ECON 546 – Spring 2010
Solution for Final Exam

Q.1. The primary goal when estimating a SEM is to obtain consistent estimates of the parameters of the structural form. A secondary goal is to ensure that these parameter estimates have maximum asymptotic efficiency. Both the single-equation estimators (such as 2SLS, LIML, and other IV estimators) achieve consistency. However, in general, they are less efficient (asymptotically) than full system estimators (such as 3SLS and FIML). So, although the full system estimators are more demanding, computationally, and require that we specify all of the structural form equations in order to estimate any one equation, they give us (asymptotically) more efficient estimates. The downside of full system estimators lies with the likelihood that there will be some mis-specification of the system. For example, if one variable is wrongly omitted from the first equation of the model, all of the FIML and 3SLS estimates of all of the coefficients in all of the equations will be inconsistent. In contrast, while the 2SLS/IV estimates will be inconsistent for the first equation, they will still be consistent in the other equations of the model.

Q. 2.

(a) $\phi_X(t) = (1 - itb)^{-c}$

$$\phi_X'(t) = (-c)(-ib)(1-itb)^{-(c+1)}$$
$$= icb(1-itb)^{-(c+1)}$$

Let $t \to 0$ & divide by i:

$E(X) = cb.$ #

$$\phi_X''(t) = -(icb)(c+1)(-ib)(1-itb)^{-(c+2)}$$
$$= + i^2 b^2 c(c+1)(1-itb)^{-(c+2)}$$

Let $t \to 0$ & divide by i^2:

$E(X^2) = b^2 c(c+1).$

So, $\text{Var.}(X) = E(X^2) - [E(X)]^2$

$$= b^2c^2 + b^2c - (cb)^2$$
$$= cb^2.$$ #

(b) By independence:

$$L(b|x) = \prod_{i=1}^{n} p(x_i|b)$$
$$= \prod_{i=1}^{n} \left\{ \frac{\left[\left(\frac{x_i}{b}\right)^{c-1} \exp(-x_i/b) \right]}{b(c-1)!} \right\}$$

$$\log L = \sum_{i=1}^{n} [(c-1)\log x_i - (c-1)\log b]$$
$$- \sum_i (x_i/b) - n\log b - n\log(c-1)!$$

$$\frac{\partial \log L}{\partial b} = -\frac{n(c-1)}{b} + \frac{\sum x_i}{b^2} - \frac{n}{b}$$

$$= -\frac{nc}{b} + \frac{\sum x_i}{b^2} = 0.$$

$$\Rightarrow \tilde{b} = \frac{1}{c}\sum_i (x_i/n) = \left(\frac{\bar{x}}{c}.\right)$$

(Note that $\tilde{b} > 0$ as $x_i > 0$ & $c > 0$.)

Check the SOC:

$$(\partial^2 \log L/\partial b^2) = \frac{n(c-1)}{b^2} - \frac{2n\bar{x}}{b^3} + n/b^2$$

$$= \frac{nc}{b^2} - \frac{2n\bar{x}}{b^3}$$

Evaluate this when $b = \tilde{b} = (\bar{x}/c)$:

$$(\partial^2 \log L/\partial b^2)\Big|_{\tilde{b}} = \left(\frac{-nc^3}{x^2}\right) < 0$$

because $c > 0$.

(c) LM Test:

$$I = -E(\partial^2 \log L/\partial b^2)$$

$$= \left(\frac{-nc}{b^2}\right) + \frac{2n}{b^3}E(\bar{x})$$

$$= \left(\frac{-nc}{b^2}\right) + \left(\frac{2n}{b^3}\right)(cb)$$

$$= (nc/b^2)$$

So, $IA = \lim_{n\to\infty}(I) = c/b^2$

$I^* = (nc/b^2)$ (So $\lim_{n\to\infty}\left(\frac{1}{n}I^*\right) = IA$)

$$D \log L_1 (\tilde{b}_R) = \left[-\frac{nc}{b} + \frac{n\bar{x}}{b^2} \right] \Bigg|_{b=1}$$

$$= -nc + n\bar{x}$$

$$= n(\bar{x} - c)$$

$$\text{So, } LM = \left[(n(\bar{x}-c))^2 \left(\frac{nc}{b^2}\right)^{-1} \right] \Bigg|_{b=1}$$

$$= \frac{[n(\bar{x}-c)]^2}{nc}$$

$$= \cancel{\left(\frac{\bar{x}-c}{c}\right)^2} = \frac{n(\bar{x}-c)^2}{c},$$ where c is known.

