ECON 546

Supplementary Exercises V – Solution

Q-1.

(a)

$$L \propto e^{-n\lambda}\lambda^{\Sigma x_i} = e^{-n\lambda}\lambda^{n\bar{x}}$$

$$p(\lambda) \propto \lambda^{\gamma-1} e^{-\lambda}$$

So, $$p(\lambda|x) \propto \lambda^{n\bar{x}+\gamma-1} e^{-(n+1)\lambda}$$

& this is Gamma, with parameters $(n\bar{x}+\gamma)$ and $(n+1)$. The prior had parameters γ and '1'.

(b) The mean of a Gamma density is γ (or in the case of the posterior, $(n\bar{x}+\gamma)$). And this is the Bayes estimator under quadratic loss. That is $\hat{\lambda} = (n\bar{x}+\gamma)$.

Mode:

$\partial p(\lambda|x)/\partial\lambda$ will be zero when $(\partial \log p/\partial\lambda) = 0$.

& $$\log[p(\lambda|x)] = \text{const.} + (n\bar{x}+\gamma-1)\log\lambda - (n+1)\lambda.$$

$$(\partial \log p/\partial\lambda) = (n\bar{x}+\gamma-1)/\lambda - (n+1) = 0$$

$$\Rightarrow \lambda^* = \left(\frac{n\bar{x}+\gamma-1}{n+1}\right)$$

This is the Bayes' estimator under 0-1 loss.

Now the data are Poisson & mean $= \lambda$.

So, $$E(\hat{\lambda}) = nE(\bar{x}) + \gamma = n\lambda + \gamma \neq \lambda.$$

(Biased)

& $$E(\lambda^*) = \left(\frac{nE(\bar{x})+\gamma-1}{n+1}\right) = \left(\frac{n\lambda+\gamma-1}{n+1}\right) \neq \lambda.$$

(Biased)

Q. 2.

(a) $L(y|\theta) = \theta^n e^{-\theta \Sigma y_i}$

$$p(\theta) = \frac{\theta^{\alpha-1} e^{-\theta/\beta}}{\beta^\alpha \Gamma(\alpha)}$$

By Bayes' Theorem,

$$p(\theta|y) \propto \frac{\theta^{n+\alpha-1} e^{-\theta[\frac{1}{\beta} + \Sigma y_i]}}{\beta^\alpha \Gamma(\alpha)}$$

$$\propto \theta^{(n+\alpha)-1} e^{-\theta/[1/\beta + \Sigma y_i]^{-1}}$$

which is a Gamma density, with parameters $(n+\alpha)$ and $[1/\beta + \Sigma y_i]^{-1}$, as required.

(b) $E(\theta|y) = (n+\alpha)/[1/\beta + \Sigma y_i]$, & this is the Bayes estimator of θ under quadratic loss.

(c) The mode is at $(n+\alpha-1)/(\frac{1}{\beta} + \Sigma y_i)$, & this is Bayes estimator of θ if $n+\alpha > 1$.

Q. 3.

(a) $\tilde{\beta} = (A + X'X)^{-1}(A\bar{\beta} + X'y)$

where $\begin{cases} p(\beta|\sigma) = N[\bar{\beta}, \sigma^2 A^{-1}] \\ p(\sigma) = I\,Gamma(\omega, c). \end{cases}$

The model is $y = X\beta + \varepsilon$; $\varepsilon \sim N[0, \sigma^2 I]$.

So: $E(\tilde{\beta}) = (A + X'X)^{-1}(A\bar{\beta} + X'E(y))$

$$= (A + X'X)^{-1}(A\bar{\beta} + X'X\beta)$$

$$\neq \beta.$$

So, the estimator is <u>biased</u>.

(b) Bayesians aren't worried about performance in "repeated samples", or "on average."

(c) The MLE for β is OLS $= b = (X'X)^{-1}X'y$, & $V(b) = \sigma^2(X'X)^{-1} \propto (X'X)^{-1}$. So, here we are going to choose $\sigma^2 A^{-1} = \sigma^2(X'X)^{-1}$.

Then, $\tilde{\beta} = \left([\sigma^2(X'X)^{-1}]^{-1} + (X'X)\right)^{-1} \cdot \left([\sigma^2(X'X)^{-1}]^{-1}\bar{\beta} + X'y\right)$

$$= \left[\frac{1}{\sigma^2}X'X + X'X\right]^{-1}\left[\frac{1}{\sigma^2}(X'X)\bar{\beta} + X'y\right]$$

$$= \left[\frac{1}{\sigma^2}X'X + X'X\right]^{-1}\left[\frac{1}{\sigma^2}(X'X)\bar{\beta} + (X'X)b\right]$$

$$= \left[\frac{1}{\sigma^2} + 1\right]^{-1}(X'X)^{-1}(X'X)\left[\frac{1}{\sigma^2}\bar{\beta} + b\right]$$

$$= \left(\frac{1}{\sigma^2}\bar{\beta} + b\right) / \left(1/\sigma^2 + 1\right)$$

& $E(\tilde{\beta}) = (1/\sigma^2\,\bar{\beta} + \beta) / (1/\sigma^2 + 1)$. #.

(d) It involves seeing the sample of X data <u>before</u> constructing the prior.

Q.4 (a) $p(x \mid \mu_0, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left[-\frac{1}{2\sigma^2}(x-\mu_0)^2\right]$

$$\propto \sigma^{-1} \exp\left[-\tfrac{1}{2}\sigma^{-2}(x-\mu_0)^2\right].$$

Let $\tau = \sigma^{-2}$:

$$p(x \mid \mu_0, \tau) \propto \tau^{1/2} \exp\left[-\tfrac{1}{2}\tau(x-\mu_0)^2\right].$$

Let $p(\tau) = \text{Gamma}(\alpha, \gamma)$

$$\propto e^{-\alpha\tau}\tau^{\gamma-1}$$

Apply Bayes' Theorem :

$$p(\tau \mid x) \propto e^{-\alpha\tau}\tau^{\gamma-1}\tau^{n/2} \exp\left[-\tfrac{\tau}{2}\sum_i (x_i-\mu_0)^2\right]$$

$$\propto \tau^{(\gamma+n/2)-1} \exp\left[-a\tau + \alpha\tau\right]$$

where $a = \frac{1}{2}\sum_i (x_i-\mu_0)^2$ (known.)

So, $p(\tau \mid x) \propto \tau^{(\gamma+n/2)-1} e^{-\delta\tau}$

where $\delta = a + \alpha$.

This is $\text{Gamma}\left[(\gamma+n/2), \alpha + \frac{1}{2}\sum_i (x_i-\mu_0)^2\right]$

so we we have conjugacy..

(b) $E(\tau \mid X) = (\gamma + n/2)$, & this is the Bayes' estimator under quadratic loss.

Following from the answer to Q. 2 (b), the mode of $p(\tau \mid X)$ is at:

$$\tau^* = \left[\frac{\gamma + n/2 - 1}{\alpha + \frac{1}{2}\sum (x_i - \mu_0)^2} \right]$$

(c) Bayes' estimators are consistent. So, in the first case, a consistent estimator of τ is $(\gamma + n/2)$ & for σ^2 it is $(\gamma + n/2)^{-2}$. In the second case, we'd have a consistent estimator of σ^2 by using

$$\left[\alpha + \tfrac{1}{2} \sum (x_i - \mu_0)^2 \right]^2 / (\gamma + n/2 - 1)^2 .$$