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Exact Asymptotic Goodness-of-Fit Testing For Discrete Circular Data, With Applications

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Abstract

We show that the full asymptotic null distribution for Watson's U_N^2 statistic, modified for discrete data, can be computed simply and exactly by standard methods. Previous approximate quantiles for the uniform multinomial case are found to be accurate. More extensive quantiles are presented for this distribution, as well as for the beta-binomial distribution and for the distributions associated with "Benford's Laws". The latter distributions are for the first one, two, or three significant digits in a sequence of "naturally occurring" numbers. A simulation experiment compares the power of the modified U_N^2 test with that of Kuiper's V_N test. In addition, four illustrative empirical applications are provided to illustrate the usefulness of the U_N^2 test.

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Keywords: Distributions on the circle; Goodness-of-fit; Watson's U_N^2; Discrete data; Benford's Law

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1. INTRODUCTION

The construction of goodness-of-fit tests when the data are distributed on the circle (or more generally the sphere) is an important statistical problem. An excellent discussion is provided, for example, by Mardia and Jupp (2000). Among the tests that have been proposed for *continuous* data are those based on Kuiper's (1959) V_N statistic and Watson's (1961) U_N^2 statistic. These tests are of the Kolmogorov-Smirnov type, being based on the empirical distribution function, and Castro-Kuriss (2011) provides a concise and recent overview of such tests. Goodness-of-fit tests on the circle in the case of *discrete* data are also of considerable practical importance, as we demonstrate with the examples provided in this paper. However, this case has received far less attention in the literature. The complication is that although Kolmogorov-Smirnov statistics are distribution-free in the continuous case, this is generally not the case when the data are discrete (Conover, 1972). In the latter case, modifications are needed.

We will be concerned with testing the null hypothesis, H_0: "The data follow a discrete circular distribution, *F*, defined by the probabilities $\{p_i\}_{i=1}^{n}$", against the alternative hypothesis, H_1: "H_0 is not true". Suppose that we have a sample of *N* observations, and let $\{r_i\}_{i=1}^{n}$ denote the sample frequencies, such that $\sum_{i=1}^{n} r_i = N$. For this general problem, Freedman (1981) proposes a modified version of Watson's U_N^2 statistic for use with discrete data. He provides Monte Carlo evidence that this test out-performs Kuiper's (1962) modified test for the discrete case, when testing the null of multinomial uniform against the alternative of a sine-curve. Freedman's test statistic is:

$$U_N^{*2} = (N/n)\left[\sum_{j=1}^{n-1} S_j^2 - \left(\sum_{j=1}^{n-1} S_j\right)^2 / n\right], \qquad (1)$$

where

$$S_j = \sum_{i=1}^{j} (r_i / N - p_i) \ ; \ j = 1, 2, \ldots, n.$$

He shows that the asymptotic null distribution of the statistic in (1) is a weighted sum of (n - 1) independent chi-squared variates, each with one degree of freedom, and with weights which are the eigenvalues of the matrix whose $(i, j)^{\text{th}}$ element is

$$(p_i / n^2)\Big(\{n - \max(i,j)\}\min(i,j) - \sum_{k=1}^{n-1} p_k \{n - \max(i,j)\}\min(j,k)\Big) \ .$$

Freedman expresses the first four moments of the asymptotic distribution of the test statistic under H_0 as functions of these eigenvalues, and uses these moments to approximate the quantiles of the asymptotic distribution by fitting Pearson curves. He confirms the quality of this approximation by Monte Carlo methods, just for the case where the population distribution is uniform multinomial.

In fact, however, the complete asymptotic null distribution of U_N^{*2} can be obtained directly and without any such approximations by using standard computational methods. Specifically, we can use those suggested by Imhof (1961), Davies (1973, 1980) and others, to invert the characteristic function for statistics which are weighted sums of chi-squared variates. There is no need to resort to approximations, curve fitting or simulation methods.

In this paper we first use this information to verify and extend Freedman's quantile calculations for the case of uniform discrete data. Then we use Davies' algorithm to compute the exact quantiles of the asymptotic distributions of U_N^{*2} when the data follow "Benford's Laws" for the first, second and third significant digits of a string of numbers. The use of these quantiles is then illustrated through two examples, one of which demonstrates that correctly allowing for the discrete nature of the data can reverse the (false) conclusion that is reached if the null hypothesis is incorrectly tested using a test that is designed for the situation where the data are continuous.

2. ASYMPTOTIC DISTRIBUTIONS

One of the important advantages of Davies' algorithm, in particular, is its numerical accuracy. Both FORTRAN and C++ code for this algorithm are freely available from Davies (2011). In what follows we use Davies' double-precision FORTRAN code, Qf.for. The integration error bound and maximum number of integration terms for the inversion of the characteristic function can be specified by the user, and these were set to 10^{-6} and 10^{3} respectively. The calculations were undertaken on a PC with an Intel Pentium 3.00 GHz processor, running Windows XP Pro.

2.1 DISCRETE UNIFORM DISTRIBUTION

Figure 1 shows the asymptotic distribution function of U_N^{*2} for the uniform discrete model under H_0, for selected values of *n*. Table 1 provides quantiles of this distribution for a wider range of *n*, and compares these with Freeman's approximate quantiles as appropriate. The case of n = 12 is of interest when testing for seasonal incidence with monthly data. Freedman's Pearson curves provide slightly more (less) accurate upper (lower) quantiles than those obtained from Monte Carlo simulation, when each are compared with our exact results.

2.2 BENFORD'S LAW(S)

As a third example, consider the discrete distribution usually referred to as "Benford's Law". Benford (1938) re-discovered Newcomb's (1881) observation that the first significant digit (d_1) of certain naturally occurring numbers follows the distribution given by

$$p_i = \Pr[d_1 = i] = \log_{10}[1 + (1/i)] \;\; ; i = 1, 2, \ldots, 9. \qquad (2)$$

The "circularity" of the d_1 values can be illustrated by considering the numbers 0.09 and 0.10. The first significant digits (9 and 1) are as "distant" as possible, yet the two numbers are numerically very close. Although we use base 10 for the logarithms in (2), and in equations (3) to (6) below, any other consistent choice of base can be made. Various mathematical justifications for "Benford's Law" have been provided by several authors, including Pinkham (1961), Cohen (1976) and Hill (1995a, b, c, 1997, 1998); and Balanzario and Sánchez-Ortiz (2010) provide sufficient conditions for Benford's Law to hold. These conditions are very general.

