$, together with a bootstrap confidence interval. The second application goes beyond the simple interpretation of the results of a semi-logarithmic model with dummy variables, and shows how to construct an appropriate hedonic price index, together with confidence intervals for each period's index value that take account of the non-standard density for $\hat{p}$ discussed in section 3. The effects of incorrectly using a normal approximation are also illustrated.

6.1 Wage Determination Equation

Our first example involves the estimation of a simple wage determination equation. The data are from the "CPS78" data-set provided by Berndt (1991). This data-set relates to 550 randomly chosen employed people from the May 1978 current population survey, conducted by the U.S. Department of Commerce. In particular, we focus on the sub-sample of 36 observations relating to Hispanic workers. The following regression is estimated by OLS:

$$\begin{aligned} ln(WAGE) = a + b_1 ED + b_2 EX + b_3 EX^2 + c_1 UNION + c_2 MANAG + c_3 PROF + \\ c_4 SALES + c_5 SERV + c_6 FE + \varepsilon \quad , \end{aligned} \tag{23}$$

where *WAGE* is average hourly earnings; *ED* is the number of years of education; and *EX* is the number of years of labour market experience. The various zero-one dummy variables are: *UNION* (if working in a union job); *MANAG* (if occupation is managerial/administrative); *PROF* (if occupation is professional/technical); *SALES* (if occupation is sales worker); *SERV* (if occupation is service worker); and *FE* (if worker is female).

The regression results, obtained using EViews 7.1 (Quantitative Micro Software, 2010), appear in Table 3. The estimated coefficients have the anticipated signs and all of the regressors are statistically significant at the 5% level. The various diagnostic tests support the model specification. Importantly, the Jarque-Bera test supports the assumption that the errors in (23) are normally distributed, as required for our analysis.

Table 4 reports estimated percentage impacts implied by the various dummy variables in the regression. These have been calculated in two ways. First, we provide naïve estimates, based on the incorrect (but frequently used) assumption that they are simply $100\,\hat{c}_j$, where $\hat{c}_j$ is the OLS estimate of the j^{th} dummy variable coefficient. Second, we report results based on the almost unbiased estimator, $100\,\hat{p}_j$. In each case, 95% confidence intervals are presented. The intervals based on the naïve estimates are constructed using the standard errors reported in Table 3, together with the Student-t critical values. The intervals based on the (almost) unbiased estimates of the percentage impacts are bootstrapped, using 999 bootstrap samples.

Table 3: (Log-)Wage Determination Equations

(Hispanic Workers)

Const.	0.8342	(5.00)	[0.00]
ED	0.0369	(2.69)	[0.01]
EX	0.0267	(3.05)	[0.00]
EX^2	-0.0004	(-2.28)	[0.02]
UNION	0.4551	(3.89)	[0.00]
MANAG	0.3811	(2.89)	[0.00]
PROF	0.4732	(4.93)	[0.00]
SALES	-0.4276	(-4.43)	[0.00]
SERV	-0.1512	(-1.82)	[0.04]
FE	-0.2791	(-3.66)	[0.00]
n	36		
$\overline{R}^2$	0.6388		
J-B {*p*}	4.2747		{0.12}
RESET {*p*}	0.7263		{0.55}
BPG {*p*}	8.3423		{0.50}
White {*p*}	7.1866		{0.62}

Note: t-values appear in parentheses. These are based on White's heteroskedasticity-consistent standard errors. One-sided p-values appear in brackets. *J-B* denotes the Jarque-Bera test for normality of the errors; *RESET* is Ramsey's specification test (using second, third and fourth powers of the predicted values); *BPG* and *White* are respectively the Breusch-Pagan-Godfrey and White nR^2 tests for homoskedasticity of the errors.

Table 4: Estimated Percentage Impacts of Dummy Variables

Dummy Variable	Naïve $(100\,\hat{c}_j)$	Almost Unbiased $(100\,\hat{p}_j)$
UNION	45.51	56.56
	[21.51 69.50]	[28.52 91.81]
MANAG	38.11	45.12
	[11.01 65.20]	[-20.64 142.74]
PROF	47.32	59.79
	[27.60 67.03]	[18.15 112.03]
SALES	-42.76	-35.09
	[-62.60 -22.92]	[-55.51 -7.45]
SERV	-15.12	-14.33
	[-32.19 1.95]	[-29.12 4.32]
FE	-27.91	-24.57
	[-43.57 -12.25]	[-36.06 -11.28]

Note: 95% confidence intervals appear in brackets. In the case of the almost unbiased percentage impacts, the confidence intervals are based on a bootstrap simulation.

