ECON 546
Supplementary Problems, II
Solution

Q.1

(a) $L = \left(\frac{1}{2\theta}\right)^n \exp\left[-1/\theta \sum_i |y_i - \mu|\right]$

$\log L = -n\log 2 - n\log\theta - 1/\theta \sum_i |y_i - \mu|$

$\frac{\partial \log L}{\partial\theta} = -n/\theta + 1/\theta^2 \sum_i |y_i - \mu| = 0$

$\Rightarrow \tilde{\theta} = 1/n \sum_i |y_i - \mu|$.

$(\partial^2 \log L / \partial\theta^2) = n/\theta^2 - 2/\theta^3 \sum_i |y_i - \mu|$

& when $\theta = \tilde{\theta}$, $\partial^2 \log L / \partial\theta = -\frac{n^3}{\sum|y_i - \mu|} < 0$.

Max.

(b) $\phi_y(t) = e^{i\mu t}[1 + \theta^2 t^2]^{-1}$

$\phi'_y(t) = i\mu e^{i\mu t}[1 + \theta^2 t^2]^{-1}$

$+ e^{i\mu t}(-1)(2t\theta^2)(1 + \theta^2 t^2)^{-2}$

& when $t = 0$, $\phi'_y(0) = i\mu$, & so

$E(y) = \frac{1}{i}\phi'_y(0) = \mu$ #.

$$\phi_y''(t) = i^2\mu^2 e^{i\mu t}(1+\theta^2 t^2)^{-1} - i\mu e^{i\mu t} 2t\theta(1+\theta^2t^2)^{-}$$

$$- \Big[i\mu e^{i\mu t}(2t\theta^2)(1+\theta^2t^2)^{-1}$$

$$+ e^{i\mu t}\Big\{ 2t\,\theta(-2)(2\theta^2 t)(1+\theta^2t^2)^{-3}$$

$$+ (1+\theta^2t^2)^{-2}\cdot 2\theta^2 \Big\}\Big]$$

& when $t = 0$:

$$\phi_y''(0) = i^2\mu^2 - 2\theta^2 = i^2(\mu^2 + 2\theta^2)).$$

& $\mathrm{Var}(y) = E(y^2) - [E(y)]^2 =$

$$= \mu^2 + 2\theta^2 - (\mu)^2 = 2\theta^2 \quad .\#.$$

Continuing in the same way, we can show that $E(y-\mu_1')^4 = 24\theta^4$,

$$\text{So kurtosis} = \frac{24\theta^4}{(2\theta^2)^2} = 6$$

& excess kurtosis $= (6-3) = 3$ #.

The MLE for the variance is $2\hat{\theta}^2$, by invariance, or

$$\frac{2}{n^2}\Big[\sum_i |y_i - \mu|\Big]^2.$$

Q.2. By independence –

$$L(\alpha \mid \underset{\sim}{y}, b) = p(\underset{\sim}{y} \mid \alpha, b) = \prod_i p(y_i \mid \alpha, b)$$

$$= \frac{(\alpha b^{\alpha})^n}{\prod_i y_i^{\alpha+1}} = \alpha^n b^{\alpha n} \left[\prod_i y_i^{\alpha+1} \right]^{-1}$$

$$\log L = n \log \alpha + \alpha n \log b - \sum_i (\alpha+1) \log y_i$$

$$\partial \log L / \partial \alpha = n/\alpha + n \log b - \sum_i \log y_i = 0$$

So, $$n/\tilde{\alpha} = \sum_i \log y_i - n \log b$$

$$\tilde{\alpha} = \left[\frac{n}{\sum_i \log y_i - n \log(b)} \right] \quad ; \quad b \text{ is known.}$$

$$(\partial^2 \log L / \partial \alpha^2) = -n/\alpha^2 < 0 \quad ; \quad \text{so } \underline{\text{max}}.$$

Q.3. Let's check the answer analytically first –

$$f(\theta) = \theta^5 - 5\theta^3$$

$$f'(\theta) = 5\theta^4 - 15\theta^2 \quad ; \quad f''(\theta) = 20\theta^3 - 30\theta$$

Setting $f'(\theta) = 0 \Rightarrow \theta^2 = 3$ or $\theta = \pm\sqrt{3}$ (or $\theta = 0, 0$).

