ECON 546

Answers to Supplementary Problems, III

Q.1 $\mathcal{L} = \theta^{\Sigma x_i} (1-\theta)^{\Sigma(1-x_i)}$

$$\log L = \sum_i x_i \log\theta + \sum_i (1-x_i)\log(1-\theta)$$

$$\partial \log L / \partial\theta = n\bar{x}/\theta + \frac{n(1-\bar{x})}{(1-\theta)}(-1) = 0$$

$$\Rightarrow n\bar{x}(1-\hat{\theta}) = n(1-\bar{x})\hat{\theta}$$

$$\Rightarrow n\bar{x} = n\hat{\theta}$$

$$\Rightarrow \hat{\theta} = \bar{x}.$$

$$(\partial^2 \log L/\partial\theta^2) = -n\bar{x}/\theta^2 - (-1)(-1)\frac{n(1-\bar{x})}{(1-\theta)^2}$$

$$= -n\bar{x}/\theta^2 - \frac{n(1-\bar{x})}{(1-\theta)^2}$$

& when $\theta = \bar{x}$: $\left(\partial^2 \log L/\partial\theta^2\right) = -\frac{n\bar{x}}{\bar{x}^2} - \frac{n(1-\bar{x})}{(1-\bar{x})^2}$

$$= -n/\bar{x} - \frac{n}{(1-\bar{x})}$$

More generally, note that for <u>any</u> θ:

$$(\partial^2 \log L/\partial\theta^2) = \frac{-n(1-\theta)^2 - n(1-\bar{x})\theta^2}{(1-\theta)^2\theta^2}$$

& the x_i's must lie in $[0, 0.5]$, so $(1-\bar{x}) > 0$.
This implies $(\partial^2 \log L/\partial\theta^2) < 0$, <u>everywhere</u>.

(a) $\widetilde{\log L_u} = \sum_i x_i \log \hat{\theta} + \sum_i (1 - x_i) \log(1 - \hat{\theta})$

$$= n\bar{x} \log \bar{x} + n(1-\bar{x}) \log(1-\bar{x})$$

$$\widetilde{\log L_R} = n\bar{x} \log_e(0.25) + n(1-\bar{x}) \log_e(0.75)$$

$$= n\bar{x}(-1.3863) + (-0.288) n (1-\bar{x})$$

$$= (-1.0986\, n\bar{x} - 0.288\, n)$$

$$LRT = -2 \log(\tilde{L}_R / \tilde{L}_u) = 2[\widetilde{\log L_u} - \widetilde{\log L_R}]$$

*

$$I(\theta) = -E[\partial^2 \log L / \partial \theta^2] = E\left[\frac{n\bar{x}}{\theta^2} + \frac{n(1-\bar{x})}{(1-\theta)^2}\right]$$

& $E(x_i) = \theta$. ($\Rightarrow E(\bar{x}) = \theta$)

So, $I(\theta) = (n\theta/\theta^2) + n(1-\theta)/(1-\theta)^2$

$$= (n/\theta) + n/(1-\theta)$$

$$IA(\theta) = \lim_{n \to \infty} [1/n \; I(\theta)] = \left(1/\theta - \frac{1}{1-\theta}\right)$$

I^* is such that limit $[1/n \; I^*] = IA$, so

$$I^* = (n/\tilde{\theta} + n/1-\tilde{\theta})$$

$$W = (\tilde{\theta} - 1/4)^2 / \left(n/\tilde{\theta} + \frac{n}{1-\tilde{\theta}}\right)^{-1} = \left[\frac{(\tilde{\theta} - 1/4)^2}{\left(\frac{n(1-\tilde{\theta}) + n\tilde{\theta}}{\tilde{\theta}(1-\tilde{\theta})}\right)^{-1}}\right]$$

$$= \frac{(\tilde{\theta} - 1/4)^2}{\tilde{\theta}(1-\tilde{\theta})}(n) \;\; * \;\; = \cancel{\frac{n(1-\tilde{\theta})}{\tilde{\theta}}} \;\; \cancel{\frac{n(1-\bar{x})}{\bar{x}}}$$

$$LM = \left[\frac{n\bar{x}}{\theta} - \frac{n(1-\bar{x})}{1-\theta}\right]^2_{(\theta=0.25)} I^{*-1}_{(\theta=0.25)}$$

$$= \{4n\bar{x} - 4n(1-\bar{x})/3\}^2 (4n + 4n/3)^{-1}$$

$$= 16n^2 (\bar{x} - (1-\bar{x})/3)^2 / \{4n(4/3)\}$$

$$= 3n(\bar{x} - (1-\bar{x})/3)^2.$$

(b) Reject H_0 if test statistic > 2.71. Assuming null distribution is $\chi^2_{(1)}$, which will be valid asymptotically.

