ECON 546: Themes in Econometrics

Two Problems Using the Characteristic Function

(a) If $Y \sim N[\mu, \sigma^2]$, then $\phi_Y(t) = \exp[it\mu - \sigma^2 t^2 / 2]$.

First, note that for *any* (not necessarily normal) random variable, *X*, the characteristic function of $W = aX$ is $\phi_W(t) = \phi_X(at)$:

$$\phi_W(t) = \phi_{aX}(t) = E[e^{it(ax)}] = E[e^{iatx}] = \phi_X(at) . \qquad \text{(i)}$$

Second, note that for any random variable, *X*, the characteristic function of $U = X + b$ is $\phi_U(t) = e^{itb}\phi_Y(t)$:

$$\phi_U(t) = \phi_{(X+b)}(t) = E[e^{it(x+b)}] = E[e^{itx}e^{itb}] = e^{itb}E[e^{itx}] = e^{itb}\phi_X(t) . \qquad \text{(ii)}$$

Combining results (i) and (ii), we have:

For any random variable, *X*, the characteristic function of $V = aX + b$ is $\phi_V(t) = e^{itb}\phi_X(at)$. (iii)

Now, we have established that if $Z \sim N[0,1]$, then $\phi_Z(t) = e^{-t^2/2}$. The relationship between *Z* and *Y* is that $Z = (Y - \mu)/\sigma$, or $Y = \sigma Z + \mu$. Using our main result, (iii), with $a = \sigma$ and $b = \mu$, we immediately have $\phi_Y(t) = \exp[it\mu - \sigma^2 t^2 / 2]$, as required.

(b) If Y_1 and Y_2 are independent, then $\phi_{Y_1+Y_2}(t) = \phi_{Y_1}(t)\phi_{Y_2}(t)$.

Note that $\phi_{Y_1+Y_2}(t) = E[e^{it(Y_1+Y_2)}] = E[e^{itY_1}e^{itY_2}]$. If the two random variables are *independent*, then

$E[e^{itY_1}e^{itY_2}] = E[e^{itY_1}]E[e^{itY_2}]$, and so in this case, $\phi_{Y_1+Y_2}(t) = E[e^{itY_1}]E[e^{itY_2}] = \phi_{Y_1}(t)\phi_{Y_2}(t)$, as required.