Intertemporal Choice

Chapter 4.2

Optimal Consumption Over Time

• So far have assumed that there is only one period
• Now consider two time periods, time 1 (today, this year, etc) and time 2 (tomorrow, a year from now, etc)
• Person has to decide about consumption today and tomorrow, or put differently how much income to spend at \( t=1, 2 \).
Assumptions

• Utility depends on consumption goods indexed by periods.
• Will assume that we have a two-stage problem.
• First decide how much to consume today and tomorrow,
• then decide in each period how much to consume of a certain good.
• Will focus only on first stage, how much to consume today and tomorrow, i.e. how much money to spend today and tomorrow.

The Model - Constraints

• In each period receive income $y_t$.
• Need to decide how much of the income received in a period consumer wants to spend on consumption ($c_1$, $c_2$) by saving or borrowing.
• Can borrow or save at interest rate $r$.
• Let $A_t$ be the amount the asset a consumer possess at the end of period $t$, with $A_t > 0$ for the saved amount.
The Model - Constraints

\[ p_1 c_1 = A_0 (1 + r) + y_1 - A_1 \]
\[ p_2 c_2 = A_1 (1 + r) + y_2 \]
\[ p_1 c_1 + \frac{p_2}{1+r} c_2 = A_0 (1 + r) + y_1 + \frac{1}{1+r} y_2 \]
\[ W_1 = A_0 (1 + r) + y_1 + \frac{1}{1+r} y_2 \]

\( W_1 \) is present value of consumer’s endowed income time-stream.

Wealth Line

\[ \frac{W_1}{p_1} \]
\[ \frac{(A_0(1+r)+y_1)}{p_1} \]
\[ \frac{y_2}{p_2} \]
\[ \frac{W_1(1+r)/p_2}{p_2/(1+r)p_1} \]
Analyzing Intertemporal Consumption

- Set up UMP as
  \[ \max_{c_1, c_2, \lambda} u(c_1, c_2) + \lambda \left( W_1 - p_1 c_1 - \frac{p_2}{1+r} c_2 \right) \]
- Note corner solution is unreasonable to assume if person wants to survive!
- Need again quasi-concave and increasing utility function for FOC’s to be necessary and sufficient.

Indifference curves in intertemporal decision problem

Consumption in \( t=1 \)

Better set

utility increases with more consumption in both periods

Consumption in \( t=2 \)
Solving for the Consumption function (Marshallian demand)

\[
\max_{c_1, c_2, \lambda} u(c_1, c_2) + \lambda \left( W_1 - p_1 c_1 - \frac{p_2}{1 + r} c_2 \right)
\]

\[
\frac{\partial u}{\partial c_1} - \lambda p_1 = 0
\]

\[
\frac{\partial u}{\partial c_2} - \lambda \frac{p_2}{1 + r} = 0
\]

Solving for the Consumption Function

- Tangency condition:
  \[
  \frac{\partial u}{\partial c_2} \bigg|_{c_1} = \frac{p_2}{(1 + r)p_1}
  \]
  - The marginal rate of substitution between consumption tomorrow and consumption today is equal to the price ratio times the discount rate. It is equal to the opportunity cost of purchasing an additional unit of the consumption good tomorrow.
  - MRS is also called the person’s rate of time preference
  - Solution in this context is called consumption function and given by \( c^* \)
  - Value function of UMP is indirect utility function \( v(p, r, y, A_0) = u(c^*_e(p, r, y, A_0), c^*_x(p, r, y, A_0)) \)
Intertemporal Consumption and Optimal Savings decision

A₁ = 0; person neither saves nor borrows

Intertemporal Consumption and Optimal Savings decision

A₁ < 0; person borrows
Intertemporal Consumption and Optimal Savings decision

$W_1/p_1$ Present consumption

$y_2/p_2$ Future consumption

$A_1 > 0$; person saves

Comparative statics

- How do changes in $p$, $r$, $y$, $A_0$ change the consumer’s welfare?
- How do these changes impact intertemporal consumption?
Changes in $p$

- An increase in the price of current consumption has the same impact in the intertemporal model as it has in our basic consumer model: Given interior solution, the person is worse off.
- Same in case of increase in price of future consumption.

Changes in $p$

- The impact of an increase in the price of current consumption on the consumption function can as usually be decomposed into a substitution effect and an income effect.
- Same for increase in price of future consumption.
Change in $r$

- Changes in the interest rate have a less straightforward impact on the consumer’s welfare than an increase in $p$.
- We can use the Envelope Theorem to say more.
- First we’ll do a graphical analysis.