Q. 2 (a) $L = \theta^n \prod_i y_i^{\theta-1}$; $\log L = n\log\theta + (\theta-1)\sum_i \log y_i$

$$(\partial \log L/\partial\theta) = n/\theta + \sum_i \log y_i = 0$$
$$\Rightarrow \tilde{\theta} = -n / \sum_i \log y_i$$

[Note that $0 < y_i < 1 \Rightarrow \log y_i < 0 \Rightarrow \tilde{\theta} > 0$]

$(\partial^2 \log L/\partial\theta^2) = -n/\theta^2$; < 0 everywhere

$$\log \tilde{L}_u = n\log\tilde{\theta} + (\tilde{\theta}-1)\sum_i \log y_i$$

$$\log \tilde{L}_R = 0 + 0 = 0$$

So, $LRT = 2[\log\tilde{L}_u - \log\tilde{L}_R]$
$$= 2n\log\tilde{\theta} + 2(\tilde{\theta}-1)\sum_i \log y_i \quad *$$

$$I(\theta) = -E[\partial^2 \log L/\partial\theta^2] = n/\theta^2$$

$$IA(\theta) = \lim_{n\to\infty}[1/n\, I(\theta)] = 1/\theta^2$$

$$I^* = n/\tilde{\theta}^2$$

$$W = (\tilde{\theta}-1)^2(n/\tilde{\theta}^2). \qquad *$$

$$LM = \frac{(n + \sum_i \log y_i)^2}{n} \cancel{(\tilde{\theta}^2/n)}. \qquad *$$

(b) As for Q. 1. (a), but critical value = 3.84.

Q. 3.

$$L = (\theta/2\pi)^{n/2} e^{n\theta}(\prod_i y_i^{-3/2}) \exp[-\theta/2 \sum_i (y_i + y_i^{-1})]$$

$$\log L = n/2 \log\theta - n/2 \log 2\pi + n\theta + \sum_i \log(y_i^{-3/2})$$

$$- (\theta/2) \sum_i (y_i + y_i^{-1})$$

$$= n/2 \log\theta - n/2 \log 2\pi + n\theta - 3/2 \sum_i \log y_i$$

$$- (\theta/2) \sum_i (y_i + y_i^{-1})$$

$$\frac{\partial \log L}{\partial\theta} = n/2\theta + n - \tfrac{1}{2}\sum_i (y_i + y_i^{-1}) = 0$$

$$\Rightarrow n/2\tilde{\theta} = [\tfrac{1}{2}\sum_i (y_i + y_i^{-1}) - n]$$

$$\Rightarrow \tilde{\theta} = \left(\frac{n}{2}\right)[\tfrac{1}{2}\sum_i (y_i + y_i^{-1}) - n)]^{-1}$$

$$(\partial^2 \log L / \partial \theta^2) = -\frac{n}{2\theta^2} < 0$$

$$I = n/2\theta^2$$

$$IA = 1/2\theta^2 \quad ; \quad I^* = n/2\tilde{\theta}^2$$

$$\log \tilde{L}_R = -n/2 \log 2\pi + n - 3/2 \sum_i \log y_i - 1/2 \sum_i (y_i + y_i^{-1})$$

$$\log \tilde{L}_u = n/2 \log \tilde{\theta} - n/2 \log 2\pi + n\tilde{\theta} - 3/2 \sum_i \log y_i$$
$$- (\tilde{\theta}/2) \sum_i (y_i + y_i^{-1})$$

So, $LRT = 2\left[n/2 \log \tilde{\theta} + n(\tilde{\theta} - 1) - \sum_i (y_i + y_i^{-1})[\tilde{\theta} - 1]/2 \right]$

$$W = n(\tilde{\theta} - 1)^2 / 2\tilde{\theta}^2$$

$$LM = \left[n/2 + n - \frac{1}{2} \sum (y_i + y_i^{-1}) \right]^2 (2/n)$$
$$= \left[1 + 2 - \frac{1}{n} \sum_i (y_i + y_i^{-1}) \right]^2$$
$$= \left[3 - \frac{1}{n} \sum_i (y_i + y_i^{-1}) \right]^2.$$

Q-4. $\quad L = \left(\frac{1}{\prod_i y_i!} \right) e^{-n\lambda} \lambda^{\sum y_i}$

$$\log L = -\sum_i \log(y_i!) - n\lambda + n\bar{y} \log \lambda$$

$$(\partial \log L / \partial \lambda) = -n + \bar{y} n / \lambda = 0$$

$$\Rightarrow \tilde{\lambda} = \bar{y}.$$

$(\partial^2 \log L / \partial \lambda^2) = -n\bar{y}/\lambda^2 < 0.$

$I = \frac{n}{\lambda^2} E(\bar{y}) = (n\lambda / \lambda^2) = n/\lambda.$

$(E(y_i) = \lambda))$

So, $I_A = 1/\lambda$ & $I^* = n/\tilde{\lambda} = (n/\bar{y}).$

a) $LRT = 2[\text{const} - n\bar{y} + n\bar{y}\log\bar{y} - \text{const} + 5n - n\bar{y}\log 5]$

$= 2[n\bar{y}(\log\bar{y} - 1 - \log_e 5) + 5n]$

$= 2[n\bar{y}(\log\bar{y} - 2.609) + 5n]$

$W = n(\bar{y} - 5)^2/\bar{y}$

$LM = 5[(n\bar{y}/5) - n]^2/n$

(b) $LRT = 2[2300(\log(4.6) - 2.609) + 2500]$ $(\bar{y} = 4.6)$

$= 18.476.$

$W = 17.391$

$LM = 16$

(Easily reject H_0 at any reasonable sig. level.)