When $\theta = +\sqrt{3}$, $f''(\theta) = 51.96 > 0 \Rightarrow$ min.

When $\theta = -\sqrt{3}$, $f''(\theta) = -51.96 < 0 \Rightarrow$ max.

When $\theta = 0$, $f''(\theta) = 0 \Rightarrow$ pt of inflexion.

[NB. $\sqrt{3} = 1.73205$].

(a) $\theta_{n+1} = \theta_n - f'(\theta_n)/f''(\theta_n)$

$\theta_0 = -2$; $\theta_1 = -2 - [20/(-100)] = -1.8$
$\theta_1 = -1.8$; $\theta_2 = -1.8 - [3.888/(-116.58)] = -1.7666$

(we are converging to the maximum of f.)

(b) $\theta_0 = 2$; $\theta_1 = 2 - [20/100] = 1.8$
$\theta_1 = 1.8$; $\theta_2 = 1.8 - [3.888/62.64] = 1.7379$

(we are converging to the minimum of f.)

If $\theta_0 = 0$; $\theta_1 = 0 - [0/0]$!
We can't continue – the algorithm fails.

Q.4
(a) $L = p(\underset{\sim}{y} \mid \theta) = \prod_i p(y_i \mid \theta)$

$= \theta^n \prod_i y_i^{\theta - 1}$

$\log L = n \log \theta + (\theta - 1) \sum_i \log y_i$

$(\partial \log L / \partial \theta) = n/\theta + \sum_i \log y_i = 0$

$\Rightarrow \tilde{\theta} = -1/n \sum_i \log y_i$. (+ve because $0 < y_i < 1$)

$(\partial^2 \log L / \partial \theta^2) = -n/\theta^2 < 0 \Rightarrow$ maximum.

(b) By invariance, the MLE of $e^{-2/\theta}$ is $e^{-2/\tilde{\theta}}$, or

$$\exp\left[\ \tfrac{2}{n} \sum_i \log y_i\right] = \left(\exp\left[\sum_i \log y_i\right]^{\frac{2}{n}}\right)$$

$$= \left(\prod_i \exp\left\{\log y_i\right\}\right)^{2/n}$$

$$= \left(\prod_i y_i\right)^{2/n} \quad = \left[\left(\prod_i y_i\right)^{1/n}\right]^2$$

which is the square of the GM of the data.

(c) $$I = -E\left[\partial^2 \log L / \partial \theta^2\right]$$
$$= -E(-n/\theta^2) = n/\theta^2.$$

$$IA = \underset{n\to\infty}{\text{Limit}}\left[\tfrac{1}{n} I\right] = \underset{n\to\infty}{\text{Limit}}\left[\tfrac{1}{n}\cdot n/\theta^2\right] = \frac{1}{\theta^2}.$$

(d) $$\sqrt{n}\,(\tilde{\theta} - \theta) \xrightarrow{d} N\left[0, \theta^2\right]. \quad ; \ IA^{-1} = \theta^2$$

(e) The asy. s.d. for $\tilde{\theta}$ itself is $\left(\frac{\theta}{\sqrt{n}}\right)$. A consistent estimate of this (which is the a.s.e) is $\left(\tilde{\theta}/\sqrt{n}\right)$ $\cancel{\frac{1}{\partial\theta}}$, ~~or (1/n) $\sum_i \log y_i$~~), which will be true, as each $y_i \in [0, 1]$.

Q.5.

