(B) **Simultaneous Equations Models:**

Consider a system where dependent variable of one equation is regressor in another equation. For example:

(1) $C_t = \alpha_1 + \alpha_2 y_t + \alpha_3 C_{t-1} + \varepsilon_{1t}$

(2) $I_t = \beta_1 + \beta_2 y_t + \beta_3 i_t + \varepsilon_{2t}$

(3) $y_t \equiv C_t + I_t + G_t$

(1) C_t random
(2) I_t random
(3) y_t random

$\Rightarrow$ random regressor (y_t) in (1), (2)

In such models, OLS is biased and inconsistent. SEM's were at heart of econometrics from outset. Search for estimators that are consistent, & preferably asymptotically efficient too.

Types of Variables:

* Endogenous – "explained by an equation"
* Exogenous
* Lagged Endogenous

$\left.\right\}$ Pre-determined

To be complete, we need as many equ'ns. as there are endogenous variables (M).

Forms of the Model:

The version of the model in example above is called the Structural Form of the model.

In general, this is:

$$\gamma_{11} y_{t1} + \cdots + \gamma_{M1} y_{tM} + \beta_{11} x_{t1} + \cdots + \beta_{K1} x_{tK} = \varepsilon_{t1}$$
$$\vdots$$
$$\gamma_{1M} y_{t1} + \cdots + \gamma_{MM} y_{tM} + \beta_{1M} x_{t1} + \cdots + \beta_{KM} x_{tK} = \varepsilon_{tM}$$

Note:
* lots of γ's and β's will be zero
* Usually $\gamma_{ii} = 1$ in equation "i"
* Usually one of x's will be intercept.

Write the model as:

$$(y_1, \ldots, y_m)_t \begin{bmatrix} \gamma_{11} & \cdots & \gamma_{1m} \\ \vdots & & \vdots \\ \gamma_{m1} & \cdots & \gamma_{mm} \end{bmatrix} + (x_1, \ldots x_K)_t \begin{bmatrix} \beta_{11} & \cdots & \beta_{1m} \\ \vdots & & \vdots \\ \beta_{K1} & \cdots & \beta_{Km} \end{bmatrix}$$

$$= (\varepsilon_1, \ldots, \varepsilon_m)_t ;$$

or $$\boxed{y_t' \Gamma + x_t' B = \varepsilon_t'} \qquad t = 1, 2, \ldots, T.$$

Note that model is linear in parameters.

(Can generalize this)

If Γ^{-1} exists, then:

$$y_t' + x_t' B \Gamma^{-1} = \varepsilon_t' \Gamma^{-1}$$

or, $$y_t' = -x_t' B \Gamma^{-1} + \varepsilon_t' \Gamma^{-1}$$

or, $$\boxed{y_t' = x_t' \Pi + v_t'} \qquad t = 1, \ldots, T$$

The latter version of the model is the Restricted Reduced Form. Use this version for forecasting, once parameters have been estimated. However, before estimating, there is a prior issue to be considered:

<u>The Identification Problem:</u>

Consider the following equilibrium model:

Demand : $Q_t^D = \alpha + \beta P_t + \varepsilon_{1t}$

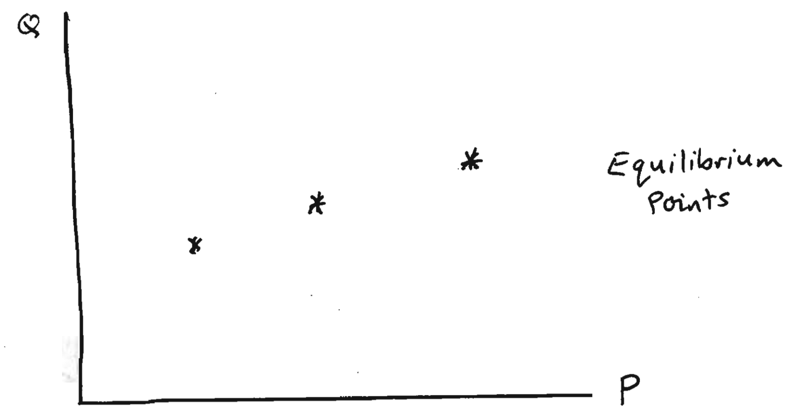
Supply : $Q_t^S = \gamma + \delta P_t + \varepsilon_{2t}$

