ECON 546 – SPRING 2010
Solution for Assignment 2

Q.1. By independence, $L(\theta) = \theta^n \prod_i x_i^{\theta-1}$, and

(a) $\log L = n\log\theta + (\theta-1)\sum_i \log x_i$.

$$(\partial \log L/\partial\theta) = n/\theta + \sum_i \log x_i = 0$$
$$\Rightarrow \tilde{\theta} = -n/\sum_i \log x_i$$

(which is positive, because $0 < x_i < 1$).

$$(\partial^2 \log L/\partial\theta^2) = -n/\theta^2 \quad (<0 \text{ everywhere})$$
$$\Rightarrow \tilde{\theta} \text{ locates a maximum of } \log L.$$

(b) By invariance of MLE's, the MLE of $\exp[-2/\theta]$ is $\exp[-2/\tilde{\theta}] = \exp[+2(\sum_i \log x_i)/n]$

$$= \exp[\log(x_1)^{2/n}] \cdot \text{----} \cdot \exp[\log(x_n)^{2/n}]$$
$$= x_1^{2/n} \cdot x_2^{2/n} \cdot \text{-----} \cdot x_n^{2/n}$$
$$= \left\{\left[\prod_i x_i\right]^{1/n}\right\}^2 = \{GM\ (x\text{'s})\}^2$$

(c) $I(\theta) = -E[\partial^2 \log L/\partial\theta^2] = (n/\theta^2)$

$IA(\theta) = \underset{n\to\infty}{\text{Limit}}\left[\frac{1}{n} I(\theta)\right] = 1/\theta^2$

(d) $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N[0, \theta^2]$.

(e) $a.s.d.[\sqrt{n}(\tilde{\theta}-\theta)] = \theta$; $a.s.d.(\tilde{\theta}) = (\theta/\sqrt{n})$

So, $a.s.e.(\tilde{\theta}) = \tilde{\theta}/\sqrt{n} = -\sqrt{n}/\sum_i \log x_i$

(& this is consistent because $\tilde{\theta}$ is the MLE.)

Q. 2. $p(y_i) = (c\, y_i^{c-1}/b^c)\exp[-(y_i/b)^c]$.

(a) $L(b, c \mid \underset{\sim}{y}) = p(\underset{\sim}{y} \mid b, c)$

$= \prod_i p(y_i \mid b, c)$ (independence)

So, $L = \left(\dfrac{c^n}{b^{nc}}\right) \prod_i y_i^{c-1} \exp\left[-\sum_i (y_i/b)^c\right]$

$\log L = n\log(c) - nc\log(b) + (c-1)\sum_i \log(y_i)$
$- \sum_i (y_i/b)^c$.

$\dfrac{\partial \log L}{\partial b} = \dfrac{-nc}{b} + \left(\dfrac{c}{b^{c+1}}\right)\sum_i y_i^c = 0$ (i)

$\dfrac{\partial \log L}{c} = \dfrac{n}{c} - n\log(b) + \sum_i \log(y_i)$
$- \sum_i \left[(y_i/b)^c \cdot \log(y_i/b)\right] = 0.$ (ii)

(Using the result that $\frac{\partial}{\partial x}(a^x) = a^x \log(a)$.)

Clearly the 2 likelihood equations, (i) & (ii), can't be solved analytically!

(b) Set $c = 1$. Now the only unknown parameter is 'b'.

Then ~~the~~ ~~2~~ likelihood equations becomes:

$$\frac{\partial \log L}{\partial b} = \frac{-n}{b} + \frac{1}{b^2} \sum_i y_i = 0$$

$$\Rightarrow \tilde{b} = \frac{1}{n} \sum_i y_i = \bar{y} .$$

Notice that in this case,

$$\frac{\partial^2 \log L}{\partial b^2} = \frac{n}{b^2} - \frac{2 \sum_i y_i}{b^3}$$

& when $b = \bar{y}$:

$$\left.\frac{\partial^2 \log L}{\partial b^2}\right|_{\tilde{b}} = \frac{n}{\bar{y}^2} - \frac{2 n \bar{y}}{\bar{y}^3}$$

$$= -n/\bar{y}^2 < 0 . \quad (\underline{\text{Maximum}})$$

So, $I(b) = -E\left[\frac{\partial^2 \log L}{\partial b^2}\right]$

$$= -n/b^2 + \frac{2}{b^3} \sum_i E(y_i)$$

Now, when $c = 1$,

$$p(y_i) = \frac{1}{b} \exp(-y_i/b)$$

which is just the density for the Exponential distribution, whose mean is just 'b'.

