

Share the Gain, Share the Pain?
Almost Transferable Utility, changes in production
possibilities, and bargaining solutions

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Abstract

We consider an n-person economy in which efficiency is independent of distribution but the cardinal properties of the agents' utility functions preclude transferable utility (a property we call "Almost TU"). We show that Almost TU is a necessary and sufficient condition for all agents to either benefit jointly or suffer jointly with any change in production possibilities under well-behaved generalized utilitarian bargaining solutions (of which the Nash Bargaining and the utilitarian solutions are special cases). We apply the result to policy analysis and to incentive compatibility.

JEL: C7, D6, D7

Keywords: Axiomatic Bargaining, Solidarity, Transferable Utility, Rotten Kid Theorem, Intrahousehold Distribution.

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1 Introduction

Solidarity is a crucially important consideration whenever individuals cooperatively decide on how to allocate resources among themselves: an agreement is unlikely to be reached if some are hurt while others benefit in the event of a foreseeable shock in the available resources. From a normative standpoint, solidarity is also a chief concern when making policy recommendations, as illustrated by the following slightly modified example found in Nash (1950).¹ Suppose Jack and Bill are siblings and have to share the following items: a book, a whip, a ball, a bat, a box, a pen, a toy, a knife, and a hat. If they can't agree, they won't be able to use any of these toys. Now suppose the parents take away the whip and the knife and replace them with a bucket and a shovel. It may be a relief for them to know that they will either disappoint both children or delight them both. In other words, the parents may want to make sure that their action does not destabilize their children's bonding by making one child better off and the other worse off and thus giving the impression that parents favor one child over the other. On a larger scale, a government may take a similar stance when it comes to family policies: it may be desirable to know that a change in family policies such as changes in parental leave policies or family taxation, – policies that clearly change a family's production possibility set – do not leave some family members worse off and others better off, which could unduly stress intrafamily relationships.

In practice, however, most bargaining situations which draw upon the results of axiomatic bargaining typically do so using some generalized utilitarian bargaining solution (GUBS). This broad class of solutions consists of maximizing an additively separable function in agents' utility gains from cooperation and includes the utilitarian and Nash bargaining solutions. Despite their appealing properties (Moulin 1988), bargaining solutions in the GUBS class typically fail to satisfy solidarity (Chun and Thomson 1988). Chun and Thomson (1988) speculate that restrictions on the utility functions of agents can lead to GUBS satisfying solidarity. One such restriction is to require utility profiles that lead to transferable utility (TU), but TU is a very strong assumption in combination with GUBS; cardinal properties of individuals' utility functions matter for GUBS in allocating resources and welfare to individuals and utility profiles leading to TU impose cardinal properties that exclude commonly observed features of individual preferences such as diminishing marginal utility and, by extension, risk aversion. Hence, if one wants to ensure that a bargaining solution satisfies the solidarity property, the use of the class of GUBS seems rather limited in applications.

We remedy the issue by establishing that well-behaved GUBS (to be defined) satisfy solidarity on a broader utility class than TU, which we define and call Almost TU. Almost TU requires the same ordinal properties as TU but allows for cardinal properties like diminishing marginal utility as well. Hence, our finding broadens the valid range of applications for bargaining solutions of the

¹We thank Phil Curry for inspiring this modification of Nash's example.

GUBS class. Also, from an implementation standpoint, policy makers may be unsure of the cardinal properties of agents' utility functions when evaluating the change in welfare due to a change in policy. Hence they may be reassured to know that the utility of the agents will change in the same direction even in the event of a misestimation of these cardinal properties, as long as the ordinal properties for TU are satisfied.²

More precisely, our main result (Theorem 2) characterizes Almost TU as the domain on which all well-behaved GUBS satisfy solidarity. This result bears useful consequences in positive economic analysis. First, Theorem 2 has important implications for policy analysis. Indeed, the solidarity property is not just normatively appealing but it is useful in positive economic analysis as well, as it allows us to receive less ambiguous results, similarly to the assumption in consumer theory that goods are normal. Of course, normality in consumer theory not only helps in determining what happens to consumer demand as the consumer's income changes, but also helps determining the impact of a price change on consumer demand. The same is true for the solidarity property. Even if policy changes impact both the joint production possibility set and the disagreement point, the solidarity property remains useful in narrowing down the ways in which a policy change can affect an agent's utility.

Second, Theorem 2 has important implications for the question of incentive compatibility. Suppose agents commit to a general formula of how to distribute goods before they have produced these goods. This formula is in the form of a well-behaved GUBS. With this agreement in place agents then choose their actions individually, maximizing their individual welfare. We show that incentive compatibility is satisfied if and only if solidarity is satisfied (Theorem 3). Almost TU takes care of both issues at once: Assuming Almost TU, the class of well-behaved GUBS satisfies the solidarity property as well as incentive compatibility.

2 Related Literature

Many works emphasize the importance of solidarity in allocation problems, be it with respect to population or to the total amount of goods available (see Moulin 1988, or Sprumont 2008, for a survey). This work belongs to the latter strand of the literature and is more closely related to Chun and Thomson (1988), which explicits the parallel between fair allocation problems and bargaining situations. Chun and Thomson (1988) show that the solidarity property holds in a one-good economy for some bargaining solutions. Our main result generalizes theirs to a many-goods production economy, possibly containing public goods, when preferences exhibit Almost TU.

In the context of axiomatic bargaining, many characterizations of bargaining

²MasColell et al. (1995, p. 831) take as given that individuals have cardinal utility functions, when they state: "[...] whereas a policy maker may be able to identify individual cardinal utility functions (from revealed risk behavior, say), it may actually do so but only up to a choice of origins and units."

solutions are motivated by at least some notion of solidarity or monotonicity arguments.³ Xu and Yoshihara (2008) offer a systematic treatment of well-known bargaining rules with respect to solidarity-type axioms.

Although we motivate the interest in the solidarity property as a normative issue, our result has also implications for policy analysis and incentive compatibility. Research on family economics frequently uses bargaining rules – most often the Nash bargaining solution – to analyze intrafamily distribution. More precisely, a husband and a wife are assumed to apply a bargaining solution to determine their utilities in marriage. In this literature parameters that change the disagreement point without changing the utility possibility set⁴ have received substantial attention (Lundberg et al. 1997, Rubalcava and Thomas 2000, Chiappori et al. 2002), because the intrafamily distributional impact of such policies is unambiguous unless both spouses' disagreement utility changes in the same direction.

We tackle the complementary problem of predicting the impact on intrafamily distribution when the utility possibility frontier changes. Policies that have the potential to affect the utility possibility set are more difficult to analyze, although there are many examples such as parental leave policies, policies subsidizing child care, or family taxation (Wrede 2003, Pollak forthcoming, Gugl 2009). Pollak (forthcoming) argues that the disagreement point does not change with a change in family taxation from individual taxation to joint taxation, but concludes that "the distributional effects of joint taxation, which operate through the feasible set, are indeterminate." Assuming a well-behaved GUBS in combination with Almost TU simplifies the analysis of such policies; by the solidarity property, the impact of those policies on the joint production possibility set can only lead to either all family members being better or all being worse off.

Theorem 3 states that well-behaved GUBS satisfy incentive compatibility if and only if they satisfy solidarity in production opportunities. The analysis of incentive compatibility in case of a well-behaved GUBS bears some striking parallels with the Rotten Kid Theorem. Becker (1974) introduced the Rotten Kid theorem as a means of reconciling the treatment of a multi-person household as one agent (the so-called unitary model) with methodological individualism. Becker defines the head of the household as somebody who "cares sufficiently about all other [household] members to transfer general resources to them" (p. 1076). He notes that "if a head exists, other members also are motivated to maximize [the head's altruistic utility function], even if their welfare depends on their own consumption alone" (p. 1080). Bergstrom (1989) restates Becker's Rotten Kid Theorem after introducing what he calls "the game rotten kids play" (p. 1145). This game consists of two-stages. In the first stage, each family member chooses an action. This vector of actions results in a specific mix of public goods for the family and family income. In the second stage, the head distributes the family income produced in the first stage to the household members

³Individual monotonicity is the property that distinguishes the Kalai-Smorodinsky solution from the Nash Bargaining solution (Kalai and Smorodinsky 1975).

⁴McElroy (1990) refers to such policies as extrahousehold environmental parameters

after observing their actions. If the Rotten Kid Theorem holds, it is possible for the head to fully compensate household members, who put themselves in an unfavorable position in the first stage through an action that benefits others, by means of these second-stage money transfers. Bergstrom then shows that the theorem only holds when utility is transferable given the assumption that the head's altruistic utility function treats each family member's utility as a normal good. Gugl (2010) shows in a companion paper that the requirement of TU can be weakened to Almost TU by imposing a stronger, yet reasonable condition on the head's altruistic utility function; it must be a generalized utilitarian social welfare function.⁵

3 The Model

Consider a population $N = \{1, 2, \dots, n\}$ of agents who produce $L \geq 2$ goods. These goods may include public goods but at least one private good.⁶ More precisely, the population faces a *production possibility set* $Y \subset \mathbb{R}_+^L$ that is a closed, convex and comprehensive set. We denote by \mathcal{Y} the class of production possibility sets. If we denote by $y \in \mathbb{R}_+^L$ a particular *product mix*, then ∂Y , the production possibility frontier of Y , and the corresponding *transformation function*, $F : \mathbb{R}_+^L \rightarrow \mathbb{R}$, are defined as follows:

$$\begin{aligned} Y &= \{y \in \mathbb{R}_+^L \mid F(y) \leq 0\}, \text{ and} \\ \partial Y &= \{y \in Y \mid F(y) = 0\}. \end{aligned}$$

For the clarity of the presentation of our results, we make the assumption that F is differentiable.