Q.5 $p(x, y) = \theta \exp[-(\beta+\theta)y][(\beta y)^x]/x!$

(a) $L = \theta^n \exp[-(\theta+\beta)\sum_i y_i] \prod_i (\beta y_i^{x_i}) / \prod_i y_i!$

$\propto \theta^n \exp[-(\beta+\theta)n\bar{y}]\beta^{\sum x_i}$

$\log L = c + n\log\theta - (\beta+\theta)n\bar{y} + n\bar{x}\log\beta$

$(\partial \log L/\partial\theta) = n/\theta - n\bar{y} = 0 \Rightarrow \tilde{\theta} = \bar{y}$.

$(\partial \log L/\partial\beta)$ $(n\bar{x}/\beta) - n\bar{y} \Rightarrow \tilde{\beta} = (\bar{x}/\bar{y})$

$(\partial^2 \log L/\partial\theta^2) = -n/\theta^2$; $(\partial^2 \log L/\partial\beta^2) = -n\bar{x}/\beta^2$

$(\partial^2 \log L/\partial\theta\partial\beta) = 0$

$$H = \begin{bmatrix} -n/\theta^2 & 0 \\ 0 & -n\bar{x}/\beta^2 \end{bmatrix}$$

(negative-definite as co-factors have signs, $-$, $+$)

So $(\tilde{\theta}, \tilde{\beta})$ locates a max. of $\log L$.

(b) By invariance MLE of $\theta/(\theta+\beta)$ is

$[\tilde{\theta}/(\tilde{\theta}+\tilde{\beta})] = [\bar{y}/(\bar{y} + \bar{x}/\bar{y})] = \bar{y}^2/(\bar{x}+\bar{y}^2)$.

(c) $p(x) = \int_0^\infty p(x,y)\,dy = \int_0^\infty \theta e^{-(\beta+\theta)y}\beta^x y^x/x!\,dy$

$= \left(\frac{\theta\beta^x}{x!}\right)\int_0^\infty e^{-(\beta+\theta)y} y^x\,dy$

Let $t = (\beta+\theta)y$; $dy = (\beta+\theta)^{-1} dt$

$$p(x) = \left(\frac{\theta\beta^x}{x!}\right)\int_0^\infty e^{-t}\left(\frac{t}{\beta+\theta}\right)^x\left(\frac{1}{\beta+\theta}\right)dt$$

$$= \frac{\theta\beta^x}{x!\,(\beta+\theta)^{x+1}}\int_0^\infty e^{-t}\,t^x\,dt$$

$$= \frac{\theta\beta^x}{x!\,(\beta+\theta)^{x+1}}\int_0^\infty e^{-t}\,t^{(x+1)-1}\,dt$$

$$= \frac{\theta\beta^x\,\Gamma(x+1)}{x!\,(\beta+\theta)^{x+1}} = \frac{\theta\beta^x\,x!}{x!\,(\beta+\theta)^{x+1}}$$

$$= \theta\beta^x/(\beta+\theta)^{x+1} \; ; \; x = 0, 1, 2, \ldots$$

$$= \delta(1-\gamma)^x$$

where $\delta = \left(\frac{1}{\beta+\theta}\right)$; $\gamma = \left(1 - \frac{\theta\beta}{\beta+\theta}\right)$

By invariance, $\hat{\gamma} = \left(1 - \frac{\hat{\theta}\hat{\beta}}{\hat{\beta}+\hat{\theta}}\right)$

or, $\hat{\gamma} = \left(\frac{\hat{\beta}+\hat{\theta} - \hat{\theta}\hat{\beta}}{\hat{\beta}+\hat{\theta}}\right) = \left(\frac{\bar{X}/\bar{y} + \bar{y} - \bar{X}}{\bar{X}/\bar{y} + \bar{y}}\right)$

$$= (\bar{x} + \bar{y}^2 - \bar{x}\bar{y})/(\bar{x}+\bar{y}^2)$$

(d) As H is diagonal, we can just focus on the (2,2) element : $-n\bar{x}/\beta^2$.

$$LRT = 2\left[c + n\log\bar{y} - (\bar{y} + \bar{x}/\bar{y})\,n\bar{y} + n\bar{x}\log(\bar{x}/\bar{y}) - c - n\log(1) + \bar{y}\,n\bar{y} - n\bar{x}\log(1) \right]$$

$$= 2\left[n\log\bar{y} - n\bar{y}^2 + n\bar{y}\bar{x} + n\bar{x}\log\bar{x} - n\bar{x}\log\bar{y} + \bar{y}^2 n \right]$$

$$= 2\left[n\bar{x}\bar{y} + n\log\bar{y}\,(1 - \bar{x}) + n\bar{x}\log\bar{x} \right]$$

Reject H_0 if $LRT > \chi^2_{(1)}$ critical value.