(a) $$L(\theta | \underset{\sim}{y}) = p(\underset{\sim}{y} | \theta) = \prod_i p(y_i | \theta)$$

$$= \left(\tfrac{\theta}{2\pi}\right)^{n/2} e^{n\theta} \prod_i y_i^{-3/2} \exp\left[-\tfrac{\theta}{2} \sum_i \left(y_i + \tfrac{1}{y_i}\right)\right].$$

* $$\log L = \tfrac{n}{2}\log\theta - \tfrac{n}{2}\log 2\pi + n\theta - \tfrac{3}{2}\sum_i \log y_i$$
$$- \tfrac{\theta}{2}\sum_i\left[y_i + \tfrac{1}{y_i}\right]$$

So,

$$(\partial \log L / \partial \theta) = \frac{n}{2\theta} + n - \frac{1}{2}\sum_i \left[y_i + \frac{1}{y_i} \right] = 0$$

$$\Rightarrow \frac{1}{2\tilde{\theta}} = \frac{1}{2n}\sum_i \left(y_i + 1/y_i \right) - 1$$

$$\Rightarrow 1/\tilde{\theta} = \frac{1}{n}\sum_i \left(y_i + 1/y_i \right) - 2$$

$$\Rightarrow \tilde{\theta} = \left[1/n \sum_i \left(y_i + 1/y_i \right) - 2 \right]^{-1}.$$

$$(\partial^2 \log L / \partial \theta^2) = -n/2\theta^2 < 0 \quad (\Rightarrow \text{maximum}).$$

(b) It is consistent, asymptotically Normal, & asymp. efficient.

Q. 6.

(a) $L(\alpha, \beta \mid \underset{\sim}{y}) = p(\underset{\sim}{y} \mid \alpha, \beta) = \prod_i p(y_i \mid \alpha, \beta)$

$$= \beta^{-n} \exp\left[\frac{1}{\beta} \sum_i (y_i - \alpha) \right] \exp\left\{ -\sum_i e^{(y_i - \alpha)/\beta} \right\}$$

$$\log L = -n \log \beta + \frac{1}{\beta}\sum_i (y_i - \alpha) - \sum_i e^{(y_i - \alpha)/\beta}$$

$$(\partial \log L / \partial \alpha) = -n/\beta - \sum_i \left\{ e^{(y_i - \alpha)/\beta} \cdot (-1/\beta) \right\}$$

$$= -n/\beta + \frac{1}{\beta}\sum_i e^{(y_i - \alpha)/\beta} = 0 \qquad (1)$$

$$(\partial \log L / \partial \beta) = -n/\beta - \frac{1}{\beta^2}\sum_i (y_i - \alpha) - \sum_i \left\{ e^{(y_i - \alpha)/\beta} \cdot (-1)(y_i - \alpha)/\beta^2 \right\}$$

$$= -n/\beta - \frac{1}{\beta^2}\sum_i (y_i - \alpha) + \frac{1}{\beta^2}\sum_i \left[(y_i - \alpha) e^{(y_i - \alpha)/\beta} \right]$$

$$= 0. \qquad (2)$$

From ①: $n = \sum_i e^{(y_i - \alpha)/\beta}$ ③

From ②: $n\beta + \sum_i (y_i - \alpha) = \sum_i \left[(y_i - \alpha) e^{(y_i - \alpha)/\beta} \right]$

or, $n\beta + n\bar{y} - n\alpha = n[n\bar{y} - n\alpha]$

$\Rightarrow \tilde{\alpha}(n^2 - 1) = \bar{y}(n^2 - 1) - \tilde{\beta}$ ④

Equations ③ & ④ need to be solved numerically for $\tilde{\alpha}$ & $\tilde{\beta}$.