The extensive bibliography by Hürlimann (2006) reflects the numerous applications of this distribution in many disciplines. Some examples include the auditing of financial data (*e.g.*, Drake and Nigrini, 2000; Geyer and Williamson, 2004; Durtschi *et al*., 2004); examining the quality of survey data (Judge and Schechter, 2009); the analysis hydrological records (*e.g.*, Nigrini and Miller, 2007); image processing (*e.g.*, Jolin, 2001; Acebo and Sbert, 2005); the α – decay half-lives of nuclei (Ni and Ren, 2008); testing for collusion and "shilling" in eBay auctions (Giles, 2007); and testing for the presence of psychological barriers in financial markets and auctions (*e.g.*, De Ceuster *et al*., 1998; Lu and Giles, 2010). In short, Benford's Law is very pervasive, and frequently encountered. For these reasons, reliable goodness-of-fit tests of this null hypothesis are of considerable interest.

Very recently Shao and Ma (2010) have linked Benford's Law to the Fermi-Dirac, Bose-Einstein and Boltzmann-Gibbs distributions that are of fundamental importance in statistical physics. Indeed, they speculate: "Thus Benford's law seems to present a general pattern for physical statistics and might be even more fundamental and profound in nature." (Shao and Ma, 2010, p.3109).

Corresponding Benford-type distributions for the higher-order significant digits are also well known. For example, the joint distributions for the first two and first three such digits (d_1, d_2 and d_3) are

$$p_{i\,j} = \Pr[d_1 = i, d_2 = j] = \log_{10}[1 + 1/(10i + j)] \ ;\ i, j = 10,\ 11,\ \ldots.,\ 99 \qquad (3)$$

and

$$p_{i\,j\,k} = \Pr[d_1 = i, d_2 = j, d_3 = k] = \log_{10}[1 + 1/(100i + 10j + k)] \ ;\ i, j, k = 100,\ 101,\ \ldots.,\ 999. \qquad (4)$$

Similarly, the marginal distributions for d_2 and d_3 are

$$p_i = \Pr[d_2 = i] = \sum_{l=1}^{9} \log_{10}[1 + 1/(10l + i)] \ ; \quad i = 0,\ 1,\ \ldots.,\ 9 \qquad (5)$$

and

$$p_i = \Pr[d_3 = i] = \sum_{l=1}^{9} \sum_{m=0}^{9} \log_{10}[1 + 1/(100l + 10m + i)] ; \quad i = 0,\ 1,\ \ldots.,\ 9. \qquad (6)$$

respectively.

In Table 2 we present quantiles for the distribution function for U_N^{*2} when testing against Benford's marginal distributions, (2), (5) and (6). Figure 2 depicts the corresponding distribution functions.

2.2 BETA-BINOMIAL DISTRIBUTION

The beta-binomial distribution is a discrete mixture distribution which can capture either under-dispersion or over-dispersion in the data. It has been used in a diverse range of applications (*e.g.*, Tong and Lord, 2007; Hunt *et al.*, 2009; Pham *et al.*, 2010). The probability mass function for a beta-binomial random variable, Y, is:

$$\Pr.(Y = y \mid \alpha, \beta, n) = \binom{n}{y} \frac{B(y + \alpha, n - y + \beta)}{B(\alpha, \beta)} \ ; \quad y = 0,\ 1,\ \ldots.,\ n \ ;\ n, \alpha, \beta > 0$$

where $B(.\ ,\ .)$ is the usual beta function. This distribution is very versatile for modeling as its p.m.f. can assume a wide range of shapes.

The asymptotic distribution function for U_N^{*2}, under the null hypothesis that the data follow the beta-binomial distribution, is illustrated in Figure 3 for n = 12, and various choices of the other parameters. The quantiles for this distribution function are given in Table 3, where the values of *n* are chosen in anticipation of applications involving daily, weekly, fortnightly, monthly, or quarterly data.

3. APPLICATIONS

3.1 CANADIAN BIRTH MONTHS

The numbers for the months of the year provide a simple example of discrete circular data, with n = 12. In one sense, December is as far from the first month of the year, January, as it can be, but in another sense it is as close as is possible. There is a substantial demographic literature relating to seasonality in the birth months of children. This literature suggests various reasons for non-uniformity, and why the seasonal pattern may vary (for sociological reasons) across countries, even those in the same hemisphere. Trovato and Odynak (1993) provide a useful discussion of seasonality in the numbers of births in Canada.

Here, we test the hypothesis of uniformity in the data for Canadian live births in 2008. These data are from Statistics Canada (2011), and are summarized in Table 4, by Province and Territory, and for Canada as a whole. These locations are for the mother at the time of birth.

Table 5 provides the results of testing for uniformity of the distribution of births across months, against the alternative of non-uniformity. When the U_N^{*2} values are compared with the tabulated critical values for n = 12 in Table 1(b), we see that the null hypothesis of uniformity is strongly rejected for Canada as a whole, and for almost all of the provinces. It cannot be rejected for Prince Edward Island or for the Yukon or Northwest Territories, at conventional significance levels. In the case of Nunavut, the null hypothesis is rejected at the 10% significance level, but not at the 5% level. Interestingly, these four exceptional cases correspond to the jurisdictions with the smallest numbers of births in 2008. In addition, three of these four jurisdictions are located in the far North, and face climatic and cultural situations somewhat different from the rest of Canada.

3.2 FIBONACCI SERIES AND FACTORIALS

Canessa (2003) has proposed a general statistical thermodynamic theory that explains, *inter alia*, why Fibonacci sequences should obey Benford's Law. See, also, Duncan (1969) and Washington (1981). However, this theory has not previously been tested empirically, so here we test the hypothesis that the distribution of the first digits of the first N numbers of the Fibonacci series, {1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, ……} follows Benford's Law, for various choices of $N \leq 20{,}000$. The alternative hypothesis is that the distribution differs from Benford's Law. We also test the null hypothesis that the distribution is discrete uniform, against the alternative of non-uniformity.