Note that $LM = \cancel{\left(\frac{\bar{x}}{c}-1\right)^2} nc(\tilde{b}-1)^2$. This makes sense, intuitively – if $\tilde{b}$ is close to 1 the test statistic is close to zero. $LM \xrightarrow{d} \chi^2(1)$ if H_0 true. So, we would calculate LM & then reject $H_0: b = 1$ if LM exceeded the $\chi^2_{(1)}(\alpha)$ critical value.

(d) Wald Test:

$$W = (\tilde{b} - 1)^2 \left(\frac{nc}{\tilde{b}^2}\right)$$

$$= \cancel{\frac{\tilde{b}(\tilde{b}-1)^2}{nc}} = \frac{nc(\tilde{b}-1)^2}{\tilde{b}^2}$$

which can also be written as ~~$\bar{x}(\bar{x}-c)^2/n$~~. $nc(\bar{x}-c)^2/\bar{x}$.

Again, if $\tilde{b}$ is close to 1, $W \to 0$ & this favours H_0. We reject $H_0: b = 1$ if $W > \chi^2_{(1)}(\alpha)$, asymptotically.

Q. 3.

(a) The linear regression model ignores the fact that the "y-values" are non-negative integers ~~integers~~ - this will make OLS inefficient.

The error term cannot be Normally distributed as the y-values are not even continuous, so the usual t-tests, F-tests will be inappropriate. OLS is just MLE with Normal errors, but the errors can't be Normal.. So, OLS is MLE with a mis-specified distribution - it will be inconsistent!

(b)
$$p(y|\mu) = \mu^y (1+\mu)^{-(y+1)}$$
$$= \exp\{ \log [\mu^y (1+\mu)^{-(y+1)}] \}$$
$$= \exp\{ y \log \mu - (y+1) \log(1+\mu) \}$$
$$= \exp\{ y \log \mu - y \log(1+\mu) - \log(1+\mu) \}$$
$$= \exp\{ y \log\left(\frac{\mu}{1+\mu}\right) - \log(1+\mu) \}$$

$$\text{So, } \begin{cases} a(\mu) = -\log(1+\mu) \\ b(y) = 0 \\ c(\mu) = \log\left(\frac{\mu}{1+\mu}\right) \end{cases}$$

(c) $a'(\mu) = -\left(\frac{1}{1+\mu}\right)$

$$c'(\mu) = \frac{1}{\mu} - \frac{1}{1+\mu}$$

$$= \left[\frac{1+\mu-\mu}{\mu(1+\mu)}\right]$$

$$= 1/[\mu(1+\mu)]$$

So, $E(Y) = \{-a'(\mu)/c'(\mu)\}$

$$= -\left\{\frac{-1}{1+\mu}\right\}\left[\frac{\mu(1+\mu)}{1}\right]$$

$$= \mu \qquad \#.$$

$$V(Y) = 1/c'\mu) = \mu(1+\mu) \qquad \#.$$

(d) $V(Y) = \mu + \mu^2$

$E(Y) = \mu$

So, $V(Y) > E(Y)$ & this distribution can be used to model "over-dispersed" data. For the Poisson Model, $E(Y) = Var(Y)$, & we can only handle "equi-dispersion".

(e) Let $\mu_i = \exp(\underset{\sim}{x}'\underset{\sim}{\beta})$. Using the exponential transformation ensures that $\mu_i > 0$, as required (regardless of the signs of the covariates).