We denote by $x_i = (x_{i1}, x_{i2}, \dots, x_{iL}) \in \mathbb{R}_+^L$ agent i 's *consumption vector*. A *distribution of y* is a list of consumption vectors, one per agent, $x = (x_1, \dots, x_n)$ such that:

$$\begin{cases} \sum_{i \in N} x_{il} = y_l & \text{for any private good, } l, \text{ and} \\ x_{ik} = x_{jk} = y_k & \text{for all } i, j \in N \text{ and any public good, } k. \end{cases}$$

For any $Y \in \mathcal{Y}$ and any product mix $y \in Y$, we denote by $X(y)$ the *set of distributions of y* and by $X(Y) = \bigcup_{y \in Y} X(y)$ the set of *feasible distributions under Y* . An *allocation* is a product mix-distribution pair $(y, x) \in Y \times X(y)$.

The preferences of each agent i are represented by a *utility function*, $u_i : \mathbb{R}_+^L \rightarrow \mathbb{R}$, which is non-decreasing, concave, differentiable, and satisfies $u_i(0) = 0$.⁷ We denote by \mathcal{U} the class of such utility functions. A *utility profile* is a

⁵There has been renewed interest in the Rotten Kid Theorem. Cornes and Silva (1999) Chiappori and Werning (2002) and Kolpin (2006) restrict their analysis of the Rotten Kid Theorem to a case of one public good and one private good with a linear technology transforming the private good into the public good. Baland and Robinson (2000) and Bommier and Dubois (2004) employ the game rotten kids play to explain the economics of child labor.

⁶Our results also apply to the case of all public goods. The problem with all public goods is trivial and therefore omitted in our analysis here.

⁷We assume differentiability for expositional purposes, but our results extend to Leontief-type preferences.

collection of utility functions, $(u_1, u_2, \dots, u_n) \in \mathcal{U}^N$, one per agent. An *economy* is a pair $(Y, u) \in \mathcal{Y} \times \mathcal{U}^N$.

We denote by $U(Y, u) = \{\psi \in \mathbb{R}_+^N \mid \psi = u(x) \text{ for some } x \in X(Y)\}$ the *utility possibility set* corresponding to the economy (Y, u) . It follows from our assumptions on an economy that $U(Y, u)$ is a closed, convex, and comprehensive set. We denote by $\partial U(Y, u)$ the Pareto frontier of $U(Y, u)$; i.e., $\partial U(Y, u) = \{\psi \in U(Y, u) \mid \psi' \geq \psi \implies \psi' \notin U(Y, u)\}$.⁸

We shall consider that agents cooperatively manage the economy, in the form of a bargaining process, with the possibility that agents disagree on how to do so. Hence, we denote by d_i agent i 's stand-alone utility level and call $d = (d_1, d_2, \dots, d_n) \in U(Y, u)$ the *disagreement point* of the bargaining process. We denote by the pair $(U(Y, u), d)$ the corresponding *bargaining problem*. Note that we take the view that the disagreement point may depend on the utility profile, u , but is independent of the cooperative production possibilities, Y . In other words, agents can guarantee themselves the same utility level under non-cooperation no matter how cooperative production possibilities change: for any $Y, Y' \in \mathcal{Y}, d(Y, u) = d(Y', u)$. We denote by $\mathcal{B} = \{(U(Y, u), d) \mid Y \in \mathcal{Y}, u \in \mathcal{U}^n \text{ and } d \in U(Y, u)\}$ the class of bargaining problems.

A bargaining solution is a function S defined on \mathcal{B} , which associates with every bargaining problem $(U(Y, u), d)$ a utility vector $S(U(Y, u), d) \in \partial U(Y, u)$ such that $S(U(Y, u), d) \geq d$. We denote by \mathcal{S} the class of bargaining solutions. A family of bargaining solutions we shall consider is that of generalized utilitarian bargaining solutions (GUBS) where, for each bargaining solution Γ in this class, there exists a list of n concave, strictly increasing, and continuous functions, $(\gamma_1, \gamma_2, \dots, \gamma_n)$, such that $\Gamma(U(Y, u), d) = \arg \max_{\psi \in \partial U(Y, u)} \sum_{i \in N} \gamma_i(\psi_i - d_i)$. Note that GUBS is a wide family of bargaining solutions; both the utilitarian solution and the Nash bargaining solution belong to GUBS, with $\gamma_i = Id$ and $\gamma_i = \ln(\cdot)$ for all i , respectively. More precisely, we denote by \mathcal{G} the subclass of GUBS for which the γ_i 's are strictly concave and differentiable; thus the Nash Bargaining solution belongs to \mathcal{G} . The utilitarian solution belongs to another subclass of GUBS, the weighted utilitarian bargaining solutions (WUBS). A bargaining solution W belonging to WUBS is characterized by a list of n non-negative weights, $\omega = (\omega_1, \omega_2, \dots, \omega_n) \in \mathbb{R}_+^N$, with $\sum \omega_i = 1$, such that $W(U(Y, u), d) \in \arg \max_{\psi \in \partial U(Y, u)} \sum_{i \in N} \omega_i(\psi_i - d_i)$.

To ensure uniqueness of the solution in case $W(U(Y, u), d)$ is not a singleton, we shall consider only the subfamily of WUBS which break ties along a non-decreasing path of \mathbb{R}_+^N . A *non-decreasing path* is a function $\pi : \mathbb{R}_+ \rightarrow \mathbb{R}_+^N$ s.t. $\pi(t) = (\pi_1(t), \pi_2(t), \dots, \pi_n(t))$ non-decreasing in each coordinate with $\pi(0) = d$ and $\sum_i \pi_i(t) = t + \sum_i d_i$. Let $\widetilde{W}(U(Y, u), d, \omega) = \arg \max_{\psi \in \partial U(Y, u)} \sum_{i \in N} \omega_i(\psi_i - d_i)$, then

$$W(U(Y, u), d) = \arg \min_{\psi \in \widetilde{W}(U(Y, u), d, \omega)} \left\| \psi - \pi \left(\max_{\psi \in \partial U(Y, u)} \sum \omega_i(\psi_i - d_i) \right) \right\|$$

⁸We adopt the usual notational convention for vector inequalities: $x \geq x'$, $x \geq x'$, and $x > x'$.

By convexity of $U(Y, u)$, $W(U(Y, u), d)$ is unique. We denote by \mathcal{W} the family of WUBS breaking ties along a non-decreasing path. We refer to $\mathcal{G} \cup \mathcal{W}$ as the class of *well-behaved GUBS*.

We say that a bargaining solution, S , satisfies *Solidarity in production opportunities* under utility profile u if one of the following vector inequalities holds:

$$S(U(Y, u), d) \geq S(U(Y', u), d) \quad \text{or} \quad S(U(Y, u), d) \leq S(U(Y', u), d)$$

for any $Y, Y' \subset \mathcal{Y}$, and any $d \in \mathbb{R}_+^N$.

4 Almost Transferable Utility

In this section we define the new concept of Almost Transferable Utility. In order to do so, we first recall what is meant by a product mix being *efficient independent of distribution*, a necessary (and sufficient) condition for (Almost) TU to hold. The conditions under which a product mix is efficient independent of distribution are well established in the literature by Bergstrom and Cornes (1983) and Bergstrom and Varian (1985). Our analysis below is not meant to reestablish these results but to apply the concept to introduce Almost TU.⁹

4.1 Efficiency Independent of Distribution¹⁰

We denote by $EX(y, u) = \{x \in X(y) | \nexists x' \in X(y), u(x') \geq u(x)\}$ the set of exchange efficient distributions in $X(y)$ relative to the utility profile u and by $P(Y, u) = \{(y, x) \in Y \times X(y) | \nexists (y', x') \in Y \times X(y'), u(x') \geq u(x)\}$ the set of (Pareto) efficient allocations in the economy (Y, u) .

Given quasi-concavity and differentiability of every u_i , for any given $y \in Y$, we say that $x \in X(y)$ is *interior exchange efficient* if and only if:

$$\frac{\frac{\partial u_i(x_i)}{\partial x_{li}}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(x_j)}{\partial x_{lj}}}{\frac{\partial u_j(x_j)}{\partial x_{mj}}} \quad \text{for all } i, j \in N, \text{ for all private goods } l, m.$$

We denote by $EX^*(y, u)$ the set of interior exchange efficient distributions of $X(y)$ relative to the utility profile u . That is, $x \in EX^*(y, u)$ if and only if the (necessary and sufficient) first order conditions of the problem of finding the utility possibility frontier for given $y \in Y$ hold with equality at x . This can be true even if some agents receive nothing of some private goods.

Next we state the conditions for interior efficiency. We say that a product mix and distribution pair $(y, x) \in Y \times X(y)$ is *interior efficient* if and only if

⁹The conditions for a product mix to be efficient independent of distribution do not depend on the differentiability of u . See Bergstrom and Cornes (1983) and Bergstrom and Varian (1985). However, assuming differentiability simplifies the exposition of this concept.

¹⁰Bergstrom and Cornes (1983) call this concept "independence of allocative efficiency from distribution."

the following holds:

$$\left\{ \begin{array}{l} \frac{\frac{\partial u_i(x_i)}{\partial x_{li}}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} = \frac{\frac{\partial F(y)}{\partial y_l}}{\frac{\partial F(y)}{\partial y_m}} \quad \text{for all } i \in N, \text{ and all private goods } l, m \\ \text{and } \sum_{i \in N} \left(\frac{\frac{\partial u_i(x_i)}{\partial x_k}}{\frac{\partial u_i(x_i)}{\partial x_{mi}}} \right) = \frac{\frac{\partial F(y)}{\partial y_k}}{\frac{\partial F(y)}{\partial y_m}} \quad \text{for any public good } k \text{ and any private good } m. \end{array} \right. \quad (1)$$

The first family of equalities states that all agents' marginal rates of substitutions (MRS) between any two private goods, $\frac{\partial u_i(x_i)}{\partial x_{li}} / \frac{\partial u_i(x_i)}{\partial x_{mi}}$, must equal the marginal rate of transformation (MRT) between these goods, $\frac{\partial F(y)}{\partial y_l} / \frac{\partial F(y)}{\partial y_m}$. The second set of equalities are the Samuelson conditions for public goods. Note that interior efficiency implies interior exchange efficiency.