The Fibonacci first digits were generated using Knott's (2010) Fibonacci number calculator. The values for N = 100 appear in Table 6, and the relative frequency distributions for N = 100, 500, and 1000 are given in Table 7. For $N \geq 50$, the test results in Table 8(a) indicate a clear rejection of uniformity (using the quantiles for n = 9 in Table 1 (b)) and an equally clear non-rejection of Benford's first-digit Law (using the quantiles in Table 2).

Sarkar (1973) demonstrates that the first digits of factorials and binomial coefficients appear to follow Benford's Law. However, he does not undertake any formal goodness-of-fit testing. The first digits of the first 100 factorials are given in Table 6, and the relative frequency distributions for N = 50, 100, and 170 appear in Table 7. The largest factorial that can be stored in computer memory is 170!. The results in Table 8(b), again using the quantiles for n = 9 from Table 1(b) and Table 2, show a strong rejection of uniformity in each case, and failure to reject Benford's distribution at conventional significance levels, for N > 50.

Given the implications of the theoretical results of Duncan (1969), Washington (1981), Canesa (2003), and Sarkar (1973), these empirical results for the Fibonacci and factorial data can be interpreted as speaking favourably to the quality of Freedman's test.

3.3 AUCTION PRICE DATA

Price data exhibit circularity. Consider two prices such as \$99.99 and \$100. Their first significant digits are as far apart as is possible, yet the associated prices are extremely close. Giles (2007) considered all of the 1,161 successful auctions for tickets for professional football games in the "event tickets" category on eBay for the period 25 November to 3 December, 2004, excluding auctions ending with the "Buy-it-Now" option, and all Dutch auctions. The winning bids should satisfy Benford's Law if they are "naturally occurring" numbers, as should be the case if there were no collusion among bidders and no "shilling" by sellers in this market.

Table 6 reports the first, second, and third digits for the first 100 observations in Giles's sample; and Table 7 provides the relative frequency distributions for the first N = 100, 500 and 1000 sample values. In Table 9 we see the results of testing these first, second and third digits using both the uniform multinomial and Benford hypotheses. Uniformity is again strongly rejected (against non-uniformity) for the first and third digits, and for the second digit in samples of size 500 or greater. At the 5% significance level, Benford's Law for the third digit is unambiguously rejected (against the non-Benford alternative), and the first digit and second digit laws are also rejected for $N > 100$. In contrast, Giles (2007) (wrongly) applied Kuiper's (1959) V_N test for continuous data to the 1,161 first-digits and marginally failed to reject Benford's Law. (He did not consider tests for the second and third digits, as we do here.) This comparison of our results with his illustrates the importance of applying a test that takes account of the discrete nature of the data.

3.4 ALCOHOL CONSUMPTION DATA

Our final application fits the beta-binomial distribution to data for the number of days in a month on which alcohol was consumed. We use a sample of 10,327 responses to the question "On how many of the past thirty days did you drink alcoholic beverages", in the *Canadian Addiction Survey* (Adlaf *et al*., 2005). In this application, the data are discrete, with $n = 30$, but they are not circular in nature. However, it is well known that Kuiper's test for goodness of fit involving continuous data has good power properties even when the data are not circular, especially if the lack of fit arises from departures in variance.

Fitting the beta-binomial distribution to the data, using R (2008) code with the VGAM package (Yee, 2009), the maximum likelihood estimates of the parameters are $\hat{\alpha} = 0.4218$ and $\hat{\beta} = 1.7021$. The goodness-of-fit of this distribution is compared with those of the binomial, negative binomial, and Poisson distributions in Figure 4. We see that….. However, testing H_0: beta-binomial, against the alternative hypothesis that the distribution is *not* beta-binomial, we have a test statistic of U_N^{*2} = 25.0148. For these values of n and the parameters, the 95th and 99th quantiles of the asymptotic distribution are 0.18445 and 0.26563 respectively, so we strongly reject the hypothesis that the data come from a beta-binomial distribution in this case.

4. POWER CONSIDERATIONS

Freedman (1981) was concerned with testing uniformity against "seasonal" fluctuations in discrete data. He provided a limited comparison of the powers of the U_N^{*2} test, Kuiper's V_N test, and Edwards' (1961) test against both sinusoidal and non-sinusoidal alternatives. The U_N^{*2} test out-performed the V_N test, and also out-performed Edwards' test in the non-sinusoidal case.

We have studied the power of the U_N^{*2} test for the two cases where the null hypothesis is the beta-binomial distribution, and where it is the first-digit distribution under Benford's law. The alternative hypothesis is that the data are (discrete) uniform on [0, 4] in the former case; and (discrete) uniform on [1, 9] in the latter case. The power of the U_N^{*2} test is compared with that of Kuiper's V_N test, even though the latter is intended for continuous distributions. Edwards' test is not considered as it is specific to alternatives representing "seasonality". Our results appear in Table 10. For the beta-binomial null hypotheses that are considered, the U_N^{*2} test out-performs the V_N test and 100% power is achieved for (approximately) $N \geq 100$ against this particular alternative. The relative performance of the U_N^{*2} is less satisfactory for very small samples in Table 10 (b), where the null hypothesis is that that the data are distributed according to Benford's first-digit law. However, both tests attain 100% power for (approximately) $N \geq 150$ against the alternative hypothesis of a discrete uniform distribution. Given that this is the most natural alternative to this null hypothesis, and that the tests are only asymptotically valid, this is actually a very satisfactory result.

5. CONCLUSIONS

When testing for goodness-of-fit, it is important to distinguish between continuous and discrete data, and also to use an appropriate test if the data are distributed on the circle, as is sometimes the case. Often, one or both of these characteristics of the problem are ignored, and inappropriate tests are used. We have shown that in fact it is a simple computational matter to test for goodness-of-fit properly when the data are circular and discrete. Freedman's (1981) test can be applied without any need to resort to approximations, contrary to the existing results in the literature. The test is asymptotically exact and is simple to apply using the accurate critical values derived in this paper for some interesting discrete distributions – uniform, beta-binomial, and those associated with "Benford's Laws". Our computational method can also be used to generate exact critical values for other discrete distributions that may be of interest.

