UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2010

MATH 122: Logic and Foundations

Instructor: Gary MacGillivray.

Check the box corresponding to your Section:

☐ [A01] (10:30 AM) CRN 14261
☐ [A02] (11:30 AM) CRN 14262

NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers should be written on the exam paper.

The exam consists of 26 questions, for a total of 70 marks. Please show all of your work and justify your answers when appropriate.

There are 11 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

The only calculator permitted is the Sharp EL 510-R

1. [2] Suppose the statement $p \to \neg q$ is FALSE. Circle all statements guaranteed to be TRUE. No justification is needed.

   - $\neg(q \leftrightarrow p)$
   - $\neg p \to q$
   - $\neg p \wedge q$
   - $\neg(p \vee q)$

2. [3] If the argument below is valid, then prove it, citing logical equivalences and inference rules. Otherwise, give a counterexample to show that the argument is invalid.

$$\begin{array}{rl} & p \to \neg q \\ & \neg r \vee q \\ & \neg r \\ \hline \therefore & \neg p \end{array}$$

3. Suppose the universe of n and t is the integers. Consider the statement:

$$\exists n, \forall t, n + t \leq 10.$$

   (a) [2] Write the negation of the statement using quantifiers and other mathematical symbols.

   (b) [2] Determine the truth value of the given statement. Briefly justify your answer.

4. [2] Let $A = \{\emptyset, \{1, 2, 3\}, \{\emptyset, \{\emptyset\}\}, 5, \{1\}\}$. Circle all TRUE statements. No justification is needed.

   - $3 \in A$.
   - $\{\{\emptyset, \{\emptyset\}\}\} \subseteq A$.
   - $\{\emptyset\} \in A$.
   - $|\mathcal{P}(A)| = 2^9$.

5. [3] Use known logical equivalences to obtain a statement logically equivalent to $p \leftrightarrow q$ that involves ONLY the statements p and q, their negations, and the logical connective $\wedge$.

6. [3] Let A and B be sets. Prove that $(A \cap B)^c = A^c \cup B^c$, where X^c is the complement of the set X.

7. [2] Let R be a symmetric, and transitive relation on the set $A = \{1, 2, 3\}$. Prove that if $(1, 2), (1, 3) \in R$, then $R = A \times A$.

8. [2] Let $f : A \to B$ be an invertible function, where A and B are two different sets. Circle all TRUE statements. No justification is needed.

- The range of f^{-1} is A.
- $f^{-1} \circ f = i_A$, the identity function on A.
- $|A| = |B|$.
- For every invertible function $g : B \to C$, the function $g \circ f$ is also invertible.

9. Define $h : \mathbb{R} \to \mathbb{R}$ by

$$h(x) = \begin{cases} 2x & \text{if } x \le 0 \\ 3x^2 & \text{if } x > 0. \end{cases}$$

(a) [2] Prove that h is onto. (Hint: Let $y \in \mathbb{R}$ and consider the cases $y \le 0$ and $y > 0$ separately.)

(b) [1] Is h 1-1? No justification is needed.

10. [2] Circle all UNCOUNTABLE sets. No justification is needed.

- $\mathbb{N} \times \mathbb{R}$
- $\{\mathbb{R}, \mathbb{Q}, \mathbb{Z}\}$
- the interval $(-\infty, 4)$
- $\{x \in \mathbb{R} : x^2 + 3 = 0\}$

11. [3] Prove that the interval $(0, 1)$ of real numbers is uncountable.

12. [2] Circle all TRUE statements. No justification is needed.

- If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.
- For $a, b, q, r \in \mathbb{Z}$, if $a = bq + r$ then $gcd(a, b) = gcd(b, r)$.
- No power of 10 is a multiple of 28.
- For all $a, b, d \in \mathbb{Z}$, if $a \equiv b \pmod{n}$ and $b \equiv d \pmod{n}$ then $a^2 \equiv bd \pmod{n}$.

13. [4] Let A be a set and R be the 'subset' relation on $\mathcal{P}(A)$, that is, $(X, Y) \in R$ if and only if $X \subseteq Y$. Prove that R is anti-symmetric and transitive.

14. [3] Let $a = 473$ and $b = 242$. Find $\gcd(a, b)$ using the Euclidean algorithm. and express it as $ax + by$ for some integers x and y.

15. (a) [2] Let $n = (abc)_5$. Show that $n \equiv a + b + c \pmod 4$.

(b) [1] Complete this statement for the integer $n = (abc)_5$ above: $4 \mid n$ *if and only if* ...

16. Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$, and $a_n = 2 + a_{n-1}$ for $n \geq 1$.

(a) [1] Compute a_1, a_2, a_3 and a_4, and then guess a formula for a_n that is valid for all $n \geq 0$.

(b) [2] Use induction to prove that your guess is correct.

17. Define a sequence b_n by $b_0 = 5, b_1 = 9$, and for $n > 1$,

$$b_n = b_{n-1} + b_{n-2}.$$

(a) [3] Use strong induction to prove that $b_n < 5 \cdot 2^n$ for all $n \geq 1$.

(b) [1] Prove that the function $f : \mathbb{N} \to \mathbb{R}$ defined by $f(n) = b_n$ is $\mathcal{O}(2^n)$.

18. [2] Prove that $\lfloor \frac{k}{2} \rfloor + \lceil \frac{k}{2} \rceil = k$ for all $k \in \mathbb{Z}$.

19. [3] Let $A = \{a_1, a_2, \ldots, a_{10}\}$ and $B = \{b_1, b_2, \ldots, b_6\}$. Complete each statement. No justification is necessary.

(a) The number of subsets of B is ____________

(b) The number of 5-element subsets of A is ____________

(c) If $|A \cap B| = 3$ then $|A \cup B| =$ ____________

(d) The number of functions $f : A \to B$ is ____________

(e) The number of 1-1, onto functions $g : A \to A$ is ____________

(f) The number of binary relations from A to B is ____________

20. [2] The twelve members of a motorcycle club all have different motorcycles. Four of them ride motorcycles made by Kawasaki, three ride Hondas and five ride BMWs. How many ways can their 12 bikes be parked in a single line at the coffee shop if the first two bikes in the line must be made by the same manufacturer.

21. [2] What is the probability that a randomly selected binary sequence of length 9 has exactly 4 ones?

22. [2] In a lunch room of 30 students, 17 have juice, 20 have sandwiches, and 12 have cookies. Exactly 8 have juice and sandwiches, 8 have juice and cookies, 7 have sandwiches and cookies, and 4 have all three items. How many students have at least one of these items?

23. Consider the statement: $3 \mid 2^{2n} - 1$ for all $n \geq 0$.

    (a) [3] Prove the given statement using mathematical induction.

    (b) [1] Prove the statement again using congruences.

24. Suppose $a, b \in \mathbb{N}$.

 (a) [2] Prove that if $c = ax + by$ for some integers x and y, then $gcd(a, b) \mid c$.

 (b) [1] Is the converse true? No justification is necessary.

25. [2] Show that $f(n) = n^3$ is not $\mathcal{O}(n^2)$.

26. [2] Consider the function $f(n) = 5\sqrt{n} - 10n\log(n) + 15\log(n)^2$. Circle all true statements. No reasons are necessary.

 (a) f is $\mathcal{O}(n^2)$.
 (b) $h(n) = \sqrt{n}$ is $\mathcal{O}(f)$.
 (c) f is $\mathcal{O}(n^n)$.
 (d) f is $\mathcal{O}(f)$.

– END OF EXAMINATION –