We say that *efficiency is independent of distribution* if there exists $\bar{y} \in Y$ such that, for all $\psi \in \partial U(Y, u)$, $\psi = u(x)$ for some $x \in X(\bar{y})$. In words, all points on the utility possibility frontier are achieved via various distributions of the same product mix. We now describe the intuition of this concept in more detail by assuming interior efficiency¹¹. It follows from (1) that \bar{y} is associated with a vector of marginal rates of substitution at the exchange efficient distributions—one per pair of goods—which is independent of the utility level achieved by any agent. Figure 1 provides a graphical illustration in the case of two private goods, strictly quasi-concave utility functions and a strictly convex production possibility set. In the figure, neither y nor y' are efficient, because the value of the MRS associated with $EX^*(y, u)$ is different from the MRT at y ; similarly for y' . The product mix associated with any efficient allocation, y'' , lies on ∂Y between y and y' , where the MRS of $EX^*(y'', u)$ equals the MRT at y'' .

In the case of public and private goods, for a product mix to be efficient independent of distribution requires that the sum of marginal rates of substitution stays the same for a given level of the public good \bar{y}_k no matter how the total amount of the private good \bar{y}_l is distributed.

4.2 Transferable Utility

A utility profile, $u \in \mathcal{U}^N$, satisfies *Transferable Utility* (Bergstrom 1989) if for any given production possibility set $Y \in \mathcal{Y}$, there exists $\lambda \in \mathbb{R}$ such that:

$$\partial U(Y, u) = \left\{ \psi \in U(Y, u) : \sum_{i \in N} \psi_i = \lambda \right\}.$$

Because λ depends on the production possibility set Y and on the utility profile u , we denote it by $\lambda(Y, u)$. We denote by $\mathcal{TU} \subset \mathcal{U}^N$ the class of utility profiles satisfying TU.

¹¹Note that interior efficiency is not necessary for a product mix to be efficient independent of distribution for a given Y .

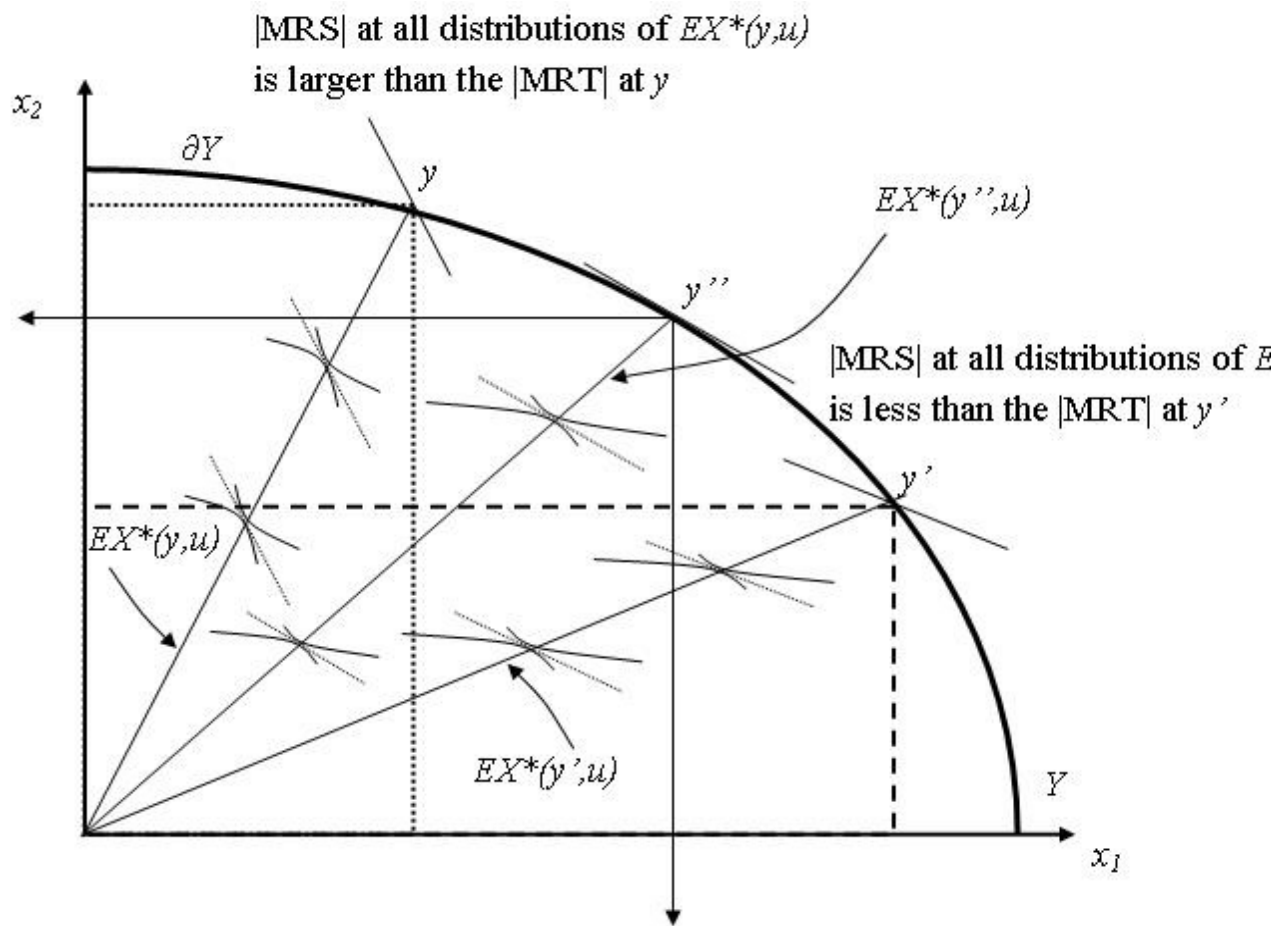


Figure 1: Efficiency independent of distribution.

Note that resource and technological constraints as given by Y only play a role in the size of λ : If TU holds, efficiency is independent of distribution (Bergstrom and Cornes 1983, and Bergstrom and Varian 1985) for every $Y \in \mathcal{Y}$. The converse is also true, if we take agents' utility to be ordinal; efficiency being independent of distribution implies TU. In the Appendix we provide the intuition for this result and an exhaustive list of utility profiles satisfying TU.

Transferable utility is sometimes equated with quasi-linear preferences. As below example 1a) shows, there are utility profiles consisting of other than quasi-linear utility functions that lead to TU. Example 1b) points to the fact that quasi-linear utility functions satisfy TU only locally, albeit for a broad range of welfare distributions. In Example 2, TU does not hold; although there exists a Y such that $U(Y, u)$ forms a simplex, it is not the case for all production possibility sets.

Example 1 *Examples of utility profiles satisfying TU.*

a) *Two private goods, two agents. Suppose $u_i = (x_{1i}x_{2i})^{1/2}$. Then $\partial U(Y, u) = \{(\psi_1, \psi_2) \in U(Y, u) : \psi_1 + \psi_2 = \lambda(Y, u)\}$, where $\lambda(Y, u) = \max_{y \in Y} (y_1 y_2)^{1/2}$.*

b) *A private and a public good, two agents. Suppose preferences over a private good and a public good are quasi-linear such that $u_i = x_{1i} + h_i(x_2)$, where $h_i(\cdot)$ is a strictly concave function. Then the segment of the utility possibility frontier at which TU holds consists of all the points on the line from $(\psi_1 = h_1(y_2), \psi_2 = y_1 + h_2(y_2))$ to $(\psi_1 = y_1 + h_1(y_2), \psi_2 = h_2(y_2))$ where the vector (y_1, y_2) is found by $\arg \max_{y \in Y} y_1 + h_1(y_2) + h_2(y_2)$, and $\lambda(Y, u) = \max_{y \in Y} y_1 + h_1(y_2) + h_2(y_2)$.*

Note that, knowing that TU holds, one can find $\partial U(Y, u)$ in two steps. First, calculate $\lambda(y, u) = \sum_{i \in N} u_i(x_i)$ such that $x \in EX^*(y, u)$. Second, find $\lambda(Y, u) = \max_{y \in \partial Y} \lambda(y, u)$.

Example 2 *Three private goods, two agents. Let u_1 and u_2 be strictly increasing, strictly quasi-concave and homogenous of degree one. Moreover, let $u_1 = u(x_{11}, x_{21})$, $u_2 = u(x_{12}, x_{32})$. That is, good 3 replaces good 2 in agent 2's utility function as compared to that of agent 1's. Let $\lambda(Y, u_1) = \max_{y \in Y} u(x_{11}, x_{21})$ and $\lambda(Y, u_2) = \max_{y \in Y} u(x_{12}, x_{32})$. Consider the production possibility set given by $Y = \{(y_1, y_2, y_3) \in \mathbb{R}_+^3 | y_1 + p_2 y_2 + p_3 y_3 \leq I\}$ with $p_2, p_3, I > 0$ and $p_2 = p_3 = p$. Then, $\lambda(Y, u_1) = \lambda(Y, u_2)$ and $\partial U(Y, u) = \{(\psi_1, \psi_2) \in U(Y, u) : \psi_1 + \psi_2 = \lambda(Y, u_1) = \lambda(Y, u_2)\}$. Suppose Y changes to Y' such that $Y' = \{(y_1, y_2, y_3) \in \mathbb{R}_+^3 | y_1 + p_2 y_2 + p_3 y_3 \leq I\}$ with $p_2 \neq p_3$. Then $\partial U(Y', u) = \{(\psi_1, \psi_2) \in U(Y', u) : \frac{\psi_1}{\lambda(Y', u_1)} + \frac{\psi_2}{\lambda(Y', u_2)} = 1\}$. TU is violated. Proof in Appendix.*

4.3 Almost Transferable Utility

Whether a product mix is efficient independently of distribution depends solely on the ordinal properties of the agents' utility functions. Therefore, if they are such that a product mix is efficient independent of distribution, but their

cardinal properties prohibit a utility representation that would lead to TU, there exist positive monotonic transformations, $f_i : \mathbb{R} \rightarrow \mathbb{R}$, such that:

$$\sum_{i \in N} f_i(\psi_i) = \bar{\lambda},$$

for some $\bar{\lambda} \in \mathbb{R}$. However, $f_i(\psi_i)$ no longer represents agent i 's utility. Hence, we say that profile $u \in \mathcal{U}^N$ exhibits *Almost Transferable Utility (Almost TU)* if there exists a profile $f = (f_1, \dots, f_n)$ of positive monotonic transformations such that:

$$\partial U(Y, u) = \left\{ \psi \in U(Y, u) : \sum_{i \in N} f_i(\psi_i) = \bar{\lambda} \right\}$$

for all $Y \in \mathcal{Y}$. We denote by $\mathcal{ATU} \subset \mathcal{U}^N$ the class of utility profiles satisfying Almost TU. Because $\bar{\lambda}$ depends on the production possibility set Y and on the utility profile u , we denote it by $\bar{\lambda}(Y, u)$. Given that the u_i 's are assumed to be concave and differentiable, it follows that f_i must be an increasing, convex and differentiable function. The intuition is that in order to recover a linear constraint, a constant "marginal utility of money" is required. Hence, given that strict concavity of u_i leads to decreasing marginal utility of money, a strictly convex transformation is required to undo this effect.