As expected from Table 1 of Halverson and Palmquist (1980), the percentage impacts in Table 4 are always algebraically larger when estimated appropriately than when estimated naïvely. These differences can be substantial – for example, in the case of the *PROF* dummy variable the naïve estimator understates the impact by 12.5 percentage points. In addition, the bootstrap confidence intervals based on $100\,\hat{p}_j$ are wider than those based on $100\,\hat{c}_j$ in four of the six cases in Table 4. In the case of the *MANAG* dummy variable the respective interval widths are 163.4 and 54.2 percentage points. For the *PROF* dummy variable the corresponding widths are 93.9 and 39.4 percentage points. Except for the *SERV* and *FE* dummy variables, the naïve approach results in confidence intervals that are misleadingly short.

6.2 Hedonic Price Index for Disk Drives

As a second example, we consider regressions for computing hedonic price indices for computer disk drives, as proposed by Cole *et al*. (1986). Their (corrected) data are provided by Berndt (1991), and comprise a total of 91 observations over the years 1972 to 1984, for the U.S.. The hedonic price regression is of the form:

$$ln(Price) = a + b_1 ln(Speed) + b_2 ln(Capacity) + \sum_{j=73}^{84} c_j D_j + \varepsilon \qquad , \qquad (24)$$

where *Price* is the list price of the disk drive; *Speed* is the reciprocal of the sum of average seek time plus average rotation delay plus transfer rate; and *Capacity* is the disk capacity in megabytes; and the dummy variables, D_j, are for the marketing years, 1973 to 1984. Some basic OLS results, obtained using EViews 7.1, appear in Table 5. The associated hedonic price indices are presented in Table 6, with 95% confidence intervals.

Two sample periods are considered – the full sample of 91 observations, and a sub-sample of 30 observations. In each case, the Jarque-Bera test again supports the assumption that the errors in (24) are normally distributed, as required for our various analytic results, and the RESET test suggests that the functional forms of the regressions are well specified. Although there is some evidence that the errors are heteroskedastic, we have compensated for this by reporting Newey-West consistent standard errors. Two 95% confidence intervals are given for the price indices in each year in Table 6. The end-points c_L^B and c_U^B relate to the bootstrap percentile intervals, based on 999 bootstrap samples, for price index values based on $\hat{p}$. The end-points for the approximate confidence intervals, obtained using $\widetilde{V}(\hat{p})$ and a normal approximation for the sampling distribution of $\hat{p}$, are denoted c_L^A and c_U^A .

Table 5: Hedonic Price Regressions

	1972 – 1984	1973 – 1976
Const.	9.4283 (9.62) [0.00]	9.6653 (7.46) [0.00]
ln(*Speed*)	0.3909 (2.15) [0.02]	0.5251 (2.04) [0.03]
ln(*Capacity*)	0.4588 (5.36) [0.00]	0.5083 (4.35) [0.00]
D_{73}	0.0160 (0.16) [0.44]	
D_{74}	-0.2177 (-1.35) [0.09]	-0.2441 (-1.84) [0.04]
D_{75}	0.3092 (-2.20) [0.02]	-0.3352 (-2.80) [0.00]
D_{76}	-0.4173 (-3.06) [0.00]	-0.4793 (-3.79) [0.00]
D_{77}	-0.4167 (-3.02) [0.00]	
D_{78}	-0.5740 (-4.05) [0.00]	
D_{79}	-0.7689 (-5.64) [0.00]	
D_{80}	-0.9602 (-6.52) [0.00]	
D_{81}	-0.9670 (-6.25) [0.00]	
D_{82}	-0.9537 (-6.14) [0.00]	
D_{83}	-1.1017 (-5.76) [0.00]	
D_{84}	-1.1812 (-5.99) [0.00]	
n	91	30
$\overline{R}^2$	0.8086	0.6592
J-B {*p*}	1.8736 {0.39}	1.3257 {0.52}
RESET {*p*}	0.9118 {0.41}	0.6429 {0.54}
BPG {*p*}	29.6478 {0.01}	14.9165 {0.01}
White {*p*}	32.2878 {0.00}	17.6305 {0.22}