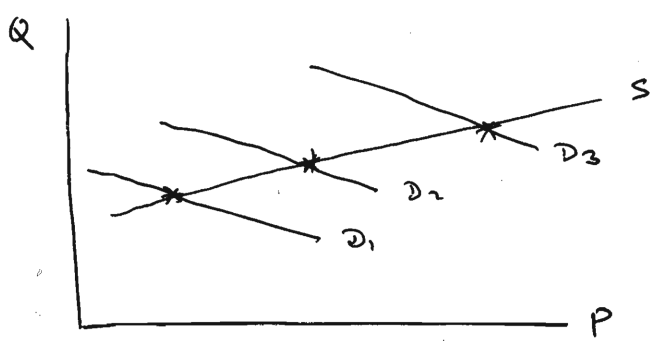
Equilibrium: $Q_t^D \equiv Q_t^S$ ($= Q_t$, say)

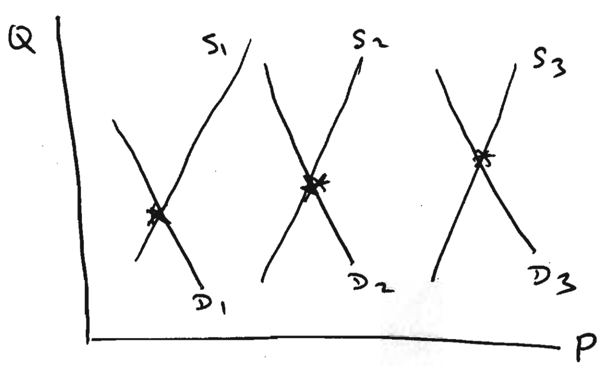
Problem : If we regress Q_t on P_t, are we estimating demand equ'n. or supply equ'n.?

Actually, its neither!

The parameters (& equations) of this model are not "identified":



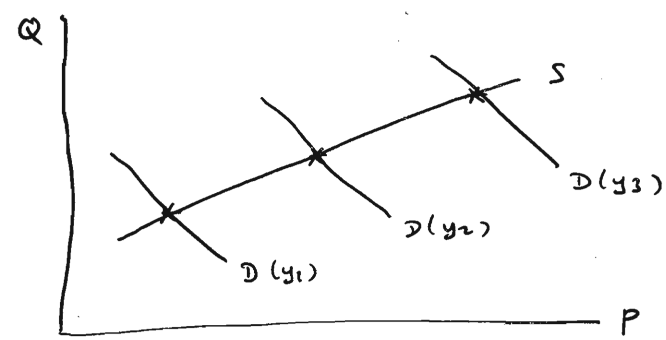




Suppose we add a variable to demand fctn.:

$$\left.\begin{aligned} Q_t^D &= \alpha + \beta P_t + \theta y_t + \varepsilon_{1t} \\ Q_t^S &= \gamma + \delta P_t + \varepsilon_{2t} \end{aligned}\right\} Q_t^S \equiv Q_t^D$$

Changes in y_t will shift the demand curve & "identify" the supply curve:



Similarly, adding a variable to the supply fctn. (but not to the demand fctn) will identify demand.

Really, the important thing is that we are omitting variables from equations.

Order Condition:

A necessary (but not sufficient) condition for an equation in a SEM to be identified is that $(K - k_i) \geq (M_i - 1)$, where k_i & M_i are no. of predetermined & endogenous variables in i'th equation.

An equation can only be estimated if it is identified (otherwise no consistent estimator exists).

A full model can be estimated only if every equation is identified.

Estimation:

The reduced form of the model can be estimated consistently by OLS.

Not very helpful – usually interested in structural form. (Usually can't solve uniquely.)

OLS is inconsistent if applied to S.F.

Other estimators:

(A) Single-equation (Limited Information):

$\left.\begin{array}{l}\text{Two Stage Least Squares} \\ \text{LIML} \\ \text{k-class}\end{array}\right\}$ All I.V.

(B) System (Full Information):

$\left.\begin{array}{l}\text{Three Stage Least Squares} \\ \text{FIML}\end{array}\right\}$ same asymp. efficiency

System estimators more efficient, but sensitive to model mis-specification.

The Two-Stage Least Squares Estimator ~

e.g. $y_{1t} = \beta_1 + \beta_2 x_{1t} + \gamma_2 y_{2t} + \varepsilon_t$

(1) Regress y_{2t} on all K predetermined variables in model & get $\hat{y}_{2t}$.