Denote by \mathcal{F} the class of convex, strictly increasing and differentiable functions on \mathbb{R} and, for any $u \in \mathcal{U}^N$, by $A(u) \subset \mathcal{U}^N$ the set of utility profiles u' such that

$$f(u') = u$$

for some $f \in \mathcal{F}^N$. Clearly, $u \in A(u)$ for any $u \in \mathcal{U}^N$. The following Theorem illustrates how much larger the class of Almost TU utility profiles is compared to that of TU:

Theorem 1 $\mathcal{ATU} = \bigcup_{u \in \mathcal{TU}} A(u)$.

Proof. Let $u \in \mathcal{TU}$, and consider $u' \in A(u)$. By definition of $A(u)$, there exists $f \in \mathcal{F}^N$, such that $f(u') = u$. Also, because u satisfies TU, it follows that, for every $Y \in \mathcal{Y}$:

$$\partial U(Y, u) = \left\{ \psi \in U(Y, u) \mid \sum_{i \in N} \psi_i = \lambda(Y, u) \right\}.$$

Rewriting, we get:

$$\partial U(Y, u) = \left\{ \psi \in U(Y, u) \mid \sum_{i \in N} f_i(f_i^{-1}(\psi_i)) = \lambda(Y, f(f^{-1}(u))) \right\}.$$

As we have seen, TU implies that efficiency is independent of distribution for all $Y \in \mathcal{Y}$. Since efficiency independent of distribution is an ordinal concept, to

any $\psi \in \partial U(Y, u)$ corresponds a $\psi' \in \partial U(Y, u')$. It follows that:

$$\partial U(Y, u') = \left\{ \psi' \in U(Y, u') \mid \sum_{i \in N} f_i(\psi'_i) = \lambda(Y, f(u')) \right\};$$

i.e., $u' \in \mathcal{ATU}$. ■

Example 3 uses utility profiles with the same ordinal properties as in Example 1, but different cardinal properties:

Example 3 *Examples of utility profiles satisfying Almost TU but not TU.*

a) Suppose $u_1 = (x_{11}x_{21})^{1/3}$ and $u_2 = (x_{12}x_{22})^{1/4}$. Then $\partial U(Y, u) = \{(\psi_1, \psi_2) \in \mathbb{R}_+^2 : \psi_1^{3/2} + \psi_2^2 = \bar{\lambda}(Y, (u_1, u_2))\}$, where $\bar{\lambda} = \max_{y \in Y} (y_1 y_2)^{1/2}$.

b) Suppose the cardinal utility function of agent i over a private good (x_{1i}) and a public good (x_2) is given by $u_i = (x_{1i} + h_i(x_2))^{\alpha_i}$, where h_i is a strictly concave function and $\alpha_i \in (0, 1)$. Then the segment of the utility possibility frontier at which Almost TU holds consists of the endpoints $(\psi_1 = (h_1(y_2))^{\alpha_1}, \psi_2 = (y_1 + h_2(y_2))^{\alpha_2})$ and $(\psi_1 = (y_1 + h_1(y_2))^{\alpha_1}, \psi_2 = (h_2(y_2))^{\alpha_2})$ and of all points (ψ_1, ψ_2) between these endpoints for which $\psi_1^{1/\alpha_1} + \psi_2^{1/\alpha_2} = \bar{\lambda}(Y, (u_1, u_2))$, where $\bar{\lambda} = \max_{y \in Y} y_1 + h_1(y_2) + h_1(y_2)$.

Any profile of utility functions that leads to TU also satisfies Almost TU. Any utility profile satisfying TU can be transformed into many utility profiles satisfying Almost TU by using different concave transformations on the utility profile satisfying TU. Since well-behaved GUBS result in different resource allocations depending on the cardinal properties of the individuals' utility functions, the domain of Almost TU is significantly larger than the domain of TU.¹²

5 Results

We now consider what happens if the production possibility set changes when Almost TU holds. The following Lemma states that a change in the production possibilities of the economy can only result in an expansion or a contraction of the utility possibility set.

Lemma 1 $\forall u \in \mathcal{ATU}, \forall Y, Y' \in \mathcal{Y}, [U(Y, u) \subseteq U(Y', u) \text{ or } U(Y', u) \subseteq U(Y, u)]$.

Proof. Let $u \in \mathcal{ATU}$, and consider a change from Y to Y' . By Almost TU, $\partial U(Y, u) = \{\psi \in U(Y, u) : \sum f_i(\psi_i) = \bar{\lambda}(Y, u)\}$ and $\partial U(Y', u) = \{\psi \in U(Y', u) : \sum f_i(\psi_i) = \bar{\lambda}(Y', u)\}$. Hence, either $\bar{\lambda}(Y, u) = \bar{\lambda}(Y', u)$, in which case $U(Y, u) = U(Y', u)$; or $\bar{\lambda}(Y, u) > \bar{\lambda}(Y', u)$ implying $U(Y, u) \supset U(Y', u)$; or $\bar{\lambda}(Y, u) < \bar{\lambda}(Y', u)$ implying $U(Y, u) \subset U(Y', u)$.

We now state our main theorem. ■

Theorem 2 *Any bargaining solution, $S \in \mathcal{G} \cup \mathcal{W}$, satisfies solidarity in production opportunities under profile u if and only if $u \in \mathcal{ATU}$.*

¹²We provide an example in the appendix illustrating this point.

Proof. For sufficiency, note that for any $S \in \mathcal{G} \cup \mathcal{W}$, a first step—in case of $S \in \mathcal{G}$, it is also the last step—to finding S is by solving the following:

$$\begin{aligned} & \max_{\psi} \sum_{i \in N} \gamma_i(\psi_i - d_i) \\ \text{s.t.} \quad & \sum_{i \in N} f_i(\psi_i) = \bar{\lambda}(Y, u) \\ & \psi_i \geq d_i \end{aligned} \quad (2)$$

In what follows, it will be useful to work with $v_i = f_i(\psi_i)$, such that the above problem becomes

$$\begin{aligned} & \max_{\psi} \sum_{i \in N} \gamma_i(f_i^{-1}(v_i) - d_i) \\ \text{s.t.} \quad & \sum_{i \in N} v_i = \bar{\lambda}(Y, u) \\ & v_i \geq f_i(d_i) \end{aligned} \quad (3)$$

and then finding the corresponding ψ_i from $\psi_i = f_i^{-1}(v_i)$. Chun and Thomson (1988) show that if agents have concave utility functions over one good only, and this good's supply increases, both agents benefit under the Nash bargaining solution. The authors remark that the result extends to any bargaining solution in \mathcal{G} (Chun and Thomson 1988, p. 19). When problem (2) is presented as (3), a change in $\bar{\lambda}(Y, u)$ due to a change in Y has the same impact on v as a change in the supply of the only good in Chun and Thomson (1988)'s one-good economy. Thus, their proof applies to our problem (3) for any $S \in \mathcal{G}$ such that $\lim_{x_i \rightarrow 0} \gamma'_i = -\infty$. In addition, our proof of sufficiency below also handles the subclass \mathcal{W} of GUBS, the possibility of corner solutions¹³, and accounts for $d \neq (0, 0)$.

Suppose Almost TU holds, and let $S \in \mathcal{G} \cup \mathcal{W}$, $Y \subset \mathbb{R}_+^L$, $u \in \mathcal{U}^N$ and $d \in U(Y, u)$. Denote by $v^* = S(\{v \in \mathbb{R}^N \mid \sum_{i \in N} v_i \leq \bar{\lambda}(Y, u)\}, d)$ the solution vector in the v -space. The proof is divided into two cases, depending on whether the utility possibility set, $U(Y, u)$, is strictly convex (Case 1) or whether $\partial U(Y, u)$ exhibits flat portions (Case 2).