So, $I(b) = -n/b^2 + \frac{2}{b^3} \cdot nb$

$= -n/b^2 + \frac{2n}{b^2}$

$= n/b^2.$

$IA(b) = \underset{n\to\infty}{\text{Limit}}\left(\frac{1}{n} I(b)\right) = 1/b^2$

So, $\sqrt{n}(\tilde{b} - b) \xrightarrow{d} N[0, b^2].$

<u>N.B.</u>: To prove that $E(y_i) = b$, use integration by parts —

$E(y_i) = \int_0^\infty y_i \frac{1}{b} e^{-y_i/b} dy_i$

$= \frac{1}{b}\int_0^\infty y_i e^{-y_i/b} dy_i.$

$= 1/b \int_0^\infty f g' dy_i$

$= \frac{1}{b}\left\{[fg]_0^\infty - \int_0^\infty f' g \, dy_i\right\}$

Let $f = y_i$; $g' = e^{-y_i/b}.$

So $f' = 1$ & $g = -b e^{-y_i/b}.$

So, $E(y_i) = \frac{1}{b}\left\{[-b y_i e^{-y_i/b}]_0^\infty + \int_0^\infty b e^{-y_i/b} dy_i\right\}$

$= \frac{1}{b}\left\{0 + b\int_0^\infty e^{-y_i/b} dy_i\right\}$

$$= \int_0^\infty e^{-y_i/b}\, dy_i$$

$$= -b\left[e^{-y_i/b}\right]_0^\infty$$

$$= -b\left[0 - 1\right] = b \qquad \#.$$

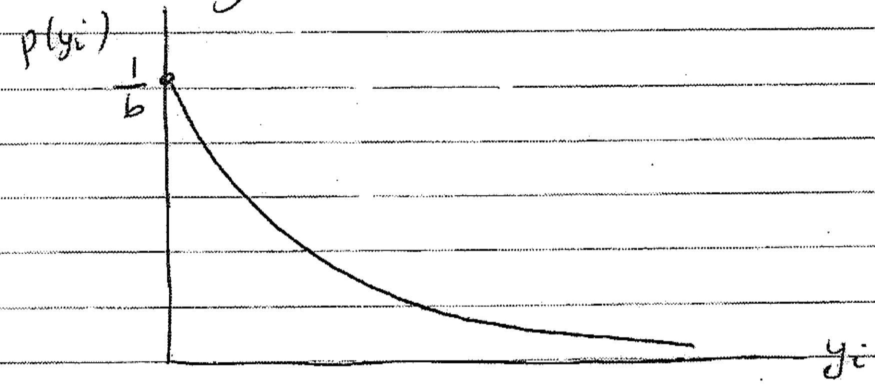
Note that you <u>don't</u> have to prove this result. You can just use $E(y_i) = b$.

(c) $$p(y_i) = \frac{1}{b} e^{-y_i/b}$$

$$\partial p(y_i)/\partial y_i = \left(\frac{1}{b}\right)\left(-\frac{1}{b}\right) e^{-y_i/b}$$

$$= -\frac{1}{b^2} e^{-y_i/b}$$

which cannot be zero for any y_i. However, the density has a mode!



Note that when $y_i = 0$, $p(y_i) = \frac{1}{b} e^0 = \frac{1}{b}$.

The modal height is $\frac{1}{b}$ & its MLE is $\frac{1}{\tilde{b}}$, or $(1/\bar{y})$. The mode is located at $y_i = 0$.