A small Monte Carlo study we demonstrate, for the first time, that when the null hypothesis is that the data are either beta-binomially distributed, or distributed according to Benford's first law, Freedman's test has excellent power against uniform alternatives. We have applied our results to four practical testing problems to show the utility and versatility of this test that takes account of both the circularity and discrete nature of certain data. In summary, we recommend the use of Freedman's U_N^{*2} test for goodness-of-fit testing with discrete, possibly circular, data.

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Table 1. Quantiles of the asymptotic null distribution function of U_N^{*2} . H_0: Uniform discrete distribution

n	(a) Left Tail Quantiles				
	1%	**2.5%**	**5%**	**10%**	**25%**
3	0.000745	0.001876	0.003800	0.007805	0.021310
4	0.002852	0.005365	0.008763	0.014592	0.030492
5	0.005346	0.008749	0.012894	0.019432	0.035733
6	0.007624	0.011513	0.015993	0.022756	0.038872
7	0.009535	0.013673	0.018285	0.025071	0.040856
8	0.011095	0.015350	0.019995	0.026724	0.042171
9	0.012361	0.016660	0.021290	0.027933	0.043077
10	0.013390	0.017694	0.022287	0.028840	0.043724
15	0.016397	0.020557	0.024928	0.031122	0.045228
20	0.017715	0.021733	0.025956	0.031961	0.045735
26	0.018483	0.022393	0.026518	0.032407	0.045995
30	0.018774	0.022639	0.026724	0.032568	0.046088
40	0.019173	0.022970	0.026999	0.032781	0.046209
50	0.019362	0.023130	0.027127	0.032880	0.046264
52	0.019388	0.023147	0.027144	0.032893	0.046272
100	0.019620	0.023336	0.027299	0.033011	0.046338
12	0.014927	0.019187	0.023687	0.030072	0.044559
	(0.0195)	(0.0218)	(0.0248)	(0.0299)	(0.0435)
	[0.015]	[0.019]	[0.024]	[0.030]	[0.045]

Table 1. (continued)

n	**(b) Right Tail Quantiles**				
	75%	**90%**	**95%**	**97.5%**	**99%**
3	0.102692	0.170572	0.221924	0.273277	0.341164
4	0.106412	0.164936	0.208604	0.252081	0.309435
5	0.106985	0.160903	0.201195	0.241375	0.294438
6	0.106860	0.158332	0.196920	0.235448	0.286360
7	0.106612	0.156670	0.194286	0.231866	0.281535
8	0.106378	0.155554	0.192561	0.229543	0.278426
9	0.106185	0.154775	0.191373	0.227953	0.276306
10	0.106031	0.154211	0.190521	0.226818	0.274706
15	0.105620	0.152858	0.188500	0.224135	0.271240
20	0.105461	0.152379	0.187792	0.223198	0.270002
26	0.105374	0.152126	0.187419	0.222707	0.269335
30	0.105344	0.152033	0.187285	0.222531	0.269121
40	0.105301	0.151914	0.187108	0.222297	0.268815
50	0.105281	0.151859	0.187026	0.222190	0.268670
52	0.105279	0.151851	0.187016	0.222174	0.268650
100	0.105256	0.151785	0.186917	0.222044	0.268481
12	0.105813	0.153470	0.189410	0.225341	0.272836
	(0.106)	(0.154)	(0.189)	(0.225)	(0.272)
	[0.107]	[0.155]	[0.191]	[0.224]	[0.264]

Note: For *n* = 12, figures in parentheses (square brackets) are Freedman's (1981) Pearson curve (Monte Carlo) estimates, each to the number of decimal places he reports. The entries for *n* = 26 and *n* = 52 are to allow for seasonal testing with fortnightly and weekly data.

Table 2. Quantiles of the asymptotic null distribution function of U_N^{*2}. H_0: Benford's marginal distributions for first, second and third digits

Quantiles (%)	**First Digit**	**Second Digit**	**Third Digit**
1	0.01024	0.01332	0.01339
2.5	0.01392	0.01760	0.01769
5	0.01794	0.02218	0.02229
10	0.02379	0.02871	0.02884
25	0.03744	0.04356	0.04372
.	.	.	.
.	.	.	.
75	0.09651	0.10576	0.10603
90	0.14313	0.15388	0.15421
95	0.17878	0.19016	0.19052
97.5	0.21485	0.22643	0.22681
99	0.26319	0.27441	0.27479

Table 3. Selected quantiles of the asymptotic null distribution function of U_N^{*2}. H_0: Beta-binomial distribution

			(a) Left Tail Quantiles			
α	***β***	**1%**	**2.5%**	**5%**	**10%**	**25%**
			***n* = 4**			
0.20	0.25	0.00151	0.00285	0.00467	0.00782	0.01660
0.70	2.00	0.00225	0.00424	0.00695	0.01165	0.02470
2.00	2.00	0.00284	0.00534	0.00871	0.01451	0.03034
600	400	0.00257	0.00485	0.00793	0.01323	0.02784
			***n* = 7**			
0.20	0.25	0.00520	0.00751	0.01036	0.01407	0.02363
0.70	2.00	0.00738	0.01071	0.01448	0.02017	0.03389
2.00	2.00	0.00944	0.01354	0.01811	0.02485	0.04055
600	400	0.00648	0.00953	0.01306	0.01844	0.03161
			***n* = 12**			
0.20	0.25	0.00847	0.01103	0.01379	0.01779	0.02721
0.70	2.00	0.01171	0.01531	0.01919	0.02480	0.03783
2.00	2.00	0.00847	0.01103	0.01380	0.01779	0.02721
600	400	0.00784	0.01067	0.01379	0.01837	0.02915
			***n* = 26**			
0.20	0.25	0.01117	0.01371	0.01642	0.02034	0.02957
0.70	2.00	0.01505	0.01847	0.02210	0.02731	0.03944
2.00	2.00	0.01767	0.02145	0.02543	0.03111	0.04422
600	400	0.00747	0.00958	0.01185	0.01511	0.02272
			***n* = 52**			
0.20	0.25	0.01222	0.01471	0.01739	0.02127	0.03042
0.70	2.00	0.01606	0.01932	0.02279	0.02782	0.03961
2.00	2.00	0.01834	0.02191	0.02571	0.03118	0.04392
600	400	0.00622	0.00774	0.00936	0.01170	0.01721