- Case 1: f_i is strictly convex in a vicinity of $f_i^{-1}(v_i^*)$ for at least $n - 1$ agents $i \in N$. It follows that v^* is the unique element of $\partial f(U(Y, u)) = \{v \in f(U(Y, u)) \mid v' \geq v \implies v' \notin f(U(Y, u))\}$ such that the following expression holds:

$$\begin{cases} \gamma'_i(f_i^{-1}(v_i) - d_i) \frac{df_i^{-1}}{dv_i}(v_i) = \gamma'_j(f_j^{-1}(v_j) - d_j) \frac{df_j^{-1}}{dv_j}(v_j) & \text{for all } i, j \in N \\ \text{such that } f_i^{-1}(v_i) \geq \max(\underline{\psi}_i, d_i) \text{ and } f_j^{-1}(v_j) \geq \max(\underline{\psi}_j, d_j) \end{cases} \quad (4)$$

where $\underline{\psi}_i$ (resp. $\underline{\psi}_j$) is the utility level of agent i (resp. agent j) when she does not receive any private good. The left hand side of (4) depends on v_i only, and the right hand side of (4) depends on v_j only. Since f_i^{-1} is increasing and concave, $\frac{df_i^{-1}}{dv_i}$ is positive and non-increasing. Similarly, γ_i

¹³The possibility of corner solutions is not a concern in the proof of Chun and Thomson (1988) as the Nash bargaining solution is necessary interior.

being concave and f_i^{-1} being increasing, $\gamma'_i(f_i^{-1}(v_i) - d_i)$ is non-increasing in v_i . Hence, the product $\frac{\partial \gamma_i}{\partial (f_i^{-1}(v_i) - d_i)} \frac{df_i^{-1}}{dv_i}$ (resp. $\frac{\partial \gamma_j}{\partial (f_j^{-1}(v_j) - d_j)} \frac{df_j^{-1}}{dv_j}$) is non-increasing in v_i (resp. v_j). Therefore, for (4) to hold as λ changes values, v_i and v_j must change in the same direction, thus proving the result.

- Case 2: f_i is strictly convex in a vicinity of $f_i^{-1}(v_i^*)$ for at most $n - 2$ agents $i \in N$ ($\partial U(Y, u)$ is flat in this vicinity). If $S \in \mathcal{G}$, v^* is the unique element of $\partial f(U(Y, u))$ for which expression (4) holds and the argument of Case 1 follows through. Now, suppose $S \in \mathcal{W}$, with $\omega \in \mathbb{R}_+^n$ its associated weights, there may be more than one $v \in \partial f(U(Y, u))$ for which expression (4) holds. Denote

$$\sigma(Y, u, \omega) = \left\{ \begin{array}{l} \psi \in \partial U(Y, u) \\ \omega_i \times \frac{\partial f_i^{-1}}{\partial v_i}(v_i) = \omega_j \times \frac{\partial f_j^{-1}}{\partial v_j}(v_j) \quad \text{for all } i, j \in N \\ \text{and } \psi_i > \max(\underline{\psi}_i, d_i) \text{ and } \psi_j > \max(\underline{\psi}_j, d_j). \end{array} \right\}.$$

We proceed to show that the fact that $\sigma(Y, u, \omega)$ may not be a singleton does not affect the comonotonicity of the utility shares. This does not follow immediately, because, despite the fact that S breaks ties along a non-decreasing path, the path may not pass through $\sigma(Y, u, \omega)$.

It follows from elementary convex optimization arguments that if $\sigma(Y, u, \omega)$ is not a singleton, then $\lim_{\hat{\omega}_t \rightarrow \omega} \sigma(Y, u, \hat{\omega}_t)$ is a singleton for any sequence of \mathbb{R}_+^n , $\{\hat{\omega}_t\}_{t \in \mathbb{N}}$, such that $\hat{\omega}_t \neq \omega$ for all t and $\lim_{t \rightarrow \infty} \hat{\omega}_t = \omega$. Therefore, an argument similar to that in Case 1, applied to the sequences $\{\hat{\omega}_t\}_{t \in \mathbb{N}}$ implies that, for any $Y' \subset \mathbb{R}_+^L$ such that, by Lemma 1, $U(Y, u) \subset U(Y', u)$ (resp. $U(Y', u) \subset U(Y, u)$), the set $\sigma(Y', u, \omega)$ dominates (resp. is dominated by) $\sigma(Y, u, \omega)$ in the following sense: for any element $\psi \in \sigma(Y, u, \omega)$, there exists $\psi' \in \sigma(Y', u, \omega)$ such that $\psi' \geq \psi$ (resp. $\psi' \leq \psi$). See Figure 2 for a two-dimensional illustration. Thus, the fact that S breaks ties along a non-decreasing path yields the desired result, regardless of whether this path passes through $\sigma(Y, u, \omega)$ or $\sigma(Y', u, \omega)$.

For necessity, let S be a generalized utilitarian bargaining solution and let $u \in \mathcal{U}^N$ be a utility profile which does not satisfy Almost TU. Consider a production possibility set, $Y_1 \subset \mathbb{R}_+^L$, and let $y \in \partial Y_1$ be an efficient product mix, so that $a \in EX^*(y, u)$ in the economy (Y_1, u) . It follows that:

$$\frac{\frac{\partial u_i(a_i)}{\partial x_{li}}}{\frac{\partial u_i(a_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(a_j)}{\partial x_{lj}}}{\frac{\partial u_j(a_j)}{\partial x_{mj}}} = \frac{\frac{\partial F_1(y)}{\partial y_l}}{\frac{\partial F_1(y)}{\partial y_m}} \quad (5)$$

for any pair of agents i and j and any pair of goods l and m .¹⁴

¹⁴For clarity, we are presenting the proof in the case of all private goods. The proof with public goods is similar, with the efficiency condition (5) being replaced by the Samuelson condition for these goods (Expression (1)).

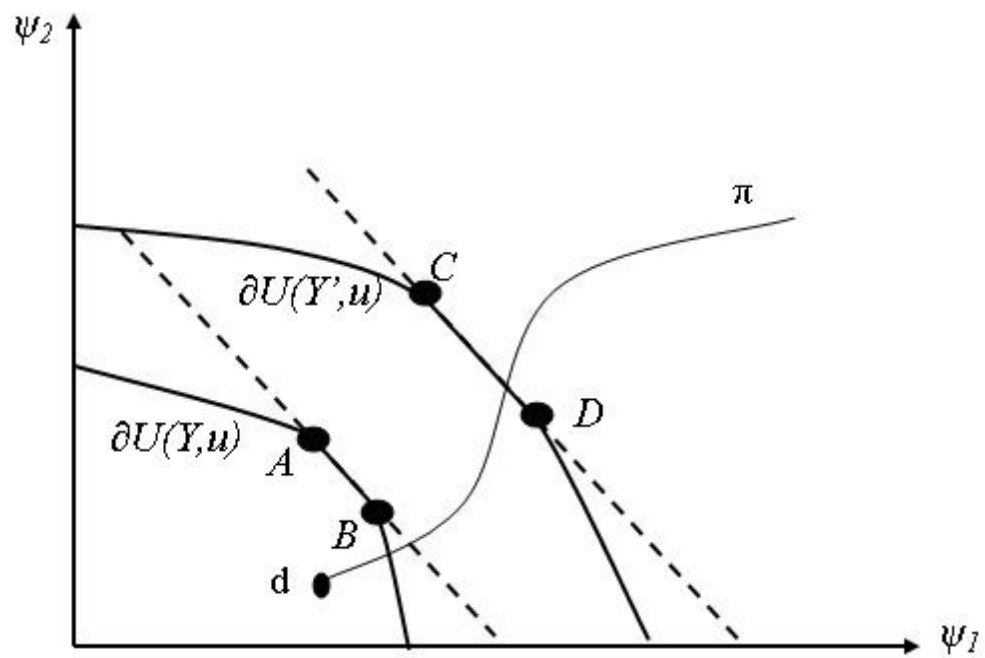


Figure 2: By Case 1, $A < C$ and $B < D$.

By continuity, and because the profile u does not satisfy Almost TU, there exists an interior exchange efficient distribution $b \in EX^*(y, u)$ such that (y, b) is not Pareto efficient in the economy (Y_1, u) . Therefore, $\frac{\frac{\partial u_i(b_i)}{\partial x_{1i}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{1j}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}}$ for all $i, j \in N$ and all $l, m \in L$ but $\frac{\frac{\partial u_i(b_i)}{\partial x_{1i}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{1j}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}} \neq \frac{\frac{\partial F_1(y)}{\partial y_l}}{\frac{\partial F_1(y)}{\partial y_m}}$ for some pair l, m of goods. Without loss of generality, suppose that

$$\frac{\frac{\partial u_i(b_i)}{\partial x_{1i}}}{\frac{\partial u_i(b_i)}{\partial x_{2i}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{1j}}}{\frac{\partial u_j(b_j)}{\partial x_{2j}}} > \frac{\frac{\partial F_1(y)}{\partial y_1}}{\frac{\partial F_1(y)}{\partial y_2}},$$

for all $i, j \in N$.

Now construct another production possibility set, $Y_2 \subset \mathbb{R}_+^L$, such that $y \in \partial Y_2$ and $\frac{\frac{\partial F_2(y)}{\partial y_l}}{\frac{\partial F_2(y)}{\partial y_m}} = \frac{\frac{\partial u_i(b_i)}{\partial x_{1i}}}{\frac{\partial u_i(b_i)}{\partial x_{mi}}} = \frac{\frac{\partial u_j(b_j)}{\partial x_{1j}}}{\frac{\partial u_j(b_j)}{\partial x_{mj}}}$ for all $l, m \in L$ and all $i, j \in N$, as shown in Figure 3 in the two-agent case.

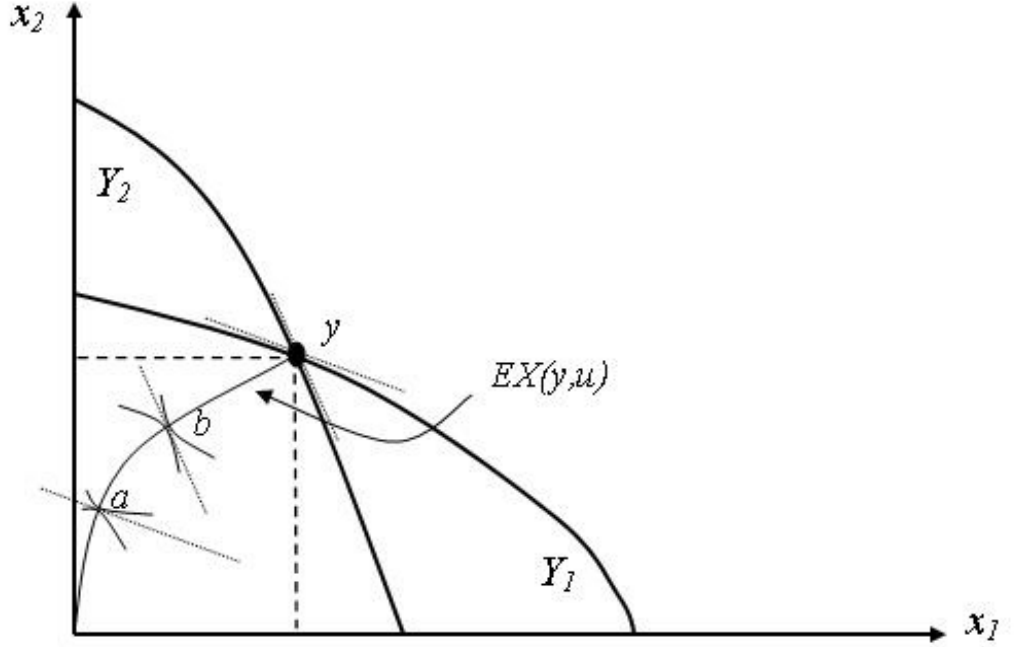


Figure 3: Allocation a (b) is efficient when the production set is Y_1 (Y_2).