(2) Estimate $y_{1t} = \beta_1 + \beta_2 x_{1t} + \gamma_2 \hat{y}_{2t} + v_t$ by OLS.

Easily shown that 2SLS is an I.V. estimator, with x_{1t} and $\hat{y}_{2t}$ as instruments for x_{1t} & y_{2t} in this example.

I.V. $\Rightarrow$ consistent estimator.

<u>Example</u>: (Klein's Model I : 1950)

$$C_t = \alpha_0 + \alpha_1 P_t + \alpha_2 P_{t-1} + \alpha_3 (W_t^p + W_t^g) + \varepsilon_{1t}$$

$$I_t = \beta_0 + \beta_1 P_t + \beta_2 P_{t-1} + \beta_3 K_{t-1} + \varepsilon_{2t}$$

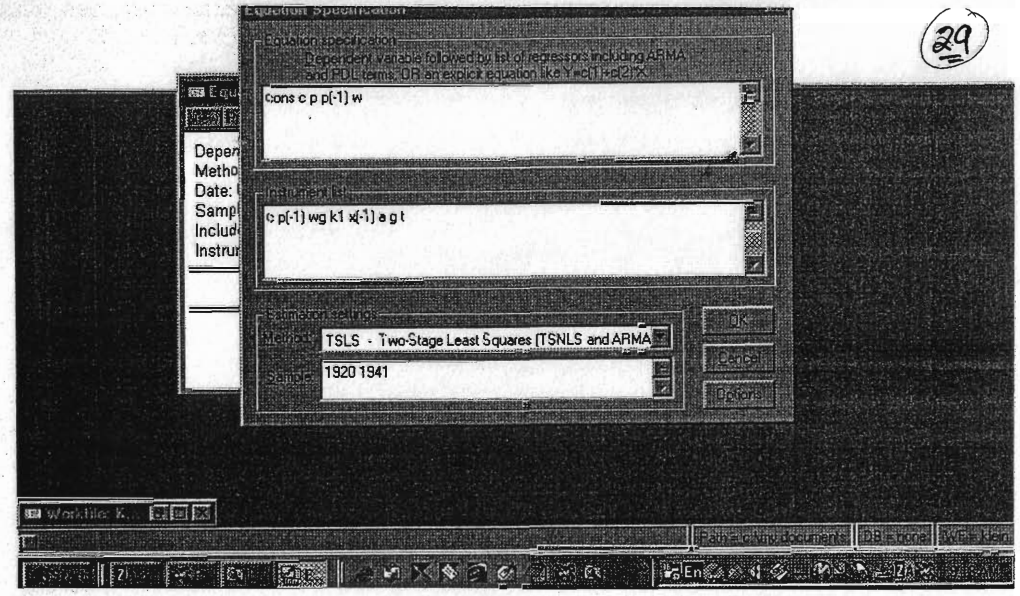
$$W_t^p = \gamma_0 + \gamma_1 X_t + \gamma_2 X_{t-1} + \gamma_3 A_t + \varepsilon_{3t}$$

$$X_t \equiv C_t + I_t + G_t$$

$$P_t \equiv X_t - T_t - W_t^p$$

$$K_t \equiv K_{t-1} + I_t$$

(Greene, p. 381)



Dependent Variable: CONS
Method: Two-Stage Least Squares
Date: 12/02/03 Time: 09:18
Sample(adjusted): 1921 1941
Included observations: 21 after adjusting endpoints
Instrument list: C P(-1) WG K1 X(-1) A G T

Variable	Coefficient	Std. Error	t-Statistic	Prob.
C	16.55476	1.467979	11.27725	0.0000
P	0.017302	0.131205	0.131872	0.8966
P(-1)	0.216234	0.119222	1.813714	0.0874
W	0.810183	0.044735	18.11069	0.0000

R-squared	0.976711	Mean dependent var	53.99524
Adjusted R-squared	0.972601	S.D. dependent var	6.860866
S.E. of regression	1.135659	Sum squared resid	21.92525
F-statistic	225.9334	Durbin-Watson stat	1.485072
Prob(F-statistic)	0.000000		

Consider FIML —

Our structural form is:

$$\underset{(1\times M)(M\times M)}{\underset{\sim}{y}_t' \Gamma} + \underset{(1\times K)(K\times M)}{\underset{\sim}{x}_t' B} = \underset{(1\times M)}{\underset{\sim}{\varepsilon}_t'} \quad ; \quad t = 1, --, T.$$

& the restricted reduced form is:

$$\underset{\sim}{y}_t' = \underset{\sim}{x}_t' \Pi + \underset{-}{v}_t' \quad ; \quad t = 1, --, T$$

where $\underset{(K\times M)}{\Pi} = -\underset{(K\times M)(M\times M)}{B\Gamma^{-1}}$, & $v_t' = \underset{\sim}{\varepsilon}_t' \Gamma^{-1}$

Suppose that $\underset{\sim}{\varepsilon}_t \sim MVN(0, \Sigma)$.

