UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2011

MATH 122: Logic and Foundations

Instructor and section (check one):			
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NAME: ____

V00#: ____

Duration: 3 Hours.

Answers should be written on the exam paper.

The exam consists of 28 questions, for a total of 70 marks. Please show all of your work and justify your answers when appropriate.

There are 11 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

The only calculator allowed is the Sharp EL 510-R.

1. [2] Show that $(p \wedge q) \vee \neg(p \vee \neg q)$ is logically equivalent to q using basic known equivalences.

2. [3] Prove the following logical argument, giving a list of statements and reasons.

$$\begin{array}{ll} & p \\ & p \to (q \vee s) \\ & s \to r \\ \hline \therefore & q \vee r \end{array}$$

#	statement	reason
1.	p	premise
2.	$p \to (q \vee s)$	premise
3.	$s \to r$	premise

3. [3] Give an assignment of truth values which shows the following argument is invalid.

$$\begin{array}{rl} & p \leftrightarrow \neg r \\ & q \to p \\ & q \vee r \\ \hline \therefore & q \end{array}$$

4. Consider the statement:

$$\forall\, S \subseteq \mathbb{N},\ [(S \neq \emptyset) \to (\exists\, x \in S, \forall\, y \in S,\ x \leq y)].$$

(a) [1] Write the statement in words, with no symbols.

(b) [1] The statement is true. What is the name it is commonly known by?

(c) [1] If $\mathbb{N}$ is changed to $\mathbb{Z}$, the statement becomes false. Why?

(d) [1] Mark the statement which is equivalent to the negation of the one obtained by changing $\mathbb{N}$ to $\mathbb{Z}$, as in (c):

- ○ $\forall\, S \subseteq \mathbb{Z},\ \neg[(S \neq \emptyset) \to (\exists\, x \in S, \forall\, y \in S,\ x \leq y)]$.
- ○ $\exists\, S \subseteq \mathbb{Z},\ [(S \neq \emptyset) \wedge (\forall\, x \in S, \exists\, y \in S,\ x > y)]$.
- ○ $\forall\, S \subseteq \mathbb{Z},\ [\neg(S \neq \emptyset) \vee (\exists\, x \in S, \forall\, y \in S,\ x > y)]$.

5. [2] Let $A = \{1, 2, \{1, 2, 3\}, \{\emptyset, \{\emptyset\}\}, \{1\}\}$. Check all TRUE statements. No justification is needed.

- ☐ $\{1, 2\} \subseteq A$.
- ☐ $\emptyset \in A$.
- ☐ $A \oplus A = \emptyset$.
- ☐ $|A| = 8$.

6. (a) [1] For a set A, define the power set $\mathcal{P}(A)$.

   (b) [1] Is it true, for all sets A and B, that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ implies $A \subseteq B$? No justification is needed.

7. [2] Prove the following one of DeMorgan's Laws for sets: $(A \cap B)^c = A^c \cup B^c$.

8. [1] Suppose $A = \{1, 2, \ldots, 6\}$, and R is a binary relation on A. The statement

If R is both symmetric and anti-symmetric, then R must equal $\emptyset$.

is (no justification needed)

- ○ true
- ○ false

9. [3] Let:
R_1 be the relation on $\mathbb{Z}$ defined by $(a, b) \in R_1$ if and only if $a \equiv b \pmod 3$,
R_2 be the relation on $\mathbb{R}$ defined by $(a, b) \in R_2$ if and only if $a \geq b$,
R_3 be the relation on $\{1, 2, 3\}$ defined by $R_3 = \{(1, 1), (2, 2), (1, 3), (3, 1)\}$.

Fill the table with check marks to indicate which relations have which properties. No justification is needed.

	R_1	R_2	R_3
reflexive			
symmetric			
anti-symmetric			
transitive			

10. [2] Let $f : A \to B$ be an invertible function, where A and B are two different sets. Check the TRUE statements. No justification is needed.

- ☐ $f^{-1} \circ f = \iota_B$, the identity on B.
- ☐ $(f^{-1})^{-1} = f$.
- ☐ $|A| = |B|$.
- ☐ For every 1-1 function $g : B \to C$, $g \circ f$ is also 1-1.

11. Define $h : \mathbb{R} \to \mathbb{Z}$ by
$$h(x) = \lfloor \sqrt[3]{x} \rfloor.$$

(a) [2] Prove that h is onto.

(b) [1] Is h 1-1?

12. (a) [2] Prove that if $f : A \to B$ is 1-1 and $g : B \to C$ is 1-1, then $g \circ f : A \to C$ is 1-1.