It follows from the construction of Y_2 that a is not an efficient allocation in (Y_2, u) because $\frac{\frac{\partial F_2(y)}{\partial y_1}}{\frac{\partial F_2(y)}{\partial y_2}} > \frac{\frac{\partial u_i(a_i)}{\partial x_{1i}}}{\frac{\partial u_i(a_i)}{\partial x_{2i}}}$ for all $i \in N$. Therefore, there exists an allocation in Y_2 which Pareto-dominates a . In other words, if we denote by ψ_{a1} the utility vector corresponding to distribution a , (recall that efficiency of a in the economy (Y_1, u) implies that $\psi_{a1} \in \partial U(Y_1, u)$) there exists another vector $\psi_{a2} \in \partial U(Y_2, u)$ such that $\psi_{a2} > \psi_{a1}$. See Figure 4.

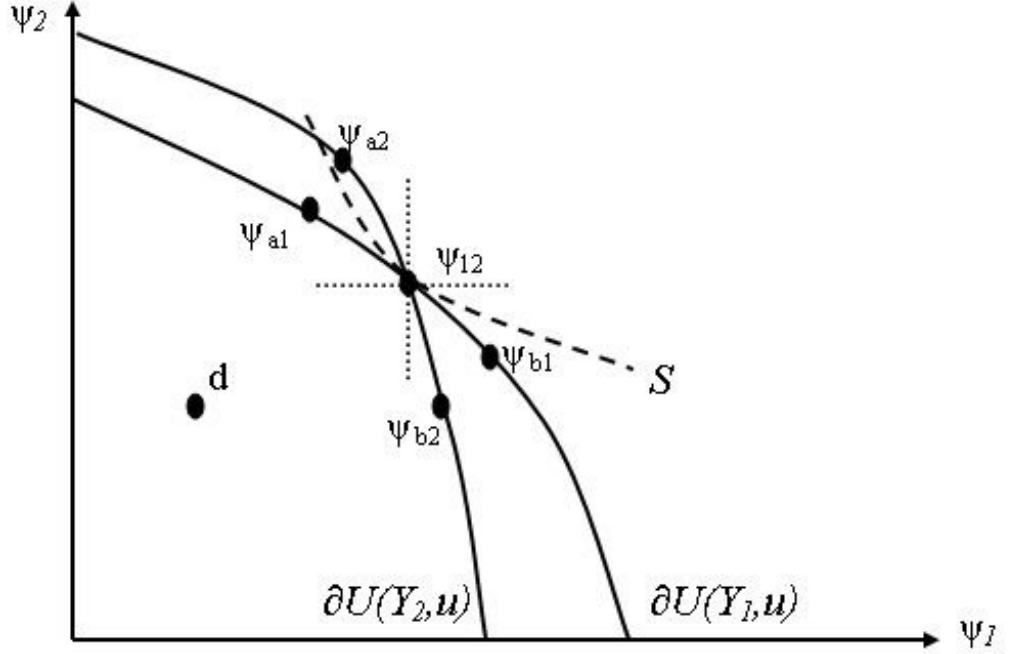


Figure 4: When utility possibility frontiers cross, solidarity may not hold

Similarly, because the allocation $(y, b) \in P(Y_2, u) \setminus P(Y_1, u)$, there exists a utility vector $\psi_{b1} \in \partial U(Y_1, u)$ which dominates $\psi_{b2} \in \partial U(Y_2, u)$; i.e. $\psi_{b1} > \psi_{b2}$.

From the two previous arguments, and from the continuity of $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$, it must be that $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$ cross at some point in the utility space. Denote by $\psi_{12} \in \partial U(Y_1, u) \cap \partial U(Y_2, u)$ such a point.

We now show that there exist bargaining situations where a change from the production possibility set Y_1 to Y_2 will benefit some agents while hurting others. Consider a disagreement point, $d \in U(Y_1, u)$, such that $S(U(Y_1, u), d) = \psi_{12}$; i.e., such that $\psi_{12} = \arg \max_{\psi \in U(Y_1, u)} \sum_i \gamma_i (\psi_i - d_i)$. Such a disagreement point exists due to the continuity, concavity, and the strict monotonicity properties of the γ_i 's, if $S \in \mathcal{G}$ or, if $S \in \mathcal{W}$, due to the fact that S breaks ties along a path. Therefore, invoking again the concavity and strict monotonicity of the

γ_i 's, and the fact that $\partial U(Y_1, u)$ and $\partial U(Y_2, u)$ cross at ψ_{12} , it follows that $S(U(Y_2, u), d) \neq \psi_{12}$.¹⁵ Finally, it follows from the fact that $U(Y_2, u)$ is convex and comprehensive that $S(U(Y_2, u), d)$ neither dominates nor is dominated by $S(U(Y_1, u), d)$. ■

Remark 1 For many GUBS not belonging to $\mathcal{G} \cup \mathcal{W}$, where the γ_i 's are not strictly concave everywhere, the proof of Theorem 2 readily applies.

Remark 2 Note that Chun and Thomson (1988)'s one-good economy is a special case of our economy; with $L = 1$, and y_1 given, it follows that $\sum_{i \in N} v_i = \sum x_{1i}$ and $\lambda(y_1, f(u)) = y_1$.

Remark 3 In the case of identical utility functions and a symmetric GUBS, if d is symmetric, the solidarity property is satisfied even if $S \notin \mathcal{G} \cup \mathcal{W}$. Indeed, by symmetry of $U(\cdot, u)$, it is impossible that $U(Y_1, u)$ and $U(Y_2, u)$ cross where $\psi_i = \psi_j$ for all $i, j \in N$; yet, a symmetric GUBS will always select a $\psi \in \partial U(\cdot, u)$ such that $\psi_i = \psi_j$ for all $i, j \in N$.

Remark 4 The Kalai-Smorodinsky solution for $n = 2$ also satisfies solidarity in production opportunities if and only if Almost TU holds. The proof for sufficiency follows again from Chun and Thomson's (1988) proof that the Kalai-Smorodinsky solution (KSS) satisfies resource monotonicity in the one-good economy. Necessity can be readily proven by an argument similar to the one presented in the proof above.

6 Applications

6.1 Policy Analysis

Theorem 2 has important implications for policy analysis. The property of solidarity in production opportunities is not just normatively appealing but is useful in positive economic analysis as well. Let's assume that there is a change in the production possibility set due to a change in government policy. Let the parameter indicating government policy be τ . Assuming a well-behaved GUBS in combination with Almost TU simplifies the analysis of such policies; by Almost TU, the impact of τ on the utility possibility set is given by

$$\partial U(Y, u) = \{\psi \in U(Y, u) : \sum_{i \in N} f_i(\psi_i) = \bar{\lambda}(\tau, u)\}$$

and a change in the policy therefore leads to a change in $\bar{\lambda}$. Assuming $\bar{\lambda}$ is differentiable in τ , we can find $\frac{\partial \bar{\lambda}}{\partial \tau}$. By the solidarity property, the impact of τ on the utility possibility set can only lead to all agents being better off or all

¹⁵Recall that we defined the disagreement point to be entirely determined by the utility profile and the agents' (exogenous) stand-alone utilities. According to this interpretation, the disagreement point is unaffected by changes in the joint production possibilities.

agents being worse off. Of course, this is only the impact of a policy change provided that the disagreement point stays the same.

Even as the disagreement point may also be affected by the same policy as the joint production possibility set, the solidarity property remains useful in this analysis. The problem in models not assuming Almost TU is that while the policies considered often have an unambiguous impact on the disagreement point, the impacts of the policies "which operate through the feasible set, are indeterminant." (Pollak, forthcoming). Let $\psi(\bar{\lambda}(\tau), d(\tau))$ be the solution vector for the bargaining problem in the presence of a government policy. Then by Almost TU, we can analyze the impact of a change in the government policy by taking the derivative for every $\psi_i(\bar{\lambda}(\tau), d(\tau))$ with respect to τ

$$\frac{\partial \psi_i}{\partial \tau} = \frac{\partial \psi_i}{\partial \bar{\lambda}} \frac{\partial \bar{\lambda}}{\partial \tau} + \sum_{j \in N} \frac{\partial \psi_i}{\partial d_j} \frac{\partial d_j}{\partial \tau}$$

By the solidarity property, we know the sign of the first term. Thus the appeal of the solidarity property in this context is simply that it allows us to receive less ambiguous results, similarly to the assumption in consumer theory that goods are normal when we consider the impact of a price change on consumer demand.

6.2 Incentive Compatibility

So far, we have considered exogenous changes in the production possibility set and assumed that agents produce efficiently the goods that they share according to a GUBS. Now suppose that agents choose their actions non-cooperatively to produce goods. In particular, denote by Z_i the action set of an agent and by $z_i \in Z_i$ a specific action taken by agent i . Denote by z_{-i} the actions taken by all the other agents except agent i . Each vector $z = (z_i, z_{-i}) \in Z_i \times \prod_{j \neq i} Z_j$ generates $y(z) \in Y(\prod_{i \in N} Z_i)$.