Table 3. (continued)

(b) Right Tail Quantiles

α	*β*	75%	90%	95%	97.5%	99%
				***n* = 4**		
0.20	0.25	0.06256	0.10224	0.13381	0.16655	0.21126
0.70	2.00	0.09183	0.14783	0.19139	0.23595	0.29625
2.00	2.00	0.10610	0.16473	0.20861	0.25242	0.31038
600	400	0.09986	0.15717	0.20057	0.24417	0.30218
				***n* = 7**		
0.20	0.25	0.06793	0.10528	0.13487	0.16551	0.20730
0.70	2.00	0.09519	0.14409	0.18169	0.21993	0.27141
2.00	2.00	0.10607	0.15597	0.19347	0.23093	0.28044
600	400	0.08997	0.13548	0.17010	0.20507	0.25187
				***n* = 12**		
0.20	0.25	0.07038	0.10664	0.13527	0.16485	0.20515
0.70	2.00	0.09486	0.14055	0.17574	0.21150	0.25962
2.00	2.00	0.10395	0.15085	0.18626	0.22169	0.26859
600	400	0.07628	0.11419	0.14367	0.17387	0.21477
				***n* = 26**		
0.20	0.25	0.07198	0.10764	0.13567	0.16457	0.20387
0.70	2.00	0.09380	0.13785	0.17189	0.20652	0.25315
2.00	2.00	0.10163	0.14707	0.18153	0.21611	0.26203
600	400	0.05726	0.08655	0.10985	0.13400	0.16698
				***n* = 52**		
0.20	0.25	0.07270	0.10810	0.13592	0.16456	0.20351
0.70	2.00	0.09313	0.13681	0.17056	0.20494	0.25125
2.00	2.00	0.10044	0.14545	0.17966	0.21406	0.25982
600	400	0.04326	0.06605	0.08440	0.10402	0.12964

Table 4. Canadian live births, 2008: relative frequency distribution (%)

Month:	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**	**11**	**12**
NL	7.4	7.4	8.4	7.8	8.8	7.8	8.8	9.7	9.6	8.9	7.8	7.6
PEI	7.3	9.0	9.0	7.6	8.6	8.2	9.6	7.6	8.3	8.4	8.7	7.6
NS	8.3	8.1	8.0	8.1	8.5	8.4	9.5	8.5	8.9	8.5	7.6	7.5
NB	8.0	7.7	8.3	7.7	8.4	8.5	8.7	9.3	9.0	8.5	7.9	7.9
QC	7.7	7.6	8.2	8.2	8.5	8.2	9.2	8.7	9.0	8.9	7.9	7.9
ON	8.2	7.8	8.2	8.4	8.6	8.4	8.8	8.6	8.9	8.6	7.9	7.8
MB	8.2	7.6	7.9	8.1	8.7	8.3	9.0	8.8	8.9	9.0	7.5	8.0
SK	8.2	8.0	8.3	8.2	8.8	8.4	8.7	8.3	9.5	8.3	7.4	7.9
AB	8.0	7.6	8.2	8.4	8.6	8.7	8.9	8.9	8.6	8.4	7.6	8.1
BC	8.0	7.6	8.1	8.2	8.8	8.4	8.9	8.7	8.9	8.4	7.8	8.2
YT	6.2	7.8	9.1	6.4	10.2	6.7	5.9	9.4	10.7	9.1	8.3	10.2
NWT	8.7	7.2	8.6	8.5	9.4	7.8	8.2	10.3	7.9	7.9	8.5	7.1
NU	7.7	7.7	9.3	8.9	9.2	9.6	8.8	8.3	8.7	6.7	7.5	7.6
CAN	8.0	7.7	8.2	8.3	8.6	8.4	8.9	8.7	8.9	8.6	7.8	7.9

Notes: The Provincial/Territorial abbreviations are: NL = Newfoundland and Labrador; PEI = Prince Edward Island; NS = Nova Scotia; NB = New Brunswick; QC = Québec; ON = Ontario; MB = Manitoba; SK = Saskatchewan; AB = Alberta; BC = British Columbia; YT = Yukon Territory; NWT = Northwest Territory; NU = Nunavut; CAN = Canada.

Table 5. Values of U_N^{*2} . H_0: Canadian birth months follow uniform discrete distribution

Province/Territory	***N***	U_N^{*2}
NL	4,898	0.771
PEI	1,483	0.038
NS	9,188	0.528
NB	7,402	0.490
QC	87,870	6.340
ON	140,791	5.681
MB	15,485	0.994
SK	13,737	0.552
AB	50,856	2.856
BC	44,276	2.093
YT	373	0.089
NWT	721	0.052
NU	805	0.168
CANADA	377,886	18.146

Note: The Provincial/Territorial abbreviations are: NL = Newfoundland and Labrador; PEI = Prince Edward Island; NS = Nova Scotia; NB = New Brunswick; QC = Québec; ON = Ontario; MB = Manitoba; SK = Saskatchewan; AB = Alberta; BC = British Columbia; YT = Yukon Territory; NWT = Northwest Territory; NU = Nunavut.

