MCMC Example

(Joint Normal)

Suppose $\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \sim N\left[\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}; \Sigma\right]$

$$\Sigma = \begin{bmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$

Now, we can show that:

$$\begin{cases} p(y_1|y_2) \sim N\left[\mu_1 + \left(\frac{\rho\sigma_1}{\sigma_2}\right)(y_2-\mu_2) \;;\; \sigma_1^2(1-\rho^2)\right] \\ p(y_2|y_1) \sim N\left[\mu_2 + \left(\frac{\rho\sigma_2}{\sigma_1}\right)(y_1-\mu_1) \;;\; \sigma_2^2(1-\rho^2)\right] \\ p(y_1) \sim N[\mu_1 ; \sigma_1^2] \\ p(y_2) \sim N[\mu_2 ; \sigma_2^2] \end{cases}$$

We'll consider the case where —

$$\begin{cases} p(y_1) \sim N[0;1] \\ p(y_2) \sim N[0;1] \\ p(y_1|y_2) \sim N[\rho y_2 \;;\; (1-\rho^2)] \\ p(y_2|y_1) \sim N[\rho y_1 \;;\; (1-\rho^2)] \end{cases}$$

Strategy:

1. Sample from $p(y_1|y_2)$ to get y_1.
2. Sample from $p(y_2|y_1)$, using y_1 value, to get y_2.
3. Return to 1 & repeat.

Eventually, $p(y_1|y_2) \rightarrow p(y_1)$ and $p(y_2|y_1) \rightarrow p(y_2)$

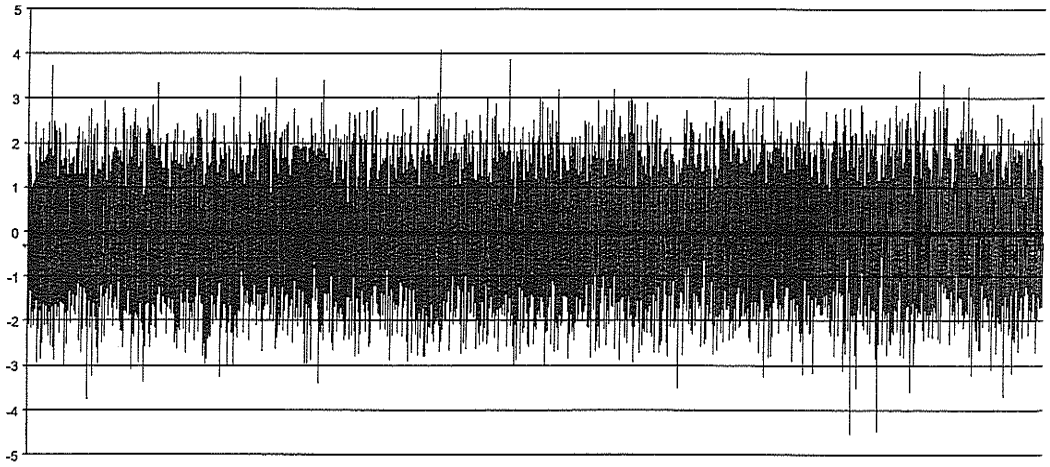
So, in this way we end up with the marginals, when all we knew to begin with was the conditionals. And no numerical integration was needed!

N.B.: * Need to initialize algorithm.