Theorem 3 (Incentive Compatibility) *If agents use a $S \in \mathcal{G} \cup \mathcal{W}$ to determine the distribution of a given product mix $y(z)$, the unique subgame perfect Nash equilibrium (SPNE) outcome of the game in which agents sequentially choose their actions is efficient if and only if Almost TU holds.*

Proof. Sufficiency: By Almost TU and Theorem 2, S satisfies solidarity. Therefore, all agents seek to maximize $\bar{\lambda}(y(z), u)$. The fact that agents play a sequential game eliminates the possibility of a coordination problem and, hence, agents will non-cooperatively reach a vector of actions $z^* \in \arg \max_{z \in \prod_{i \in N} Z_i} \bar{\lambda}(y(z), u)$.

Necessity follows from Theorem 2: Only if Almost TU holds does a solution in $\mathcal{G} \cup \mathcal{W}$ guarantee that agents have a common goal (i.e., to maximize $\bar{\lambda}(y(z), u)$).

■

Remark 6 While we establish incentive compatibility in a context where agents act sequentially (Theorem 3), it is worth noticing that a similar result holds

in a simultaneous-move game. Indeed, using $S \in \mathcal{G} \cup \mathcal{W}$ ensures that the agents' interests are aligned; hence, once coordination issues have been resolved, say, by cooperatively agreeing on a production plan *ex ante*, unilateral deviations from the efficient plan are unprofitable.

7 Conclusion

Many normatively appealing properties are also crucial in determining positive questions.¹⁶ The solidarity property is no exception. We showed that for well-behaved General Utilitarian Bargaining Solutions the solidarity property is satisfied if and only if Almost TU holds. We then showed that if the agents can agree on how to distribute goods once they are produced, but choose their actions individually to produce these goods, incentive compatibility is satisfied if and only if the GUBS satisfies the solidarity property.

Almost TU is an important subdomain of all utility profiles and we believe Almost TU, combined with GUBS, to be a useful approach to modelling joint decisions in a variety of economic situations. For example, the issue of incentive compatibility arises in a model in which spouses need to take individual actions in order to produce goods that they will later distribute among themselves using a bargaining solution (see e.g. Gugl 2009). Although there is no bargaining problem at the centre of Gary Becker (1974)'s Rotten Kid Theorem, our results are also relevant in this context. Bergstrom (1989) formalizes the game that rotten kids play with their altruistic parent: children are now the agents; each child's action impacts the production possibility set of the family. In the second stage, the parent has a fixed amount of money at her disposal and, after observing her kids' actions, determines monetary transfers to her offspring by maximizing her altruistic utility function. Children thus take into account how the parent will react to their actions when they choose their own actions. The Rotten Kid Theorem states that even if the children are completely selfish and care only about their own consumption, they will behave as if they are maximizing the parent's altruistic utility function. Bergstrom (1989)'s proof of the Rotten Kid Theorem requires TU, because he assumes that the parent treats every child's utility as a normal good in her altruistic utility function: Only if any action by a child, given the actions of all the other children results in a restricted utility possibility set in the form of a simplex are all the children guaranteed to benefit from taking efficient actions. In comparison to Bergstrom, we can weaken the requirement of TU to Almost TU by imposing a stronger, yet reasonable condition on the parent's altruistic utility function in the form of a general utilitarian welfare function. Gugl (2010) provides this analysis in a companion paper.

We are also aware that one may take the opposite view, seeing our result as a damnation of GUBS because Almost TU seems to be rarely satisfied in practice. In that case, the question becomes which class of bargaining solutions

¹⁶See Moulin (1988) for a link between strong monotonicity in voting rules and strategy proofness.

should take their place. The one that comes to mind immediately is the egalitarian solution (i.e., the solution that equally splits utility gains) or any other solution that plots a monotonic path through the disagreement point and pays no attention to the shape of the utility possibility set. It is obvious that such solutions satisfy the solidarity property and therefore incentive compatibility regardless of whether the utility profile leads to Almost TU or not.

However, such specifications are not without their own drawbacks. For example, consider a two stage game, in which agents can choose actions in the first stage that impact their disagreement utility as well as their joint production possibilities in the second stage. The second stage consists of the joint production and distribution of goods as modelled in this paper. The more bargaining solutions emphasize the disagreement point the more will agents inefficiently invest in their disagreement utility (Anbarci et al. 2002).

8 References

Anbarci, N, S. Skaperdas and C. Syropoulos, 2002, Comparing Bargaining Solutions in the Shadow of Conflict: How Norms against Threats Can Have Real Effects, *Journal of Economic Theory* 106, 1-16.

Baland, J-M and J. A. Robinson, 2000, Is Child Labor Efficient? *Journal of Political Economy* 108, 663-79.

Becker, Gary S., 1974, A Theory of Social Interactions, *Journal of Political Economy* 82, 1063-94.

Bergstrom, Th. C., 1989, A Fresh Look at the Rotten Kid Theorem – and Other Household Mysteries, *Journal of Political Economy* 97, 1138-59.

Bergstrom, Th. C. and R. C. Cornes, 1983, Independence of Allocative Efficiency from Distribution in the Theory of Public Goods, *Econometrica* 51, 1753-66.

Bergstrom, Th. C. and R. C. Cornes, 1981, Gorman and Musgrave are Dual: An Antipodean Theorem on Public Goods, *Economic Letters* 7, 371-378.

Bergstrom, Th. C. and H. R. Varian, 1985, When do market games have transferable utility? *Journal of Economic Theory* 35, 222-33.

Bommier, A. and P. Dubois, 2004, Rotten Parents and Child Labor, *Journal of Political Economy* 112, 240-248.

Chiappori, PA., B. Fortin, and G. Lacroix, 2002, Marriage Market, Divorce Legislation, and Household Labor Supply, *Journal of Political Economy* 110, 37-72.

Chiappori, PA. and I. Werning, 2002, Comment on rotten kids, purity, and perfection, *Journal of Political Economy* 110: 475-480.

Cornes, R.C. and E.C.D. Silva, 1999, Rotten kids, purity, and perfection, *Journal of Political Economy* 107: 1034-40.

Chun, Y. and W. Thomson, 1988, Monotonicity Properties of Bargaining Solutions When Applied to Economics” *Mathematical Social Sciences* 15, 11-27.

Gugl, E., 2009, Income splitting, specialization, and intra-family distribution, *Canadian Journal of Economics* 42, 1050-71.

Gugl, E., 2010, The Rotten Kid Theorem and Almost Transferable Utility, mimeo, University of Victoria.

Kalai, E. and M. Smorodinsky, 1975, Other Solutions to Nash's Bargaining Problem, *Econometrica* 43, 513-18.

Kolpin, V., 2006, The modelling and analysis of rotten kids, *Social Choice and Welfare* 26: 23-30.

Lundberg, Shelly J., Robert A. Pollak and Terence J. Wales, 1997, Do Husbands and Wives Pool Their Resources? Evidence from the United Kingdom Child Benefit, *Journal of Human Resources* 32, 463-480.

Mas-Colell, A., M. D. Whinston and J. R. Green, 1995, *Microeconomic Theory* (New York: Oxford University Press).

Moulin, H., 1988, *Axioms of Cooperative Decision Making* (Cambridge: Cambridge University Press).

Nash, J. F. Jr., 1950, The Bargaining Problem, *Econometrica* 18, 155-62.

Robert A. Pollak, forthcoming, Family Bargaining and Taxes: A Prolegomenon to the Analysis of Joint Taxation, in *Taxation and the Family*, Cignon A., P. Pestieau, and R. Rees (eds.) (Cambridge: MIT Press).

Rubalcava, Luis and Duncan Thomas, 2000, Family Bargaining and Welfare, Working Paper CCPR-007-00, California Center of Population Research.

Rubinstein, A., Z. Safra, and W. Thomson, 1992, On the Interpretation of the Nash Bargaining Solution and Its Extensions to Non-Expected Utility Preferences, *Econometrica* 60, 1171-81.

Sprumont, Y., 2008, Monotonicity and Solidarity Axioms in Economics and Game Theory, in *Rational Choice and Social Welfare*, Pattanaik, P.K.; Tadenuma, K.; Xu, Y.; Yoshihara, N. (eds.) (Berlin and Heidelberg: Springer) 71-94.

Xu, Y. and N. Yoshihara, 2008, The Behaviour of Solutions to Bargaining Problems on the Basis of Solidarity, *Japanese Economic Review* 59,133-138.

9 Appendix

9.1 Efficiency independent of distribution implies TU

In this section we provide the intuition for the result that if utility is ordinal, efficiency independent of distribution implies TU. This can be seen fairly easily in the case of all private goods by invoking the second fundamental theorem of welfare economics.¹⁷ Along the way we will also explicit the class of utility profiles leading to TU in case of only private goods. This characterization is due to Bergstrom and Varian (1985).

¹⁷The second fundamental theorem of welfare economics states the conditions under which any point on the utility possibility frontier can be reached by the competitive equilibrium with appropriate (re)distribution of the initial endowment. These conditions are met in our model. (See Mas Collet et al. 1995, p. 552-555.)