Interest rate changes, $A_0 = 0$

Person who neither saved nor borrowed is better off as $r$ increases (red b.c.) or decreases (blue b.c.)
Interest rate changes, $A_0 = 0$

Person who borrowed is worse off as $r$ increases (red b.c.) and better off as $r$ decreases (blue b.c.)

Person who saved is better off as $r$ increases (red b.c.) and worse off as $r$ decreases (blue b.c.)
Interest rate changes, $A_0 > 0$

Person who neither saved nor borrowed is better off as $r$ increases (red b.c.) but worse off as $r$ decreases (blue b.c.)

Person who borrowed is as well off as $r$ increases (red b.c.) and better off as $r$ decreases (blue b.c.)
Interest rate changes, $A_0 > 0$

Person who saved is better off as $r$ increases (red b.c.) and worse off as $r$ decreases (blue b.c.)

Interest rate changes, $A_0 < 0$

Person who neither saved nor borrowed is worse off as $r$ increases (red b.c.) and better off as $r$ decreases (blue b.c.)
Interest rate changes, $A_0 < 0$

Person who borrowed is worse off as $r$ increases (red b.c.) and better off as $r$ decreases (blue b.c.)

Person who saved is as well off as $r$ increases (red b.c.) and worse off as $r$ decreases (blue b.c.)
Systematically analyzing changes in interest rate

- Whether person is better off or worse off with an increase in the interest rate depends on where the person was consuming before the interest rate change and the sign and relative importance of $A_0$.
- If $A_0 = 0$ and she was a borrower an increase in the interest rate may hurt her. But a further increase may benefit her.
- Utility can go up or down with an increase in $r$!

Using the Envelope Theorem

- Let’s investigate the impact of $r$ on $v(p,r,y,A_0)$ further by taking the derivative of $v$ wrt $r$.

$$L(c(p,r,y,A_0),\lambda(p,r,y,A_0)) = u(c^*) + \lambda^* \left[ A_0(1+r) + y_1 + \frac{y_2}{1+r} - p_1 c_1^* - \frac{p_2}{1+r} c_2^* \right]$$

$$\frac{\partial v}{\partial r} = \frac{\partial L}{\partial r} = \lambda^* \left[ A_0 - \frac{y_2}{(1+r)^2} + \frac{p_2}{(1+r)^2} c_2^* \right]$$

$$= \lambda^* \left[ A_0 + \frac{A_1}{(1+r)} \right]$$
Using the Envelope Theorem when $A_0 = 0$

Person is worse off with a small increase in $r$ as long as she keeps borrowing, since $A_1 < 0$.

Person is better off with a small increase in $r$ if she is a saver, since $A_1 > 0$.

Unlike an increase in the wage rate in the labour supply model, an increase in the interest rate can hurt a person!

Is there a worst interest rate?

- What if $A_1 = 0$? What does it mean that utility doesn’t change in this case?
- It means that we have reached the interest rate that is the worst for the person.
- Think about it. In this case the person does not improve her utility compared to the case where she has no access to the credit market.
Decomposing the interest rate effect with $A_0 = 0$

- Income and Substitution effect go in opposite directions for consumption today if the person is a saver and consumption today is a normal good.
- Income and Substitution effect go in the same direction for consumption today if the person is a borrower and consumption today is a normal good.
- The amount saved can increase or decrease with an increase in $r$, but the amount borrowed must decrease with an increase in $r$ if consumption today is a normal good.

CMP and the Slutsky equation

- Again, the Slutsky equation is extremely useful to pin down the income and substitution effects as well as helping us come up with testable predictions of the substitution matrix.
- Need to solve the expenditure minimization problem first.
Note that we minimize wealth at $t=1$!

\[
\begin{align*}
\min_{c_1,c_2,u} & \ p_1 c_1 + \frac{p_2}{1+r} c_2 + \mu(u - u(c_1,c_2)) \\
p_1 - \mu \frac{\partial u}{\partial c_1} &= 0 \\
p_2 \frac{p_2}{1+r} - \mu \frac{\partial u}{\partial c_2} &= 0
\end{align*}
\]

Solution to CMP

- Find Hicksian demand this way. Denoted by $h_t(p,r,u)$
- Value function of EMP is minimized discounted wealth function $c(p,r,u)=h_1 p_1 + h_2 p_2/(1+r)$
- Again, we can use the envelope theorem to find the impact of a change in the interest rate on the minimum discounted wealth required to reach utility level $u$. 
Finding $h_2$ from cost function

\[ \frac{\partial c(p,r,u)}{\partial r} = -\frac{p_2}{(1+r)^2} h_2 \]

- The higher the interest rate, the lower the discounted wealth at time $t=1$. 