Now, $p(y_i|\mu) = \exp(x_i'\beta)^{y_i}(1+\exp(x_i'\beta))^{-(y_i+1)}$

Assuming independent sample observations,

$$L = \prod_i p(y_i|\mu) = \prod_{i=1}^{n}\left[\exp(x_i'\beta)^{y_i}(1+\exp(x_i'\beta))^{-(y_i+1)}\right]$$

& $\log L = \sum_i \left[y_i(x_i'\beta) - (y_i+1)\log(1+\exp(x_i'\beta))\right]$

$$\frac{\partial \log L}{\partial \underset{\sim}{\beta}} = \sum_i \left[y_i \underset{\sim}{x}_i - \frac{(y_i+1)\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})\underset{\sim}{x}_i}{1+\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}\right]$$

$$= \sum_i \left[y_i - \frac{(y_i+1)\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}{1+\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}\right]\underset{\sim}{x}_i$$

$$= \sum_i \left[\frac{y_i + y_i\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta}) - y_i\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta}) - \exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}{1+\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}\right]\underset{\sim}{x}_i$$

$$= \sum_i \left[\frac{y_i - \exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}{1+\exp(\underset{\sim}{x}_i'\underset{\sim}{\beta})}\right]\underset{\sim}{x}_i$$

& setting this (k-element) vector equation equal to zero gives us the (k) "likelihood equations". These equations are highly non-linear in $\underset{\sim}{\beta}$, & must be solved numerically, just as for the Poisson model.

Q.4.

(a) $\hat{\theta} = [(y+a)/(a+b+n)]$

(i) $E(\hat{\theta}) = [(n\theta+a)/(a+b+n)]$

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$$

$$= \left(\frac{n\theta+a}{a+b+n}\right) - \theta$$

$$= \frac{n\theta+a-\theta(a+b+n)}{a+b+n}$$

$$= \frac{a - a\theta - b\theta}{a+b+n}$$

$$= \frac{a - \theta(a+b)}{a+b+n}$$

(ii) $$\text{Var.}(\hat{\theta}) = \frac{\text{var}(y+a)}{(a+b+n)^2}$$

$$= \frac{\text{var.}(y)}{(a+b+n)^2}$$

$$= \frac{n\theta(1-\theta)}{(a+b+n)^2}$$

(b) $\text{MSE}(\hat{\theta}) = \text{Var.}(\hat{\theta}) + \text{Bias}^2(\hat{\theta})$

$$= \frac{n\theta(1-\theta)}{(a+b+n)^2} + \frac{[a-\theta(a+b)]^2}{(a+b+n)^2}$$

$$= \frac{n\theta(1-\theta) + [a-\theta(a+b)]^2}{(a+b+n)^2}$$

(c) Now, we want this risk (MSE) to be independent of θ. We can re-write the MSE as:

$$MSE = \frac{[n\theta - n\theta^2 + a^2 + \theta^2(a+b)^2 - 2a\theta(a+b)]}{(a+b+n)^2}$$

$$= \frac{\theta[n-2a(a+b)] + \theta^2[(a+b)^2-n] + a^2}{(a+b+n)^2}$$

& this will be constant for all θ, if & only if

$$\left.\begin{array}{l} [n-2a(a+b)] = 0 \\ \& \ [(a+b)^2-n] = 0 \end{array}\right\}$$

(d) We now need to find the values of 'a' & 'b' that will satisfy the above 2 equations simultaneously.