As $v_t = \Gamma^{-1\prime} \varepsilon_t$, $V(v_t) = \Gamma^{-1\prime} \Sigma \Gamma^{-1}$

$= \Omega$; say.

We can stack the equations of the model side by side:

$$\underset{(T\times M)(M\times M)}{Y\Gamma} + \underset{(T\times K)(K\times M)}{X B} = \underset{(T\times M)}{E} \qquad \text{S.F.}$$

$$\underset{(T\times M)}{Y} = \underset{(T\times K)(K\times M)}{X\Pi} + \underset{(T\times M)}{V} \qquad \text{R.R.F.}$$

Using the normality of the errors the log-likelihood fctn. can be obtained using the R.R.F. :

$$\log L = -T/2 \, [M \log(2\pi) + \log|\Omega| + tr.\{\tfrac{1}{T}(Y - X\Pi)\Omega^{-1}(Y - X\Pi)'\}]$$

& substituting for Ω & for Π :

$$\log L = -T/2 \, [M \log(2\pi) + \log|\Gamma^{-1\prime}\Sigma\Gamma^{-1}| + tr.\{\tfrac{1}{T}(Y + XB\Gamma^{-1})(\Gamma\Sigma^{-1}\Gamma')(Y + XB\Gamma^{-1})'\}]$$

$$= -T/2 \, [M \log(2\pi) + \log|\Gamma^{-1\prime}\Sigma\Gamma^{-1}| + tr.\{\tfrac{1}{T}\Gamma\Sigma^{-1}\Gamma'(Y + XB\Gamma^{-1})'(Y + XB\Gamma^{-1})\}]$$

Now,

$$-T/2 \log|\Gamma^{-1\prime}\Sigma\Gamma^{-1}| = -T/2 \log|\Sigma| + T\log|\Gamma|$$