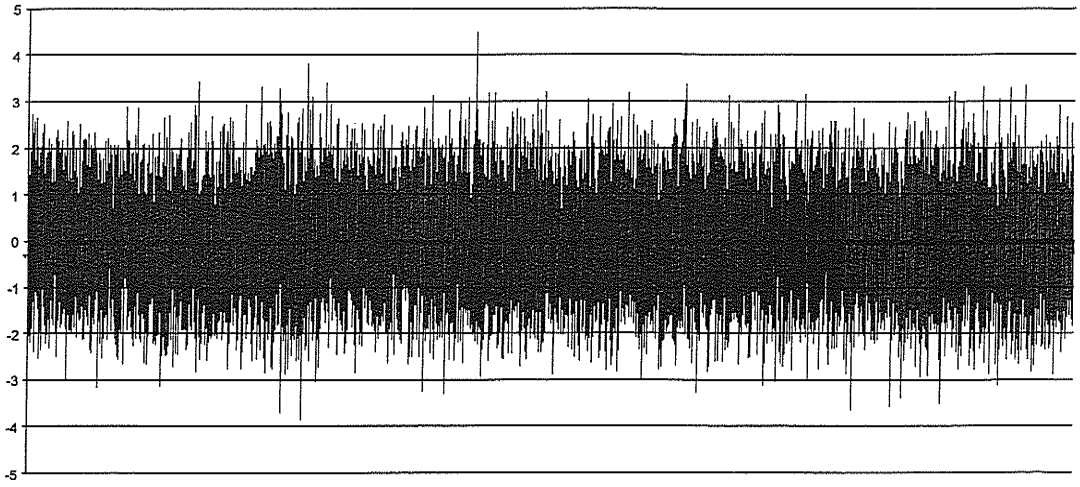
** Need a "burn-in" period.

```
      program mcmc
      integer idum /-1/
      dimension yy1(50000), yy2(50000)
      open(unit=4,file='mcmc.out')
      write(*,*) 'Enter value of rho '
      read(*,*) rho
      write(*,*) 'Enter the number of repetitions ( < 50,000) '
      read(*,*) n
      sd=sqrt(1-rho**2)
      z1=gasdev(idum)
      z2=gasdev(idum)
      y1=sd*z1
      y2=sd*z2
      do 2 i=1,n
      z1=gasdev(idum)
      z2=gasdev(idum)
      y1=sd*z1+rho*y2
      y2=sd*z2+rho*y1
      yy1(i)=y1
      yy2(i)=y2
    2 continue
    3 format(2f15.6)
      write(4,3) (yy1(i),yy2(i), i=1,n)
      write(*,*) 'Finished !'
      write(*,*) 'Output is in file MCMC.OUT'
      end
      FUNCTION GASDEV(IDUM)
      DATA ISET/0/
      IF (ISET.EQ.0) THEN
    1   V1=2.*RAN1(IDUM)-1.
        V2=2.*RAN1(IDUM)-1.
        R=V1**2+V2**2
        IF(R.GE.1.)GO TO 1
        FAC=SQRT(-2.*LOG(R)/R)
        GSET=V1*FAC
        GASDEV=V2*FAC
        ISET=1
      ELSE
        GASDEV=GSET
        ISET=0
      ENDIF
      RETURN
      END
      function ran1(IDUM)
      DIMENSION R(97)
      PARAMETER (M1=259200,IA1=7141,IC1=54773,RM1=3.8580247E-6)
      PARAMETER (M2=134456,IA2=8121,IC2=28411,RM2=7.4373773E-6)
      PARAMETER (M3=243000,IA3=4561,IC3=51349)
      DATA IFF /0/
      IF (IDum.LT.0.OR.IFF.EQ.0) THEN
        IFF=1
        IX1=MOD(IC1-IDum,M1)
        IX1=MOD(IA1*IX1+IC1,M1)
        IX2=MOD(IX1,M2)
        IX1=MOD(IA1*IX1+IC1,M1)
        IX3=MOD(IX1,M3)
        DO 11 J=1,97
          IX1=MOD(IA1*IX1+IC1,M1)
          IX2=MOD(IA2*IX2+IC2,M2)
          R(J)=(FLOAT(IX1)+FLOAT(IX2)*RM2)*RM1
   11   CONTINUE
        IDum=1
      ENDIF
      IX1=MOD(IA1*IX1+IC1,M1)
      IX2=MOD(IA2*IX2+IC2,M2)
      IX3=MOD(IA3*IX3+IC3,M3)
      J=1+(97*IX3)/M3
      IF(J.GT.97.OR.J.LT.1)PAUSE
      RAN1=R(J)
      R(J)=(FLOAT(IX1)+FLOAT(IX2)*RM2)*RM1
      END
```

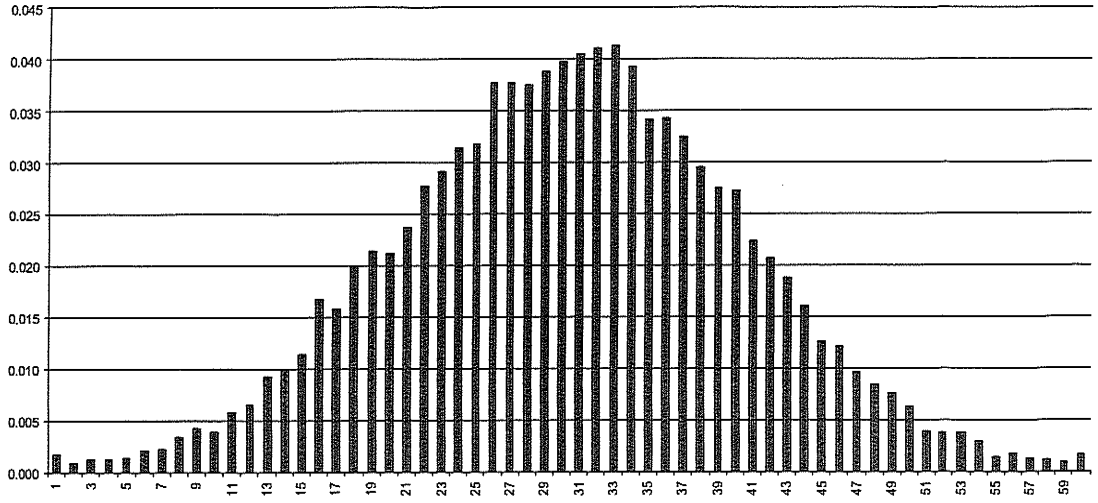
Values for Y1 (Rho = 0.25)
(N = 20,000; Burn-in = 1,000)