We have : $2a(a+b) = n = (a+b)^2$

$$\Rightarrow 2a^2 + 2ab = a^2 + 2ab + b^2$$

$$\Rightarrow a^2 = b^2 \qquad .$$

$$\Rightarrow a = b \quad (\text{because } a, b > 0).$$

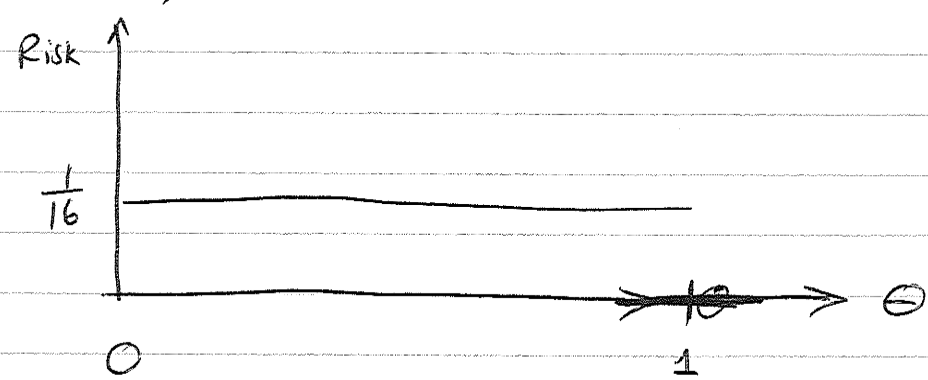
So, $n = 2a^2 + 2ab = 2a^2 + 2a^2 = 4a^2$

$\Rightarrow \quad a = (n/4)^{1/2} = b$

In this case, $\text{Risk}(\hat{\theta}) = MSE(\hat{\theta}) = \dfrac{a^2}{(a+b+n)^2}$,

so, $\text{Risk}(\hat{\theta}) = \dfrac{n/4}{\left[\left(\frac{n}{4}\right)^{\frac{1}{2}} + \left(\frac{n}{4}\right)^{1/2} + 1\right]^2}$

& when $n = 1$, $\text{Risk}(\hat{\theta}) = 1/16$



Q.5.

(a) The estimated coefficients are identical because each equations contains the same regressors. So, in this case, the OLS & SUR estimates coincide.

However, the standard errors differ – this is because the estimator for the covariance matrix of 'b' in OLS is $\hat{\sigma}^2 (X'X)^{-1}$, while for SUR it is of the form $[X'(\hat{\Sigma} \otimes I)^{-1} X]^{-1}$. Hence, the t-statistics differ too.

(b) I would anticipate that all of the estimated coefficients (apart from the intercept) will be +ve, & they are. Accordingly, the p-values should be halved (1-sided alternative hypothesis). The largest p-value is then 7.08% – everything is significant at the 10% level, and all but one coefficient are significant at the 5% level.

(c) We need to test H_0: Covariance matrix is diagonal vs. H_A: Not diagonal. If H_0 is rejected we would use SUR, otherwise we would use the OLS version of the model.

The likelihood ratio test statistic is:

$$LR = T \log [|\hat{\Sigma}| / \tilde{\Sigma}]$$

where $\tilde{\Sigma}$ is the SUR estimator & $\hat{\Sigma}$ is the OLS estimator. T is the # of observations per equation. $LR \xrightarrow{d} \chi^2_v$ where $v = \frac{M(M-1)}{2}$ & M = # equations, if H_0 is true.

$$\text{So,} \quad LR = 616 \log [5.669095 / 5.450788] = 24.19 \qquad (v = 1)$$

The 5% critical value is 3.84 so we Reject H_0 (at any reasonable significance level) & we would use OUTPUT 3 (SUR).

(d) Output 4 give Wald test results of the hypothesis $H_0: c_4 = c_{14}$ against $H_A: c_4 \neq c_{14}$. (i.e. is the UNION effect the same for earnings and for benefits, ceteris paribus?)

The p-value of 0.53 suggests that being the member of a union impacts (positively) on both earnings and benefits to the same degree – more correctly, although $\hat{c}_4 = 0.704$ & $\hat{c}_{14} = 0.465$, implying increases of 70.4 cents/hour & 46.5 cents/hour, these values are not significantly different.

(e) EARN1 gives us the predicted earnings for an individual with an average education & level of experience, who is a single, white, male, union member. EARN2 gives us the corresponding value if the person is non-white. The difference of \$0.86/hour is the "marginal effect (on earnings) of being white", within single-male-union member group.