(b) $p(y) = \frac{1}{\beta} \exp\left[\left(\frac{y-\alpha}{\beta}\right) - \exp\left(\frac{y-\alpha}{\beta}\right) \right]$

$(\partial p/\partial y) = \frac{1}{\beta} \exp\left[\left(\frac{y-\alpha}{\beta}\right) - \exp\left(\frac{y-\alpha}{\beta}\right) \right]$

$\cdot \left[1/\beta - e^{(y-\alpha)/\beta} \cdot 1/\beta \right] = 0$; for mode

$\Rightarrow \exp\left[\left(\frac{y-\alpha}{\beta}\right) - \exp\left(\frac{y-\alpha}{\beta}\right) \right] \left(1 - e^{(y-\alpha)/\beta}\right) = 0$

Now $e^z > 0$ for any z, so $(\partial p/\partial y) = 0$ iff

$(1 - e^{(y-\alpha)/\beta}) = 0$; i.e. iff $e^{(y-\alpha)/\beta} = 0$

i.e. iff $(y-\alpha)/\beta = 0$, or $y = \alpha$ #.

So, the mode is at $y = \alpha$. And in this case the value of $p(y)$ is

$p(\alpha) = \frac{1}{\beta} \exp[0 - \exp(0)] = \frac{1}{\beta} e^{-1}$

$= (0.3679/\beta)$ #

The MLE for the mode is then $(0.3679/\tilde{\beta})$, by invariance.

(c) $\text{var.}(y) = (\pi^2\beta^2/6)$, so $\text{s.d.}(y) = (\pi\beta/2.449)$

& the MLE for this is $(\pi\tilde{\beta}/2.449)$ & this is a consistent & asy. efficient estimator of the s.d.

Q.7

(a) $L(\theta|\underline{y}) = p(\underline{y}|\theta) = \prod_i p(y_i|\theta)$

$$= (\theta/2\pi)^{n/2} \prod_i y_i^{-3/2} \exp - \theta \Sigma(1/2y_i)$$

$$\log L = n/2 \log \theta - n/2 \log 2\pi - 3/2 \sum_i \log y_i - \theta \sum_i \left(\frac{1}{2y_i}\right)$$

$$(\partial \log L/\partial\theta) = \frac{n}{2\theta} - \sum_i \left(\frac{1}{2y_i}\right) = 0$$

$$\Rightarrow n/\tilde{\theta} = \sum_i (1/y_i)$$

$$\Rightarrow \tilde{\theta} = [n / \textstyle\sum_i (1/y_i)] = \left[\frac{1}{n} \sum_i \left(\frac{1}{y_i}\right)\right]^{-1}$$

which is the harmonic mean of the y_i's.

$$(\partial^2 \log L/\partial\theta^2) = -n/2\theta^2 < 0 \quad \text{(maximum)}.$$

(b) $p(y|\theta) = \theta^{\frac{1}{2}} (2\pi)^{-1/2} e^{-\theta/2y} y^{-3/2}$

$$(\partial p/\partial y) = \theta^{1/2}(2\pi)^{-1/2}\left[e^{-\theta/2y}(-3/2)y^{-5/2} + y^{-3/2}e^{-\theta/2y}\frac{\theta}{2y^2}\right]$$

$= 0$, for mode

So, $\left[-\frac{3}{2} e^{-\theta/2y} y^{-5/2} + \frac{\theta}{2} e^{-\theta/2y} y^{-7/2} \right] = 0$

or, $3y^{-5/2} = \theta y^{-7/2}$

or, $\theta = 3y$

or, $y = \theta/3$.

The value of the density at the mode is –

$$m = (\theta/2\pi)^{1/2} e^{-\theta/2(\theta/3)} (\theta/3)^{-3/2}$$

$$= \theta^{1/2} \cdot (2\pi)^{-1/2} e^{-3/2} \theta^{-3/2} (3)^{3/2}$$

$$= \left(\frac{1}{\theta}\right) . (0.46254)$$

The MLE of m is $\tilde{m} = (0.46254/\tilde{\theta})$

(c) $I = -E\left[\partial^2 \log L/\partial\theta^2\right] = -E\left[-n/2\theta^2\right] = n/2\theta^2$

$IA = \underset{n\to\infty}{\text{Limit}} \left[\frac{1}{n} I \right] = 1/2\theta^2$

(d) $\sqrt{n}(\tilde{\theta} - \theta) \xrightarrow{d} N\left[0, 2\theta^2\right]$.