Q. 3.

(a) Using independence —

$$\mathcal{L}(p) = p(\underset{\sim}{y} \mid p) = \prod_i p(y_i \mid p)$$

$$= c\, p^{nx} (1-p)^{\sum_i y_i}$$

where $c = \prod_i \binom{x + y_i - 1}{y_i}$ is a "constant" as it does not involve p.

$$\log L = \log c + nx \log(p) + \sum_i y_i \log(1-p)$$

$$(\partial \log L / \partial p) = (nx/p) - n\bar{y}/(1-p) = 0$$

$$\Rightarrow \tilde{p} = x/(x+\bar{y}). \quad (\text{Clearly, } 0 < \tilde{p} < 1)$$

$$(\partial^2 \log L / \partial p^2) = -(nx/p^2) - n\bar{y}/(1-p)^2$$

When $p = x/(x+\bar{y})$ this 2nd. derivative is clearly negative. (In fact, always negative.) So, we have <u>maximized</u> the likelihood fctn.

(b) $$I(p) = -E[\partial^2 \log L / \partial p^2]$$

$$= \frac{nx}{p^2} + \frac{n}{(1-p)^2} \frac{1}{n} \sum_i E(y_i)$$

From part (d), $E(y_i) = x(1-p)/p$.

So, $I(p) = \frac{nx}{p^2} + \frac{n}{(1-p)^2} \cdot \frac{x(1-p)}{p}$

$$= \frac{nx}{p^2(1-p)^2} \left\{ (1-p)^2 + p(1-p) \right\}$$

$$= \frac{nx}{p^2(1-p)}$$

$$IA(p) = \lim_{n \to \infty} \left[\frac{1}{n} I(p) \right] = \frac{x}{p^2(1-p)}$$

& so $\sqrt{n}(\tilde{p} - p) \xrightarrow{d} N\left[0, \left[\frac{x}{p^2(1-p)}\right]^{-1}\right]$

$$\xrightarrow{d} N\left[0, \frac{p^2(1-p)}{x}\right].$$

(c) The true asymptotic variance for $\tilde{p}$ is $\left[\frac{p^2(1-p)}{nx}\right]$; the asymptotic std. deviation for $\tilde{p} = \frac{p\sqrt{1-p}}{\sqrt{nx}}$; and the a.s.e. $(\tilde{p})$ is $\frac{\tilde{p}\sqrt{1-\tilde{p}}}{\sqrt{nx}}$. A 95% C.I. for p that is asymptotically valid would be

$$\tilde{p} \pm 1.96 \left(\frac{\tilde{p}\sqrt{1-\tilde{p}}}{\sqrt{nx}} \right).$$

(d) $\phi_y(t) = p^x\left[1 - q\exp(it)\right]^{-x}$

$$\phi_y'(t) = p^x(-x)(-q)\,i\exp(it)\left[1-q\exp(it)\right]^{-(x+1)}$$

$$= xqp^x\,i\,e^{it}\left[1-qe^{it}\right]^{-(x+1)}$$

Setting $t=0$:

$$\left(\phi_y'(0)/i\right) = E(y) = xqp^x(1-q)^{-(x+1)}$$

$$= (xq/p). \quad \#$$

$$\phi_y''(t) = xqip^x\left\{\left[1-qe^{it}\right]^{-(x+1)} \cdot ie^{it} + e^{it}\left(-(x+1)(-q)ie^{it}\right)\left(1-qe^{it}\right)^{-(x+2)}\right\}$$

$$= e^{it}xqi^2p^x\left\{\left[1-qe^{it}\right]^{-(x+1)} + q(1+x)\left[1-qe^{it}\right]^{-(x+2)}\right\}$$

So, $\left(\phi_y''(0)/i^2\right) = E(y^2)$

$$= xqp^x\left\{p^{-(x+1)} + q(1+x)p^{-(x+2)}\right\}$$

$$= (xq/p) + xq^2(1+x)/p^2.$$

$$\mathrm{Var.}(y) = E(y^2) - \left[E(y)\right]^2$$

$$= (xq/p) + (xq^2/p^2) + (x^2q^2/p^2) - (x^2q^2/p^2)$$

$$= \frac{xq}{p^2}\left[p+q\right] = (xq/p^2) \ \#$$

(e) By invariance the MLE for $E(Y)$ is $x\tilde{q}/\tilde{p}$ $= x(1-\tilde{p})/\tilde{p}$; and the MLE for var. (Y) is $x(1-\tilde{p})/\tilde{p}^2$.