(b) [1] Is the converse to this statement true? No justification is necessary.

13. [3] Consider the set W of all infinite sequences of the form $\mathbf{w} = w_1 w_2 w_3 w_4 \ldots$, where each w_i is a letter in $\{\mathtt{a}, \mathtt{b}, \mathtt{c}, \mathtt{d}\}$. Prove that W is uncountable.

14. [2] Check the COUNTABLY INFINITE sets. No justification is needed.

- ☐ $\{k^2 : k \in \mathbb{N}\}$
- ☐ $\{\pi x : x \in \mathbb{Q}\}$
- ☐ the interval $[0, \frac{1}{100})$
- ☐ $\mathcal{P}(\mathbb{N})$

15. [1] Which of the following is the <u>remainder</u> when the division algorithm is applied to positive integers a and b (with, as usual, b being the divis<u>or</u>)?

○ $\lfloor \frac{a}{b} \rfloor$ ○ $a - b\lfloor \frac{a}{b} \rfloor$ ○ $a - b\lceil \frac{a}{b} \rceil$ ○ $b\lceil \frac{a}{b} \rceil - a$

16. (a) [2] Suppose $d \mid a$ and $d \mid b$. Prove that $d \mid ax + by$ for any integers x and y.

(b) [1] Suppose there exist integers x, y such that $ax + by = 1$. Prove that a and b are relatively prime.

17. [2] Let $a = 396$ and $b = 154$. Find $\gcd(a, b)$ using the Euclidean Algorithm.

18. Let $n = (a_2 a_1 a_0)_6$.

(a) [1] Give a formula for n in terms of a_0, a_1, a_2.

(b) [2] Show that $n \equiv a_0 \pmod 3$.

19. [2] Suppose $m = 2^a 3^b$, $n = 10^c$, and $\text{lcm}(m, n) = 360$. Solve for the integers a, b, c.

20. [1] Let n be a positive integer. Use congruences to prove that $7 \mid 16^n - 9^n$.

21. [3] Consider the sequence $a_0, a_1, a_2, a_3, \ldots$ defined by $a_0 = 0$ and $a_n = a_{n-1} + (2n - 1)$ for $n \geq 1$. Prove $a_n = n^2$ for all $n \geq 0$.

22. (a) [3] Prove that $n^2 < 3 \cdot 2^n$ for all $n \geq 1$.

(b) [1] Use part (a) to argue that the function $f(n) = n^2$ is $O(2^n)$.

23. [2] Let $f : \mathbb{N} \to \mathbb{R}$ be defined by

$$f(n) = \begin{cases} n^2 \cdot 2^n - 100^n + n! + 10^{10} \log n & \text{if } n \text{ is even} \\ 1 & \text{if } n \text{ is odd.} \end{cases}$$

Check the TRUE statements. No justification is needed.

- ☐ f is $O(2^n)$.
- ☐ f is $O(n!)$.
- ☐ f is $O(f)$.
- ☐ $g(n) = \log n$ is $O(f)$.

24. [3] Define the sequence $c_0, c_1, c_2, \ldots$ by $c_0 = 1$, $c_1 = 3$, and, for all $n \geq 2$, $c_n = 2c_{n-1} + 3c_{n-2}$. Prove that $c_n = 3^n$ for all $n \geq 0$.

25. [3] In a lunch room of 40 students, 17 have juice, 20 have sandwiches, and 12 have cookies. Exactly 8 have juice and sandwiches, 8 have juice and cookies, 7 have sandwiches and cookies, and 4 have all three items. What is the probability that a chosen student chosen has at least one of these items?

26. [2] Twelve people are lining up for an eye exam. Five of the people have brown eyes, four have green eyes, and three have blue eyes. In how many of the possible line-ups do the first two people have the same eye colour?

27. [3] How many arrangements of BANANABREAD have no two consecutive "A"s?

28. [3] Let A be a set with $|A| = 7$. Place the following numbers in the blanks provided.

$$2^7, \quad 2^{49}, \quad \binom{7}{2}, \quad 7!, \quad \frac{7!}{(7-3)!}, \quad 7^3.$$

(a) The number of permutations of A is ____________.

(b) The number of functions from $\{1, 2, 3\}$ to A is ____________.

(c) The number of binary relations on A is ____________.

(d) The number of 1-1 functions from $\{1, 2, 3\}$ to A is ____________.

(e) The number of subsets of A is ____________.

(f) The number of unordered pairs from A is ____________.

– END OF EXAMINATION –