Table 6. Illustrative data: digits when $N = 100$

Fibonacci numbers - first digits

1, 1, 2, 3, 5, 8, 1, 2, 3, 5, 8, 1, 2, 3, 6, 9, 1, 2, 4, 6, 1, 1, 2, 4, 7, 1, 1, 3, 5, 8, 1, 2, 3, 5, 9, 1, 2, 3, 6, 1, 1, 2, 4, 7, 1, 1, 2, 4, 7, 1, 2, 3, 5, 8, 1, 2, 3, 5, 9, 1, 2, 4, 6, 1, 1, 2, 4, 7, 1, 1, 3, 4, 8, 1, 2, 3, 5, 8, 1, 2, 3, 6, 9, 1, 2, 4, 6, 1, 1, 2, 4, 7, 1, 1, 3, 5, 8, 1, 2, 3

Factorials – first digits

1, 2, 6, 2, 1, 7, 5, 4, 3, 3, 3, 4, 6, 8, 1, 2, 3, 6, 1, 2, 5, 1, 2, 6, 1, 4, 1, 3, 8, 2, 8, 2, 8, 2, 1, 3, 1, 5, 2, 8, 3, 1, 6, 2, 1, 5, 2, 1, 6, 3, 1, 8, 4, 2, 1, 7, 4, 2, 1, 8, 5, 3, 1, 1, 8, 5, 3, 2, 1, 1, 8, 6, 4, 3, 2, 1, 1, 1, 9, 8, 5, 4, 3, 3, 2, 2, 2, 1, 1, 1, 1, 1, 1, 1, 1, 9, 9, 9, 9, 9

Auction prices – first digits

6, 9, 5, 4, 6, 3, 1, 8, 7, 9, 3, 2, 2, 2, 1, 1, 4, 2, 1, 1, 1, 4, 1, 3, 3, 9, 3, 6, 1, 1, 7, 7, 8, 1, 1, 2, 2, 7, 7, 1, 2, 2, 1, 1, 2, 1, 1, 4, 3, 7, 4, 2, 2, 2, 1, 2, 9, 2, 3, 1, 2, 1, 1, 1, 7, 5, 2, 2, 2, 3, 1, 9, 5, 2, 7, 4, 7, 2, 2, 1, 5, 5, 3, 3, 5, 1, 2, 3, 1, 2, 1, 1, 1, 7, 2, 1, 1, 2, 5, 6

Auction prices – second digits

6, 4, 0, 5, 2, 7, 0, 1, 0, 2, 0, 3, 8, 0, 1, 1, 3, 5, 2, 4, 5, 8, 5, 0, 4, 2, 0, 3, 1, 8, 6, 8, 0, 7, 7, 9, 5, 1, 8, 9, 9, 0, 2, 8, 2, 8, 9, 6, 8, 0, 4, 5, 4, 5, 1, 6, 6, 8, 2, 3, 0, 6, 5, 7, 1, 1, 2, 0, 7, 1, 3, 6, 1, 3, 5, 7, 6, 2, 8, 1, 1, 0, 4, 3, 1, 0, 8, 0, 6, 0, 6, 0, 4, 6, 3, 5, 3, 0, 3, 1

Auction prices – third digits

0, 0, 0, 5, 0, 5, 2, 1, 1, 9, 5, 8, 0, 5, 2, 9, 5, 5, 2, 7, 7, 5, 0, 5, 0, 0, 7, 0, 7, 2, 0, 0, 0, 7, 5, 0, 5, 0, 0, 2, 5, 2, 2, 2, 0, 2, 2, 0, 5, 0, 9, 0, 0, 6, 9, 0, 5, 5, 0, 1, 1, 2, 2, 0, 0, 0, 7, 2, 5, 0, 1, 0, 0, 7, 9, 2, 0, 2, 5, 0, 0, 0, 2, 5, 0, 2, 0, 0, 7, 2, 2, 0, 2, 0, 2, 7, 4, 2, 0, 0

Table 7. Illustrative data: relative frequency distributions

Digit	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
N					**Benford's Law – first digits**				
	0.301	0.176	0.125	0.097	0.080	0.067	0.058	0.051	0.046
					Fibonacci numbers - first digits				
100	0.300	0.180	0.130	0.090	0.080	0.060	0.050	0.070	0.040
500	0.302	0.176	0.126	0.094	0.080	0.066	0.058	0.054	0.044
1000	0.301	0.177	0.125	0.096	0.080	0.067	0.056	0.053	0.045
					Factorials – first digits				
50	0.240	0.220	0.160	0.060	0.080	0.120	0.020	0.100	0.000
100	0.300	0.180	0.130	0.070	0.070	0.070	0.020	0.100	0.060
170	0.306	0.182	0.124	0.070	0.076	0.059	0.029	0.082	0.071
					Auction prices – first digits				
100	0.300	0.250	0.110	0.060	0.070	0.040	0.100	0.020	0.050
500	0.326	0.226	0.104	0.076	0.072	0.048	0.066	0.048	0.034
1000	0.326	0.198	0.133	0.078	0.071	0.051	0.060	0.048	0.035

Table 7. (continued)

Digit	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
N	**Benford's Law – second digits**									
	0.120	0.114	0.109	0.104	0.100	0.097	0.093	0.090	0.088	0.085
	Auction prices – second digits									
100	0.180	0.140	0.090	0.100	0.070	0.100	0.110	0.060	0.110	0.040
500	0.194	0.156	0.102	0.066	0.066	0.146	0.106	0.040	0.058	0.066
1000	0.202	0.145	0.101	0.078	0.058	0.135	0.111	0.047	0.060	0.063
	Benford's Law – third digits									
	0.102	0.101	0.101	0.101	0.100	0.100	0.099	0.099	0.099	0.098
	Auction prices – third digits									
100	0.390	0.050	0.220	0.000	0.010	0.170	0.010	0.090	0.010	0.050
500	0.406	0.032	0.166	0.016	0.014	0.182	0.018	0.084	0.022	0.060
1000	0.416	0.032	0.154	0.022	0.014	0.191	0.023	0.090	0.020	0.038

Table 8 (a). Values of U_N^{*2} . H_0: Fibonacci first digits follow uniform discrete distribution; *or* H_0: Fibonacci first digits follow Benford's distribution

N	U_N^{*2}	
	H_0: Uniform discrete	**H_0: Benford**
50	0.42831	0.00486
100	0.79613	0.00342
250	1.91658	0.00175
500	3.84342	0.00063
1000	7.71638	0.00042
2000	13.35437	0.00021
5000	38.44199	0.00012
10000	76.84573	0.00007
20000	153.54990	0.00003

