Some Notes on the Negative Binomial Distribution

The negative binomial distribution can be defined in terms of the random variable Y = number of failures in independent Bernoulli trials (with probability of "success, p) before the r^{th} success.

The probability mass function for Y is:

$$\Pr.[Y = y] = \binom{r+y-1}{y} p^r (1-p)^y \,; \qquad y = 0,\, 1,\, 2,\, \ldots\ldots;\ \ r = 1,\, 2,\, \ldots;\ \ 0 < p < 1. \tag{1}$$

The negative binomial distribution gets its name from the following relationship:

$$\binom{r+y-1}{y} = (-1)^y \binom{-r}{y} = (-1)^y \frac{(-r)(-r-1).....(-r-y+1)}{(y)(y-1)(y-2).....(2)(1)}\,, \tag{2}$$

which defines the usual binomial coefficients in the case of *negative* integers.

Then, using the usual binomial expansion for a negative power (remember this from high school?), namely:

$$(1+x)^{-r} = \sum_{i=0}^{\infty} \binom{-r}{i} x^i = \sum_{i=0}^{\infty} (-1)^i \binom{r+i-1}{i} x^i \,, \tag{3}$$

it follows immediately that

$$\sum_{y=0}^{\infty} \Pr.[Y = y] = 1\,. \tag{4}$$

The mean of the distribution can be obtained as follows:

$$\begin{aligned} E(Y) &= \sum_{y=0}^{\infty} y \binom{r+y-1}{y} p^y (1-p)^y = \sum_{y=1}^{\infty} \frac{(r+y-1)!}{(y-1)!(r-1)!} p^r (1-p)^y \\ &= \frac{r(1-p)}{p} \sum_{y=1}^{\infty} \binom{r+y-1}{y-1} p^{r+1} (1-p)^{y-1} = \frac{r(1-p)}{p} \sum_{k=0}^{\infty} \binom{r+1+k-1}{k} p^{r+1} (1-p)^k \\ &= \frac{r(1-p)}{p} = r[(1/p)-1] \end{aligned} \tag{5}$$

In the same way, we can show that

$$E(Y^2) = \frac{r(1-p)[1+r(1-p)]}{p^2}\,, \tag{6}$$

so that

$$Var.(Y) = E(Y^2) - [E(Y)]^2 = \frac{r(1-p)}{p^2}\,. \tag{7}$$

Some other characteristics of this distribution are as follows:

$$Skew(Y) = \frac{(2-p)}{\sqrt{r(1-p)}} \qquad \text{; which is always positive} \tag{8}$$

$$Excess\ Kurtosis\,(Y) = \frac{6(1-p)+p^2}{r(1-p)} \qquad \text{; which is always positive} \tag{9}$$

$$Mode\,(Y) = \frac{(r-1)(1-p)}{p} \;; \qquad \text{if } r > 1 \quad \text{(otherwise the mode = 0),} \tag{10}$$

and the characteristic function for the negative binomial distribution is

$$\phi_Y(t) = \left(\frac{p}{1-(1-p)e^{it}}\right)^r. \tag{11}$$

You know that the Poisson is a limiting case of the Negative Binomial distribution. This comes about by re-parameterizing the latter distribution in terms of the mean, $\mu = r[(1/p)-1]$, derived above. Then the probability mass function for the Negative Binomial distribution becomes

$$\Pr.[Y = y] = \frac{\lambda^y}{y!}\frac{(y+r-1)!}{(r-1)!(r+\lambda)^y}\frac{1}{(1+\lambda/r)^r}\,. \tag{12}$$

Taking the limit as $r \to \infty$, this p.m.f collapses to that for a Poisson-distributed random variable:

$$\Pr.[Y = y] = \frac{\lambda^y}{y!}\exp(-\lambda)\,. \tag{13}$$

The Geometric distribution is also a special case of the Negative Binomial distribution. In equation (1), set r = 1:

$$\Pr.[Y = y] = p(1-p)^y \quad ; \qquad y = 0, 1, 2, \ldots\ldots; \qquad 0 < p < 1. \tag{13}$$