# UNIVERSITY OF VICTORIA
# DECEMBER EXAMINATIONS 2009

## MATH 122: Logic and Foundations

| Instructor and section (check one): | | | |
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NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers should be written on the exam paper.

The exam consists of 26 questions, for a total of 70 marks. Please show all of your work and justify your answers when appropriate.

There are 11 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [2] Suppose the statement $p \leftrightarrow q$ is FALSE. Circle all statements guaranteed to be TRUE. No justification is needed.

   - $\neg q \leftrightarrow p$
   - $\neg p \rightarrow q$
   - $\neg(p \wedge q)$
   - $\neg(p \vee q)$

2. [3] If the argument below is valid, then prove it, citing logical equivalences and inference rules. Otherwise, give a counterexample to show that the argument is invalid.

$$\begin{array}{ll} & p \leftrightarrow \neg q \\ & r \rightarrow q \\ & p \\ \hline \therefore & \neg r \end{array}$$

3. Consider the statement:

*There exists an integer $n$ which is larger than all other integers.*

   (a) [2] Write it using quantifiers and other mathematical symbols.

   (b) [2] Write its negation in words.

4. [2] Let $A = \{\emptyset, \{1, 2, 3\}, \{\emptyset, \{\emptyset\}\}, 5, \{1\}\}$. Circle all TRUE statements. No justification is needed.

   - $\{1\} \subseteq A$.
   - $\{\emptyset\} \in A$.
   - $\{\emptyset\} \in \mathcal{P}(A)$.
   - $|A| = 4$.

5. [3] Let $A, B, C$ and $D$ be sets. Prove that if $A \subseteq C$ and $B \subseteq D$, then $A \cap B \subseteq C \cap D$.

6. [3] Use the laws of set theory and known equalities to express the symmetric difference $A \oplus B$ using ONLY the operations of intersection and complement.

7. [2] Let $R$ be a reflexive relation on a set $A$ and let $D = \{(a, a) : a \in A\}$. Order the following sets by set inclusion: $R, D$ and $A \times A$. (Write your answer in the form $X \subseteq Y \subseteq Z$; no justification is needed.)

8. [2] Let $f : A \to B$ be an invertible function, where $A$ and $B$ are two different sets. Circle all TRUE statements. No justification is needed.

- The range of $f$ is $B$.
- $f \circ f^{-1} = i_A$, the identity on $A$.
- $(f^{-1})^{-1} = f$.
- For every 1-1 function $g : B \to C$, $g \circ f$ is also 1-1.

9. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} x & \text{if } x \leq 0 \\ x^2 & \text{if } x > 0. \end{cases}$$

(a) [2] Prove that $h$ is onto. (Hint: Let $y \in \mathbb{R}$ and consider the cases $y \leq 0$ and $y > 0$ separately.)

(b) [1] Is $h$ 1-1? No justification is needed.

10. [2] Circle all COUNTABLY INFINITE sets. No justification is needed.

- $\{2k + 1 : k \in \mathbb{N}\}$
- $\mathbb{Q} \times \mathbb{Z}$
- the interval $[2, 3)$
- the finite-length binary sequences $\{0, 1, 00, 01, 10, 11, 000, \ldots\}$

11. [3] Consider the set $T$ of all infinite ternary sequences, that is, sequences of the form $\mathbf{t} = (t_1, t_2, t_3, t_4, \ldots)$, where each term $t_i \in \{0, 1, 2\}$. Prove that $T$ is uncountable.

12. [2] Circle all TRUE statements. No justification is needed.

- If $p$ is prime, then $\gcd(p, n) = 1$ or $p$, for any $n \in \mathbb{Z}$.
- For all $a, b, d \in \mathbb{Z}$, if $d \mid ab$ then $d \mid a$ or $d \mid b$.
- $-3 + 6 \cdot 8 \equiv 10 \pmod 5$.
- The Division Algorithm implies that every integer is congruent (mod 6) to one of 0, 1, 2, 3, 4, or 5.

13. Let $R$ be the 'divides' relation on $\mathbb{Z}$ defined by $(a, b) \in R$ if and only if $a \mid b$.

    (a) [3] Prove that $R$ is transitive.

    (b) [1] Is $R$ anti-symmetric? Why or why not?

14. [3] Let $a = 378$ and $b = 112$. Find $\gcd(a, b)$ and $\operatorname{lcm}(a, b)$.

15. (a) [2] Let $n = (abc)_9$. Show that $n \equiv c \pmod 3$.

(b) [1] Complete this statement for the integer $n = (abc)_9$ above: $9 \mid n$ *if and only if* ...

16. [2] Which theorem implies that the equation $7^x = 3^y$ has no positive integer solutions $x, y$? Why?

17. [2] Give a recursive definition for the geometric sequence $a_0, a_1, a_2, a_3 \ldots = 3, -6, 12, -24, \ldots$.

18. Define a sequence $b_n$ by $b_1 = 2$, and for $k > 1$,

$$b_k = b_{\lfloor k/2 \rfloor} + b_{\lceil k/2 \rceil}.$$

(a) [1] Find $b_{10}$.

(b) [1] Guess a formula for $b_n$ that is valid for all $n \geq 1$.

(c) [3] Use strong induction to prove that the formula you guessed above is correct. The fact that $\lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$ *for all* $k \in \mathbb{N}$ can be used without proof.

19. [2] Suppose that we throw red checkers and black checkers into 15 jars. What is the minimum number of checkers that must be used in order to guarantee that there are two checkers OF THE SAME COLOUR in some jar? Explain.

20. [3] Let $A = \{a_1, a_2, \ldots, a_7\}$ and $B = \{b_1, b_2, \ldots, b_{13}\}$. Complete each statement. No justification is necessary.

(a) The number of subsets of $A$ is ____________

(b) The number of 3-element subsets of $B$ is ____________

(c) If $A$ and $B$ are disjoint then $|A \cup B| =$ ____________

(d) The number of functions $f : A \to B$ is ____________

(e) The number of 1-1, onto functions $g : A \to A$ is ____________

(f) The number of binary relations from $A$ to $B$ is ____________

21. [2] Find the number of six-digit positive integers that can be formed using the digits 1, 2, 3, 4, and 5 (each of which may be repeated) if the number formed must begin with two even digits or begin with two odd digits.

22. [2] How many arrangements of the letters in NEWSPAPERS have all vowels together?

23. [3] In a lunch room of 30 students, 17 have juice, 20 have sandwiches, and 12 have cookies. Exactly 8 have juice and sandwiches, 8 have juice and cookies, 7 have sandwiches and cookies, and 4 have all three items. TWO students are selected at random. What is the probability that both students chosen have at least one of these items?

24. [3] Use induction to prove that $2^n \leq n!$ for $n \geq 4$.

25. Define $f : \mathbb{N} \to \mathbb{R}$ by

$$f(n) = \begin{cases} 3n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

(a) [2] Show that $f$ is $\mathcal{O}(n)$.

(b) [1] Is it true that $g(n) = n$ is $\mathcal{O}(f)$? Why or why not?

26. [2] Let $f_1(n) = 2^{(n^2)}$, $f_2(n) = n!$, $f_3(n) = n^n$, $f_4(n) = 100^n$. Order these functions as $f_{i_1} \prec f_{i_2} \prec f_{i_3} \prec f_{i_4}$, that is, so that each function has strictly smaller order than those on its right. No justification is required.

– END OF EXAMINATION –