(b). Values of U_N^{*2} . H_0: Factorials first digits follow uniform discrete distribution; *or* H_0: Factorials first digits follow Benford's distribution

N	U_N^{*2}	
	H_0: Uniform discrete	**H_0: Benford**
50	1.16915	0.27684
100	1.47179	0.08815
170	1.56025	0.04822

Table 9. Values of U_N^{*2} . H_0: Football ticket price digits follow uniform discrete distribution; *or* H_0: Football ticket price digits follow Benford's distribution

	U_N^{*2}					
	Uniform discrete			**Benford**		
N	**Digit 1**	**Digit 2**	**Digit 3**	**Digit 1**	**Digit 2**	**Digit 3**
50	0.4574	0.1242	0.3952	0.0463	0.1094	0.3883
100	1.1306	0.0476	0.2490	0.0778	0.0195	0.2407
250	3.3508	0.7566	2.7800	0.2673	0.5128	2.7390
500	5.6113	1.1440	4.8389	0.2539	0.6876	4.7680
750	8.3334	1.4935	6.9105	0.3210	0.8473	6.7987
1000	10.6368	2.1118	9.2640	0.2919	1.2482	9.1235
1161	11.7730	2.4803	11.1671	0.2258	1.4664	10.9962

Table 10. Illustrative powers (%) of the U_N^{*2} and V_N tests.

(a) H_0: Beta-binomial (n = 4) ; H_1: Discrete Uniform [0, 4]

N	**10%**		**5%**		**1%**	
	U_N^{*2}	V_N	U_N^{*2}	V_N	U_N^{*2}	V_N
			***α* = 0.7; *β* = 2.0**			
25	78.00	52.62	78.00	33.96	41.44	18.72
50	98.72	90.08	96.94	84.76	88.26	62.98
75	99.78	99.18	99..68	98.34	98.32	93.94
100	100.00	99.94	100.00	99.88	99.86	98.84
			***α* = 0.2; *β* = 0.25**			
25	78.00	47.24	78.00	43.56	61.04	27.84
50	96.94	91.38	93.80	85.60	88.26	68.64
75	99.68	99.30	99.28	97.94	96.58	93.70
100	100.00	99.94	99.86	99.82	99.34	99.36

(b) H_0: Benford's first-digit ; H_1: Discrete Uniform [1, 9]

N	**10%**		**5%**		**1%**	
	U_N^{*2}	V_N	U_N^{*2}	V_N	U_N^{*2}	V_N
50	68.28	84.38	50.70	75.92	17.40	52.48
75	92.92	97.98	79.74	95.98	53.86	86.88
100	98.62	99.74	94.78	99.42	77.70	96.92
150	99.98	99.98	99.82	99.96	98.46	99.90

Figure 1: Exact Asymptotic Distribution of Freeman's Statistic for the Uniform Discrete Distribution Under the Null Hypothesis

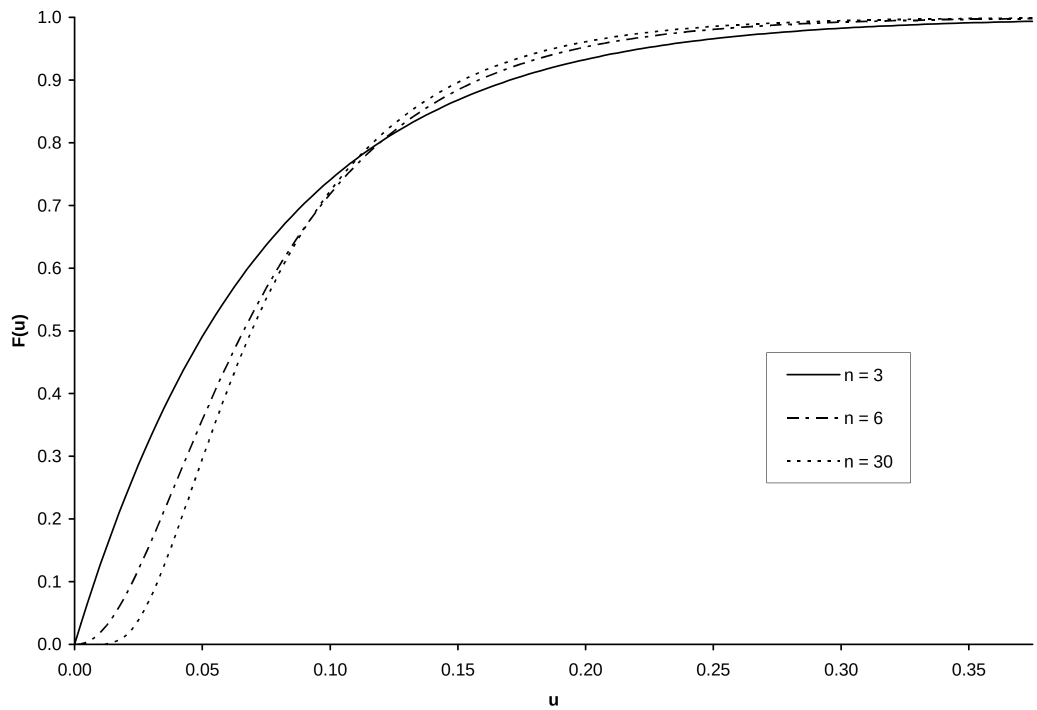


Figure 2: Exact Asymptotic Distributions of Freeman's Statistic for Benford's Distributions for First and Second Digits Under the Null Hypthesis