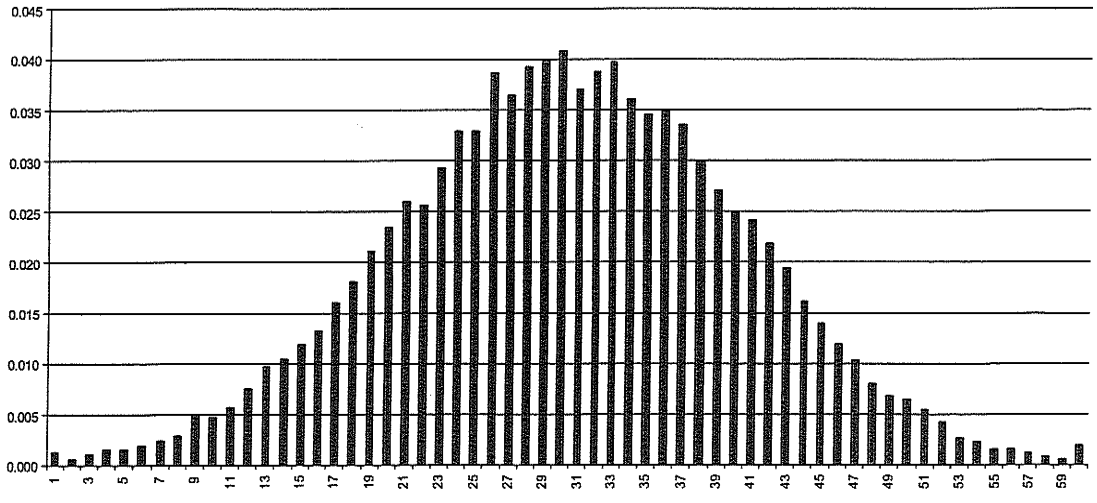
Values for Y2 (Rho = 0.25)
(N = 20,000; Burn-in = 1,000)



Marginal Density for Y1 (Rho = 0.25)
(N = 20,000; Burn-in = 1,000)



Marginal Density for Y2 (Rho = 0.25)
(N = 20,000; Burn-in = 1,000)



Analysis of $p(y_1)$ & $p(y_2)$

Date: 03/21/06 Time: 13:55
Sample: 1001 20000

	Y1	Y2
Mean	0.008684	-0.000847
Median	0.015196	0.001186
Maximum	4.169973	4.054235
Minimum	-4.234128	-4.429364
Std. Dev.	0.988655	0.994341
Skewness	-0.019612	-0.025258
Kurtosis	3.023010	3.038568
Jarque-Bera	1.637095	3.197806
Probability	0.441072	0.202118
Sum	164.9987	-16.08524
Sum Sq. Dev.	18570.36	18784.58
Observations	19000	19000

Empirical Distribution Test for Y1
Hypothesis: Normal
Date: 03/21/06 Time: 14:02
Sample: 1001 20000
Included observations: 19000

Method	Value	Adj. Value	Probability
Lilliefors (D)	0.006086	NA	> 0.1
Cramer-von Mises (W2)	0.097658	0.097661	0.1212
Watson (U2)	0.088190	0.088193	0.1355
Anderson-Darling (A2)	0.558699	0.558721	0.1490

Method: Maximum Likelihood - d.f. corrected (Exact Solution)

Parameter	Value	Std. Error	z-Statistic	Prob.
MU	0.008684	0.007172	1.210762	0.2260
SIGMA	0.988655	0.005072	194.9308	0.0000

Log likelihood	-26742.54	Mean dependent var.	0.008684
No. of Coefficients	2	S.D. dependent var.	0.988655

Empirical Distribution Test for Y2
Hypothesis: Normal
Date: 03/21/06 Time: 14:02
Sample: 1001 20000
Included observations: 19000

Method	Value	Adj. Value	Probability
Lilliefors (D)	0.004705	NA	> 0.1
Cramer-von Mises (W2)	0.053611	0.053612	0.4594
Watson (U2)	0.047302	0.047303	0.5074
Anderson-Darling (A2)	0.349967	0.349981	0.4729

Method: Maximum Likelihood - d.f. corrected (Exact Solution)

Parameter	Value	Std. Error	z-Statistic	Prob.
MU	-0.000847	0.007214	-0.117359	0.9066
SIGMA	0.994341	0.005101	194.9308	0.0000

Log likelihood	-26851.50	Mean dependent var.	-0.000847
No. of Coefficients	2	S.D. dependent var.	0.994341