& $\Gamma'(Y + XB\Gamma^{-1})' = \Gamma'Y' + B'X'$

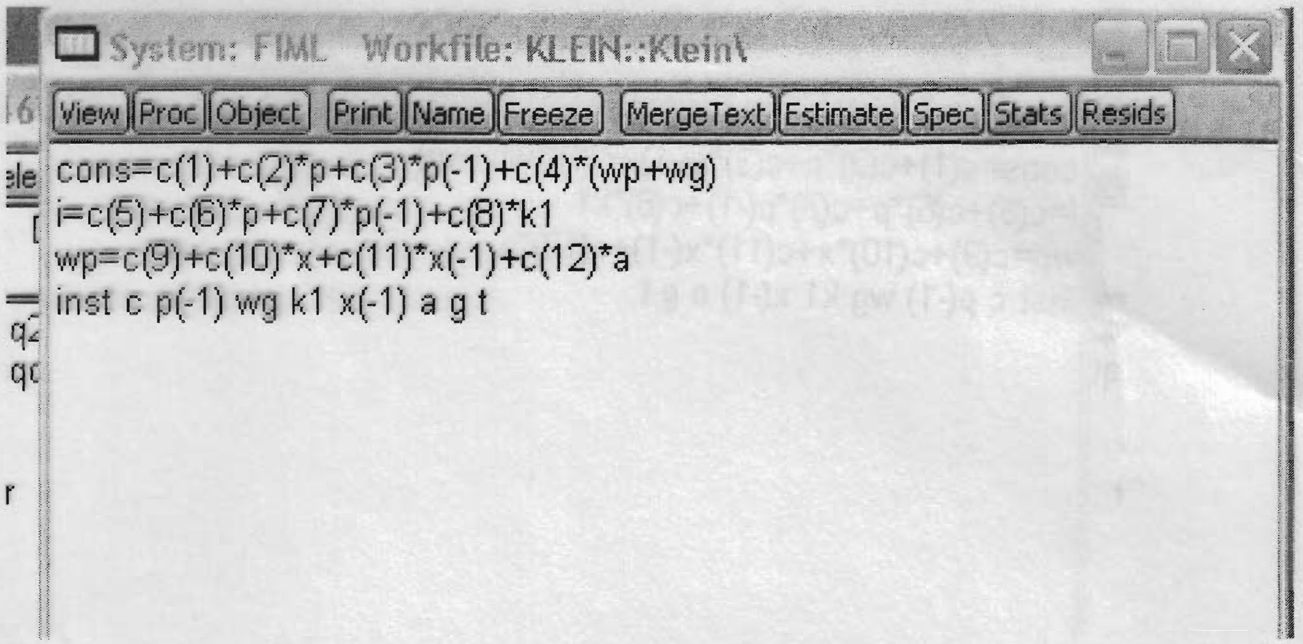
so, with some re-arranging -

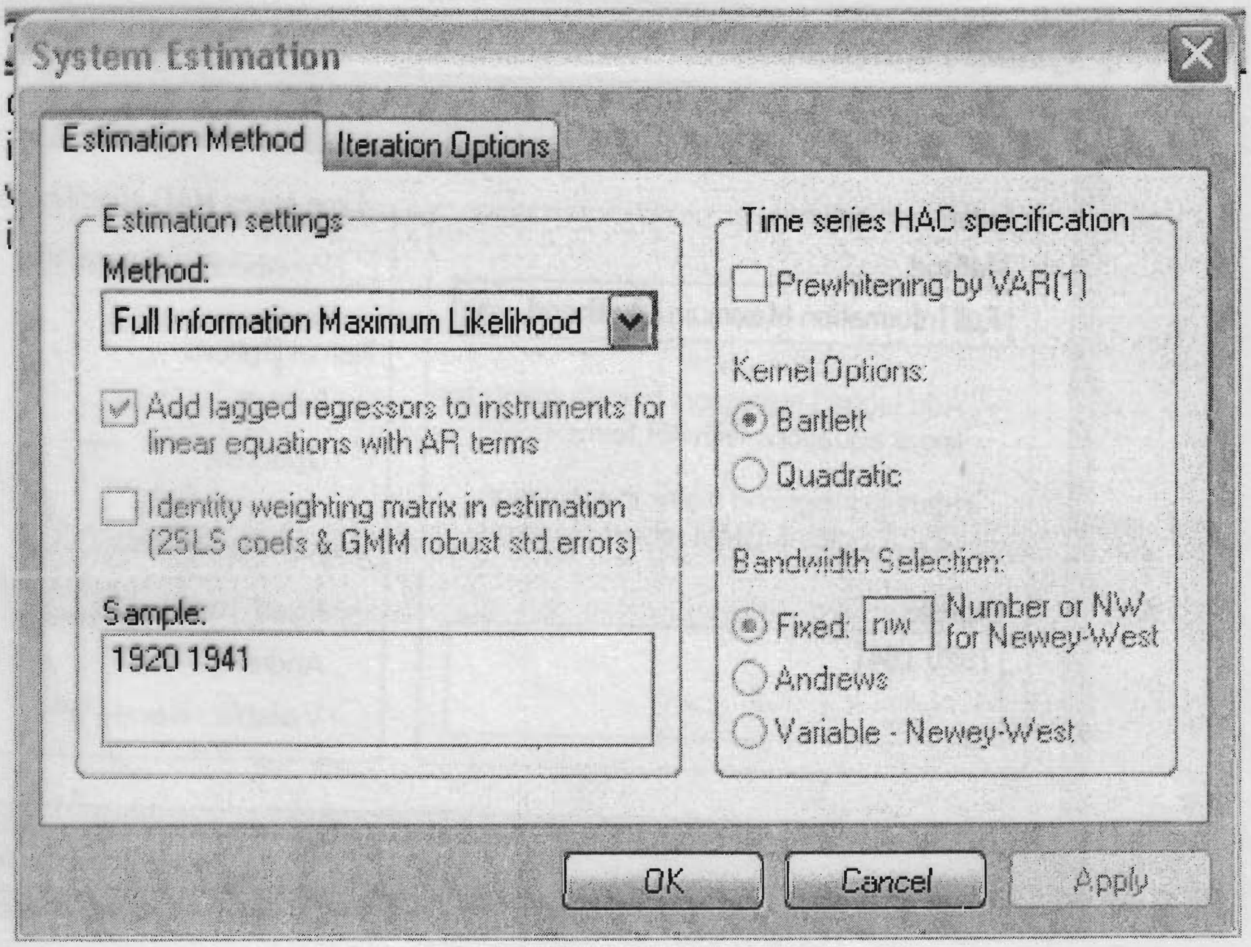
$$\log L = -T/2 \, [M \log(2\pi) - 2\log|\Gamma| + \log|\Sigma| + tr.\{\tfrac{1}{T}\Sigma^{-1}(Y\Gamma + XB)'(Y\Gamma + XB)\}]$$