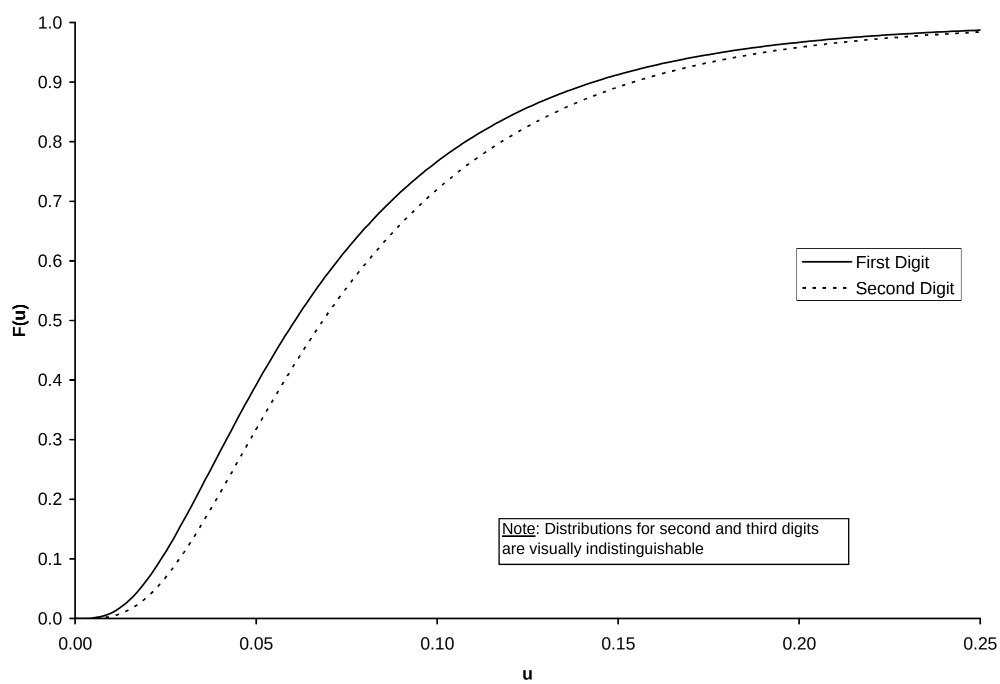


Figure 3: Exact Asymptotic Distribution of Freeman's Statistic for the Beta-Binomial Distribution With n = 12 Under the Null Hypothesis

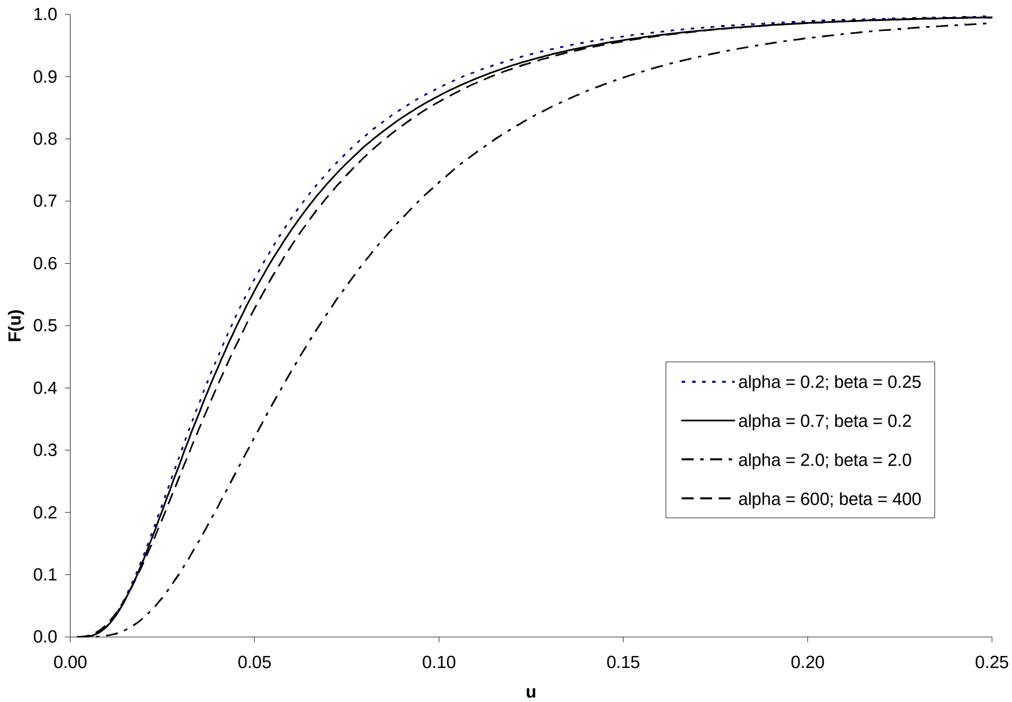
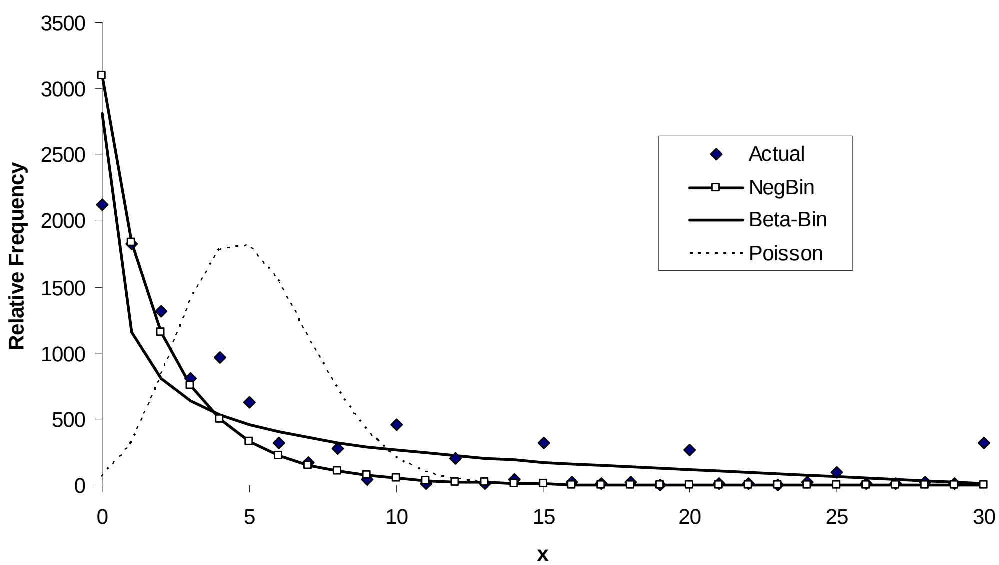


Figure 4: Fitted Distributions for Alcoholic Beverages Data



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