Slutsky Equation

\[ c_i^*(p,r,y,A_0) = h_i\left( p,r,v\left(p,r,y,A_0\right)\right) \]

\[ \frac{\partial c_i}{\partial r} = \frac{\partial h_i}{\partial r} + \frac{\partial h_i}{\partial u} \frac{\partial v}{\partial r} \]

recall \[ \frac{\partial v}{\partial r} = \lambda^* \left[ A_0 + \frac{A_1}{(1+r)} \right] \]

\[ c_i^\ast\left(p,\frac{p_2}{1+r},W_i\right) = h_i\left( p,r,v\left(p,\frac{p_2}{1+r},W_1\right)\right) \]

\[ \frac{\partial c_i}{\partial W_i} = \frac{\partial h_i}{\partial u} \frac{\partial v}{\partial W_i} \]
Slutsky Equation

• By Envelope Theorem \( \frac{\partial v}{\partial W_i} = \lambda^* \)
• Hence \( \frac{\partial h_i}{\partial u} = \frac{1}{\lambda^*} \frac{\partial c_i}{\partial W_1} \)

Putting all this together

\[
\frac{\partial c_i}{\partial r} = \frac{\partial h_i}{\partial r} + \frac{\partial c_i}{\partial u} \frac{\partial v}{\partial r} \\
\frac{\partial c_i}{\partial r} = \frac{\partial h_i}{\partial r} + \frac{\partial c_i}{\partial W_i} \frac{\lambda^*}{\lambda^*} \left[ A_0 + \frac{A_1}{1 + r} \right] \\
\frac{\partial h_i}{\partial r} = \frac{\partial c_i}{\partial r} - \frac{\partial c_i}{\partial W_i} \left[ A_0 + \frac{A_1}{1 + r} \right]
\]
Consumption tax vs. Income tax

• In one-period model, there is no difference between a consumption tax and an income tax:
• For any consumption tax \( T \), we can find an income tax \( t \), such that

\[
\begin{align*}
  p(1 + \tau) y &= wz + M \\
  py &= (1 - t)(wz + M) \\
  \Rightarrow (1 - t) &= \frac{1}{1 + \tau} \\
  \Rightarrow \tau &= \frac{t}{1 - t}
\end{align*}
\]

Difference shows up in intertemporal setting

• A tax on saving has the same effect as lowering the interest rate; will change intertemporal budget set.
  – Assume no credit constraints
  – Assume borrowing interest rate is same as saving (lending) interest rate
  – Imagine government imposes a tax of \( t \) on interest income
    • Affects slope of wealth line to left of endowment point
• A consumption tax taxes consumption today and tomorrow the same way and therefore does not change the slope of the wealth line.
Difference bw consumption tax and income tax

• Consumption tax

\[(1 + \tau)p_1c_1 + (1 + \tau)p_2 \frac{c_2}{(1 + r)} = W_1\]

• Income tax

\[p_1c_1 = (1 - t)y_1 + A_0(1 + r) - trA_0 - A_1\]
\[p_2c_2 = (1 - t)y_2 + (A_1(1 + r) - trA_1)\]
\[p_1c_1 + \frac{p_2}{(1 + (1 - t)r)} = (A_0(1 + (1 - t)r) + (1 - t)\left(\frac{y_1 + \frac{y_2}{1 + (1 - t)r}}{1 + (1 - t)r}\right) = W_1(t)\]

Effect of Tax on Interest Income

• Tax on interest income lowers relative return to lenders
• If \(p_2/(1+\tau)\) is relative price of consumption tomorrow, then taxing income lowers relative price of consumption today
  – Substitution effect: consume more today
  – Income effect (you’re poorer now): consume less today (assuming \(c_1\) is normal good)
  – Overall effect on lenders is ambiguous; if substitution effect dominates, saving falls
  – No effect on people who were borrowers before the tax was applied (the change in the intertemporal be doesn’t affect them)
**Tax on Interest Income (with deductibility of interest payments)**

- In some cases, interest payments are tax deductible
  - Interest payments on Student Loans in Canada
  - Business interest payments in Canada and US
- This causes whole budget constraint to pivot around the endowment point, reflecting the fact that borrowing becomes cheaper as a result of the deductibility of interest income

**DWL with Income Tax**

- Income tax by taxing capital gains (i.e. interest) changes the opportunity cost of spending an additional $ today. This causes a substitution effect (unless consumption in each period are perfect complements)
- Thus the income tax creates DWL.
- Economists favour consumption tax over income tax for that reason.
  - Explains opposition of conservative economists to Harper government’s decrease in GST.