ECON 546: Extra Bayes Exercises

Question 1

Suppose that we have n independent observations from a Poisson distribution, whose p.m.f. is:

$$p(y_i \mid \lambda) = \lambda^{y_i} \exp\{-\lambda\} / y_i! \quad ; y_i = 0,1,2,........; \quad \lambda > 0$$

Suppose that, *a priori*, we are totally ignorant about the value of λ.

Prove that the Bayes estimator of λ, when the loss function is quadratic, is $\bar{y} = (\sum_{i=1}^{n} y_i / n)$.

[**Hint:** The p.d.f. for a Gamma distribution is $p(x) \propto x^{\alpha-1} \exp\{-\beta x\}$; α, β, $x > 0$. The mean of this distribution is (α / β).]

Question 2

Suppose that Y is ***uniform*** on the interval $[0 , \theta]$, and that we have a random sample of n observations on Y.

Let the prior p.d.f. for θ be:

$$p(\theta) = a k^a \theta^{-(a+1)} \quad ; \quad \theta \geq k ; \quad a > 0 .$$

(This is the p.d.f. for a Pareto distribution.)

(a) Show that this is the natural conjugate prior for θ.

(b) Obtain the full posterior density for θ, including the normalizing constant.

(c) Find the mean of the prior distribution.

(d) What is the Bayes estimator of θ if the loss function is quadratic?

Question 3

You probably know that J. M. Keynes made many important contributions to probability theory and statistics (as well as to economics, of course). His *Treatise on Probability* is a classic work that makes seminal contributions to the "subjective" theory of probability used by Bayesians. He also provided Keynes, 1911) the first modern treatment of "Laplace's (1774) first law" - if we have an odd number of observations, "n", then the value of θ that minimizes the expression $\sum_{i=1}^{n} | y_i - \theta |$ is the median of the y_i's. (An odd number is needed to ensure that the median is unique.) Now suppose that we have a random sample of "*n*" (which you can assume to be an odd number of) observations from a Laplace (or "double exponential") distribution. That is, the density function for an individual y_i is:

$$p(y_i \mid \theta, \lambda) = (2\lambda)^{-1} \exp\{-\mid y_i - \theta \mid / \lambda\}; \quad -\infty < y_i < \infty \quad ; \quad \lambda > 0 .$$

(a) Prove that the MLE for θ, say $\hat{\theta}$, is the median of the sample, and that the MLE for λ is

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} \mid y_i - \hat{\theta} \mid .$$

(b) It can be shown that $E(y_i) = \theta$ and $E(y_i^2) = (2\lambda^2 + \theta^2)$. Suppose that we want to provide a unitless measure of the variability of the data. One such measure is the "coefficient of variation", $cv = \sqrt{\text{var.}(y_i)} / E(y_i)$. Provide a consistent estimator for *cv*. What else can you say about the asymptotic properties of your estimator?

(c) Now, suppose we ***know*** θ, but that we are totally ***ignorant*** about λ. Obtain the posterior density for λ, and derive the Bayes' estimator of this parameter when we have a zero-one loss function. Do this estimator and the MLE converge in probability to λ at the same rate as $n \to \infty$?

References: Keynes, J. M. (1911), "The principal averages and the laws of error which lead to them", *Journal of the Royal Statistical Society, Series A*, 74, 322-328.

Laplace, P. (1774), "Mémoire sur la probabilité des causes par les èvénemens", *Mémoires de Mathématique et de Physique*, 6, 621-656.