Note: t-values appear in parentheses. These are based on Newey-West HAC standard errors. One-sided p-values appear in brackets. *J-B* denotes the Jarque-Bera test for normality of the errors. *RESET* is Ramsey's specification test (using second and third powers of the predicted values); *BPG* and *White* are respectively the Breusch-Pagan-Godfrey and White nR^2 tests for homoskedasticity of the errors.

Table 6: Hedonic Price Indices for Disk Drives

(Base = 100)

	1972 – 1984 (*n* = 91)			**1973 – 1976 (*n* = 30)**		
	(c_L^B	**Price Index**	c_U^B **)**	**(** c_L^B	**Price Index**	c_U^B **)**
	[c_L^A		c_U^A **]**	**[** c_L^A		c_U^A **]**
1972		100.000				
1973	(79.800	101.092	125.863)		1 00.000	
	[81.139		121.050]			
1974	(63.489	79.404	101.005)	(57.220	77.658	103.183)
	[54.554		104.255]	[57.584		97.732]
1975	(58.648	72.679	90.565)	(52.239	71.011	95.308)
	[52.768		92.590]	[54.430		87.592]
1976	(52.384	65.270	81.111)	(46.710	61.428	80.577)
	[47.906		82.635]	[46.262		76.594]
1977	(51.543	65.298	83.026)			
	[47.687		82.909]			
1978	(39.134	55.762	76.434)			
	[40.348		71.176]			
1979	(33.591	45.925	63.474)			
	[33.721		58.130]			
1980	(27.082	37.870	52.892)			
	[26.996		48.744]			
1981	(26.653	37.569	51.454)			
	[26.249		48.890]			
1982	(27.049	38.071	52.725)			
	[26.550		49.592]			
1983	(23.167	32.628	46.585)			
	[20.515		44.741]			
1984	(20.535	30.101	44.329)			
	[18.582		41.619]			

First, consider the results in Table 6 for the period 1973 to 1976. All of the approximate confidence intervals are shorter (and misleadingly "more informative") than those computed using the bootstrap to mimic the true sampling distribution of $\hat{p}$. For 1975, for example, the approximate interval is of length 33.2, while the appropriate interval has length 43.1. The results for the period 1972 to 1984 exhibit the same phenomenon in seven of the twelve years. These results also demonstrate another unsettling feature of the approximate intervals. Consider the values of the price index in 1973 and 1974. The appropriate 95% confidence interval for 1974, namely (63.489 , 101.005), does not (quite) cover the point estimate of the index in 1973, namely 101.092. This suggests that the measured fall in the price index from 101.092 to 79.404 is statistically significant at the 5% level. We reach the same conclusion by comparing the appropriate confidence interval for 1973 with the point estimate of the index in 1974. In contrast, we come to exactly the opposite conclusion if we make such comparisons using the *approximate* confidence interval for 1974 and the point estimate for the index in 1973: the notional 21.45% drop in prices from 1973 to 1974 is not statistically different from zero.

7. Conclusions

The correct interpretation of estimated coefficients of dummy variables in a semi-logarithmic regression model has been discussed extensively in the literature. However, incorrect interpretations are easy to find in empirical studies. We have explored this issue by extending the established results in several respects. First, we have derived the exact finite-sample distribution for Kennedy's (1981) widely used (almost) unbiased estimator of the percentage impact of such a dummy variable. This is found to be positively skewed for small samples, and non-normal even for quite large sample sizes. Second, we have demonstrated the effectiveness of constructing bootstrap confidence intervals for the percentage impact of interest, based on the correct underlying distribution. Together, these contributions fill a gap in the known results for the sampling properties of the correctly estimated percentage impact. Finally, two empirical examples illustrate that with modest sample sizes, very misleading results can be obtained if the dummy variables' coefficients are not interpreted correctly; or if the non-standard distribution of the implied percentage changes is ignored, and a normal approximation is blithely used instead.

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