Q.4 (a) The expression is squared to ensure that the variance is non-negative. If all of the elements of α, except for the first one, are set equal to zero then the variance will be $(\alpha_1 \cdot 1)^2 = \alpha_1^2$, which is constant. This will correspond to the case of homoskedastic errors.

(b)
$$\begin{cases} y_i = x_i'\beta + \varepsilon_i \quad ; \quad \varepsilon_i \sim N[0, \sigma_i^2] \\ \sigma_i^2 = (z_i'\alpha)^2 \end{cases}$$

$$p(y_i) = \frac{1}{\sigma_i\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma_i^2}(y_i - x_i'\beta)^2\right]$$
$$= \frac{1}{(z_i'\alpha)\sqrt{2\pi}} \exp\left[-\frac{1}{2(z_i'\alpha)^2}(y_i - x_i'\beta)^2\right]$$

Assuming independence:

$$L(\underset{\sim}{\beta}, \underset{\sim}{\alpha} \mid \underset{\sim}{y}) = p(\underset{\sim}{y} \mid \underset{\sim}{\beta}, \underset{\sim}{\alpha})$$
$$= \prod_{i=1}^{n} \frac{1}{(z_i'\alpha)\sqrt{2\pi}} \exp\left[-\frac{1}{2(z_i'\alpha)^2}(y_i - x_i'\beta)^2\right]$$
$$= (2\pi)^{-n/2} \cdot \left[\prod_{i=1}^{n} (z_i'\alpha)^{-1}\right] \exp\left[-\frac{1}{2}\sum_{i=1}^{n} \frac{(y_i - x_i'\beta)^2}{(z_i'\alpha)^2}\right]$$

&

$$\log L = -\frac{n}{2}\log(2\pi) + \sum_i \log(z_i'\alpha)^{-1} - \frac{1}{2}\sum_i \frac{(y_i - x_i'\beta)^2}{(z_i'\alpha)^2}$$
$$= -n/2 \log(2\pi) - \sum_i \log(z_i'\alpha) - \frac{1}{2}\sum_i \frac{(y_i - x_i'\beta)^2}{(z_i'\alpha)^2}$$

[Recall that $(\partial \underset{\sim}{a}'\underset{\sim}{x} / \partial \underset{\sim}{x}) = \underset{\sim}{a}$; so $(\partial \underset{\sim}{x}_i'\underset{\sim}{\beta} / \partial \underset{\sim}{\beta}) = \underset{\sim}{x}_i$]

(c)

$$(\partial \log L/\partial\beta) = -\frac{1}{2}\sum_i \frac{(y_i - x_i'\beta)\cdot 2(-x_i)}{(z_i'\cancel{\beta})^2}$$

$$= \sum_i \left(\frac{y_i - x_i'\beta}{z_i'\alpha}\right)^2 \cdot x_i \qquad (1)$$

(Note that y_i, $(z_i'\alpha)$ and $(x_i'\beta)$ are scalars, but x_i is a (1xK) column vector.)

Also, $$(\partial \log L/\partial\alpha) = \sum_i \left(\frac{1}{z_i'\alpha}\right) z_i - \frac{1}{2}\sum_i (y_i - x_i'\beta)^2(-2) z_i \left(\frac{1}{z_i'\alpha}\right)^3$$

$$= \sum_i \left(\frac{1}{z_i'\alpha}\right) z_i + \sum_i \frac{(y_i - x_i'\beta)^2}{(z_i'\alpha)^3} z_i$$

$$= \sum_i \left[\frac{1}{z_i'\alpha} + \frac{(y_i - x_i'\beta)^2}{(z_i'\alpha)^3}\right] z_i \qquad (2)$$

(Again, note that z_i is a (1xp) column vector & the other terms are scalars.

Set (1) & (2) equal to zero to get the likelihood equations for $\underset{\sim}{\beta}$ and $\underset{\sim}{\alpha}$. This is a set of $(k+p)$ equations in the same number of unknowns. These equations <u>cannot</u> be solved analytically for the MLE's. A numerical solution must be obtained.

(d) See the EViews workfile. There is a "SOLUTION" text object inside the file.