If efficiency is independent of distribution for every $Y \in \mathcal{Y}$, there must exist infinitely many Y for which interior efficiency holds. Applying the second fundamental theorem of welfare economics in those cases, implies that the equilibrium price vector is independent of distribution of the efficient product mix. To see this more formally, let $p \in \mathbb{R}_+^L$ be the equilibrium price vector and let $\omega \in X(y)$ be the initial endowment vector. For every interior efficient distribution of goods constituting a competitive equilibrium we must have

$$\sum_{i \in N} x_i(p, \omega_i) = y(p).$$

and by utility maximization

$$\frac{\frac{\partial u_i(x)}{\partial x_k}}{\frac{\partial u_i(x)}{\partial x_l}} = \frac{p_k}{p_l} \text{ for every } k, l \text{ and every } i$$

By efficiency being independent of distribution, for every $\omega \in X(y)$, the equilibrium price vector will always be the same. This means, reallocation of initial endowments at a fixed price vector and fixed product mix amounts to redistributing fixed income among agents. Then aggregate demand for each private good is independent of the distribution of income and it depends on the price vector and aggregate income only. That is,

$$\sum_{i \in N} x_i(p, I_i) = x(p, I)$$

where $I_i = p\omega_i$ and $I = \sum_{i \in N} I_i$. It is well known in consumer theory that this property is satisfied if and only if the income expansion paths of all consumers are parallel, straight lines. This property is satisfied if and only if agent's preferences admit indirect utility functions of the Gorman form (Mas Collé et al. 1995). An indirect utility function is in Gorman form if preferences for an individual i admit an indirect utility function written as

$$v_i(p, I) = a_i(p) + b(p)I$$

where p denotes the competitive price vector and I denotes the fixed income. (See e.g. Mas Collé et al. 1995, chapter 4). Since the Gorman form exhibits constant marginal utility of money, there is a constant trade-off between any two agents' utility on the utility possibility frontier whenever one unit of income is taken away from one agent and given to the other. Moreover, since the marginal utility of money, $b(p)$, is the same for all agents, the utility possibility set must be a simplex.

9.2 Utility Functions Satisfying TU

Bergstrom and Cornes (1981 and 1983) also give an exhaustive list of agents' utility functions that lead to TU in case of many $(L - 1)$ public goods and

one private good (x_1). Agents' utility functions must allow a form dual to the Gorman form of

$$u_i(x_{1i}, x_2, \dots, x_L) = A(x_2, \dots, x_L) x_{1i} + B_i((x_2, \dots, x_L))$$

For a different mix of public and private goods, i.e. let there be $L - K$ private goods and K public goods

$$u_i(x_{1i}, \dots, x_{(L-K)i}, x_{L-K+1}, \dots, x_L) = A(x_{L-K+1}, \dots, x_L) m(x_{1i}, \dots, x_{(L-K)i}, x_{L-K+1}) + B_i((x_{L-K+1}, \dots, x_L))$$

where $m(x_{1i}, \dots, x_{(L-K)i})$ must allow the indirect utility representation of the Gorman Form.

Remark 5 Chun and Thomson (1988) establish resource monotonicity in an l -good economy with two agents, when agents' utility functions are additively separable. This restriction is not enough to guarantee that the solidarity in production opportunities holds. While most utility functions that admit the indirect utility representation of the Gorman form are separable additive, there are many utility profiles of separably additive utility functions that do not lead to TU. For example, suppose all agents have separably additive utility functions that are also concave and homogenous of degree 1 over L private goods. In order to satisfy TU agents' utility functions also need to be identical.

9.3 Proof of Example 2

To find $(y, x) \in P(Y, u)$, first note that any efficient allocation must satisfy the following:

$$\begin{cases} y_1 = x_{11} + x_{12} \\ y_2 = x_{21} \\ y_3 = x_{32} \end{cases}$$

To find $P(Y, u)$ and $\partial U(Y, u)$ we solve:

$$\begin{aligned} & \max_{x_{11}, x_{12}, x_{21}, x_{32}} u(x_{11}, x_{21}) \\ & \text{s.t.} \quad \psi_2 = u(x_{12}, x_{32}) \\ & \quad \quad x_{11} + x_{12} + p_2 x_{21} + p_3 x_{32} = I \end{aligned}$$

This problem is equivalent to simultaneously

$$\begin{aligned} & \max_{x_{11}, x_{12}} u(x_{11}, x_{21}) \\ & \text{s.t.} \quad x_{11} + x_{12} + p_2 x_{21} = I_1 \end{aligned}$$

and

$$\begin{aligned} & \max_{x_{21}, x_{32}} u(x_{12}, x_{32}) \\ & \text{s.t.} \quad x_{11} + p_3 x_{32} = I_2 \end{aligned}$$

subject to

$$I_1 + I_2 = I.$$

It is obvious that the higher agent 1's share of I , the higher her utility. Thus each $\psi \in \partial U(Y, u)$ must be associated with a different distribution of I where agent 1 gains utility with every increase in her share and agent 2 loses utility with every decrease in his share. Homogeneity of degree one implies homotheticity which in turn implies that it is optimal for agent 1 to consume (x_{11}, x_{21}) in the same proportion as before as her share of I increases. The same is true for agent 2 with respect to (x_{12}, x_{32}) . By homogeneity of degree 1 of the utility functions, we also know that an increase in the share of I causes a proportional increase in u_i . Hence, the indirect utility function of agent 1 writes as follows

$$v(p_2, I_1) = \tilde{v}(p_2) I_1$$

where

$$\begin{aligned} \tilde{v}(p_2) &= \max_{x_{11}, x_{21}} u(x_{11}, x_{21}) \\ \text{s.t.} \quad &x_{11} + p_2 x_{21} = 1 \end{aligned}$$

Similarly, we can write the indirect utility function of agent 2 as

$$v(p_3, I_2) = \tilde{v}(p_3) I_2$$

where

$$\begin{aligned} \tilde{v}(p_3) &= \max_{x_{12}, x_{32}} u(x_{12}, x_{32}) \\ \text{s.t.} \quad &x_{12} + p_3 x_{32} = 1 \end{aligned}$$

Differentiating with respect to I_i :

$$\frac{\partial v_i}{\partial I_i} = \tilde{v}(p_{i+1}).$$

This implies that as agent 1's share of I increases by one unit, her utility increases by $\tilde{v}(p_2)$, and agent 2's utility decreases by $\tilde{v}(p_3)$. Independent of how many units of I are already allocated to person 1, the decrease in person 2's utility and the increase in person 1's utility will always be the same as person 1 receives an additional unit of I . Therefore any (ψ_1, ψ_2) is found by

$$\psi_2 = \lambda(Y, u_2) - \frac{\tilde{v}(p_3)}{\tilde{v}(p_2)} \psi_1.$$

Also note that

$$\frac{\tilde{v}(p_3)}{\tilde{v}(p_2)} = \frac{\tilde{v}(p_3) I}{\tilde{v}(p_2) I} = \frac{\lambda(Y, u_2)}{\lambda(Y, u_1)}.$$

Thus

$$\partial U(Y, u) = \left\{ \psi \in \mathbb{R}_+^2 \mid \frac{\psi_1}{\lambda(Y, u_1)} + \frac{\psi_2}{\lambda(Y, u_2)} = 1 \right\}.$$

Only in the special case in which $p_2 = p_3 = p$ such that $Y = \{y \in \mathbb{R}_+^3 | y_1 + p(y_2 + y_3) = I\}$ we have

$$\lambda(Y, u_1) = \tilde{v}(p)I = \lambda(Y, u_2).$$

Then

$$\partial U(Y, u) = \{\psi \in \mathbb{R}_+^2 | \psi_1 + \psi_2 = \lambda(Y, u_1) = \lambda(Y, u_2)\}.$$

9.4 Example illustrating the difference in resource allocation under NBS as cardinal properties of utility functions change

Consider a utility profile of two agents over two private goods given by

$$u_i(x_{1i}, x_{2i}) = x_{1i}^{1/3} x_{2i}^{2/3}$$

for all $i = 1, 2$.

Let there be three different production possibility sets give by Y_1, Y_2 , and Y_3 , respectively. Let

$$\lambda_s = \max_{y_1, y_2 \in Y_s} x_{1i}^{1/3} x_{2i}^{2/3}$$

where s denotes the specific production possibility set; $s = 1, 2, 3$. Suppose the efficient product mix for each production possibility set is given by

	Y_1	Y_2	Y_3
y_1	100	80	30
y_2	100	160	120
λ	100	126.99	75.595
MRS/MRT	$\frac{1}{2}$	1	2

The table also gives the corresponding value for λ and the marginal rate of substitution/ transformation at which interior efficiency holds. Then TU leads to a utility possibility frontier of

$$\psi_1 + \psi_2 = \lambda_s.$$

Take, for example, the Nash Bargaining Solution

$$\arg \max_{\psi_i} \sum \ln(\psi_i - d_i) \text{ s.t. } \sum_{i \in N} \psi_i = \lambda_s$$

and assume

$$d_1 = 1, d_2 = 3.$$

This leads to the following distribution of welfare and goods

	Y_1	Y_2	Y_3
ψ_1	49	62.495	36.798
ψ_2	51	64.495	38.798
x_{11}, x_{21}	49, 49	39.369, 78.738	14.603, 58.412
x_{12}, x_{22}	51, 51	40.631, 81.262	15.397, 61.588

Now consider two strictly concave transformations of the utility function given above. To be specific, assume

$$\begin{aligned} u_1(x_{11}, x_{21}) &= x_{11}^{1/6} x_{21}^{1/3}; \\ u_2(x_{12}, x_{22}) &= \frac{\ln x_{12} + 2 \ln x_{22}}{3} \end{aligned}$$

Applying the same transformations on the disagreement utility of each agent,

$$\begin{aligned} d_1 &= 1 \\ d_2 &= \ln 3 = 1.0986 \end{aligned}$$

Almost TU leads to a utility possibility frontier of

$$\psi_1^2 + \exp \psi_2 = \lambda_s$$

The Nash Bargaining solution is given by

$$\arg \max_{\psi_i} \sum \ln(\psi_i - d_i) \text{ s.t. } \psi_1^2 + \exp \psi_2 = \lambda_s$$

	Y_1	Y_2	Y_3
ψ_1	7.7354	8.8167	6.6282
ψ_2	3.6930	3.897	3.4551
x_{11}, x_{21}	59.836, 59.836	48.969, 97.938	17.435, 69.74
x_{12}, x_{22}	40.164, 40.164	31.031, 62.062	12.565, 50.26

This example illustrates how different resource allocation looks like when there is a change in the cardinal properties of the utility functions. Note that person 2 ends up with less of each good than person 1 after the concave transformations of the original utility functions, but received more of both goods than person 1 with the original utility profile. However, one can see that welfare for each person changes in the same direction as the production possibility set changes.