As usual, maximizing $\log L$ w.r.t. B, Γ & Σ involves non-linear optimization. Keep in mind that B & Γ will have lots of zero's as elements. The identifying restrictions are already incorporated into the problem.

Notice that unless the identities are "substituted out", Σ will be singular, & $\log L$ won't be defined.

The FIML estimator: consistent and the asymptotically most efficient estimator of the structural form parameters.


System: FIML Workfile: KLEIN::Klein\
View Proc Object Print Name Freeze MergeText Estimate Spec Stats Resids
cons=c(1)+c(2)*p+c(3)*p(-1)+c(4)*(wp+wg)
i=c(5)+c(6)*p+c(7)*p(-1)+c(8)*k1
wp=c(9)+c(10)*x+c(11)*x(-1)+c(12)*a
inst c p(-1) wg k1 x(-1) a g t


System Estimation
Estimation Method
Iteration Options
Estimation settings
Method:
Full Information Maximum Likelihood
Add lagged regressors to instruments for linear equations with AR terms
Identity weighting matrix in estimation (2SLS coefs & GMM robust std.errors)
Sample:
1920 1941
Time series HAC specification
Prewhitening by VAR(1)
Kernel Options:
Bartlett
Quadratic
Bandwidth Selection:
Fixed: nw
Number or NW for Newey-West
Andrews
Variable - Newey-West
OK
Cancel
Apply

System: FIML Workfile: KLEIN::Klein

View Proc Object Print Name Freeze MergeText Estimate Spec Stats Resids

System: FIML
Estimation Method: Full Information Maximum Likelihood (Marquardt)
Date: 03/16/07 Time: 14:45
Sample: 1921 1941
Included observations: 21
Total system (balanced) observations 63
Convergence achieved after 23 iterations

	Coefficient	Std. Error	z-Statistic	Prob.
C(1)	15.83817	4.160045	3.807210	0.0001
C(2)	0.301113	0.409639	0.735070	0.4623
C(3)	0.043195	0.164138	0.263160	0.7924
C(4)	0.780210	0.078098	9.990133	0.0000
C(5)	16.10446	14.15952	1.137359	0.2554
C(6)	0.378498	0.345710	1.094843	0.2736
C(7)	0.412547	0.250269	1.648417	0.0993
C(8)	-0.139586	0.069613	-2.005166	0.0449
C(9)	2.083677	5.108942	0.407849	0.6834
C(10)	0.370717	0.130062	2.850311	0.0044
C(11)	0.207179	0.090796	2.281811	0.0225
C(12)	0.184241	0.101529	1.814668	0.0696

Log Likelihood	-69.25975
Determinant residual covariance	0.146979

Equation: CONS=C(1)+C(2)*P+C(3)*P(-1)+C(4)*(WP+WG)
Observations: 21

R-squared	0.979252	Mean dependent var	53.99524
Adjusted R-squared	0.975591	S.D. dependent var	6.860866
S.E. of regression	1.071909	Sum squared resid	19.53280
Durbin-Watson stat	1.257530		

Equation: I=C(5)+C(6)*P+C(7)*P(-1)+C(8)*K1
Observations: 21

R-squared	0.925343	Mean dependent var	1.266667
Adjusted R-squared	0.912169	S.D. dependent var	3.551948
S.E. of regression	1.052667	Sum squared resid	18.83785
Durbin-Watson stat	1.886288		

Equation: WP=C(9)+C(10)*X+C(11)*X(-1)+C(12)*A
Observations: 21

R-squared	0.982887	Mean dependent var	36.36190
Adjusted R-squared	0.979867	S.D. dependent var	6.304401
S.E. of regression	0.894535	Sum squared resid	13.60327
Durbin-Watson stat	2.024993		