UNIVERSITY OF VICTORIA

DECEMBER EXAMINATIONS 2013

MATH 122: Logic and Foundations

Instructor and section (check one):

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NAME: ____

V00#: ____

Duration: 3 Hours

Answers should be written on the exam paper.

The exam consists of **20** questions, for a total of 80 marks.

Please show all of your work and justify your answers when appropriate.

There are **10** pages (numbered), not including covers.

Count the pages before beginning and report any discrepancy immediately to the invigilator.

The only calculator permitted is the Sharp EL 510-R or EL-510RNB

Total: $\dfrac{\quad}{\mathbf{80}} = \dfrac{\quad}{\mathbf{40}}$

1. [5] Check the correct box (T = True, F = False) to the left of each statement. No reasons are necessary.

☐ T ☐ F If the universe is $\mathbb{R}$, the statements $\forall x,\ \exists y,\ xy \geq 0$ and $\exists y,\ \forall x,\ xy \geq 0$ are logically equivalent.

☐ T ☐ F The negation of the statement "*all dogs bite*" is "*all dogs don't bite.*"

☐ T ☐ F If A and B are sets and $A \neq B$, then $A \oplus B \neq \emptyset$.

☐ T ☐ F Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $\mathcal{R}$ is not reflexive, then $(a, a) \notin \mathcal{R}$.

☐ T ☐ F For any $x \in \mathbb{R}$, $\lceil x \rceil = \lfloor x \rfloor + 1$.

☐ T ☐ F For functions f and g, $(g \circ f)(x) = g(x)f(x)$ whenever both sides are defined.

☐ T ☐ F Any two non-empty open intervals of real numbers have the same cardinality.

☐ T ☐ F If X is a countable set and $x \notin X$, then $X \cup \{x\}$ is countable.

☐ T ☐ F If a, b, c are nonzero integers such that $c|a$ and $c|b$, then $\gcd(a, b)|c$.

☐ T ☐ F If a, b, c, n are integers such that $ac \equiv bc \pmod{n}$, then either $a \equiv b \pmod{n}$ or $c \equiv 0 \pmod{n}$.

2. [3] Let $A = \{a, b, \{a, b, c\}, \{\emptyset, \{\emptyset\}\}, \{c\}\}$. Check the correct box to the left of each statement. No reasons are necessary.

☐ T ☐ F $\{a, b\} \subseteq A$

☐ T ☐ F $\{a, b\} \in A$

☐ T ☐ F $\{\emptyset\} \in A$

☐ T ☐ F $|A| = 8$

☐ T ☐ F $|\mathcal{P}(A)| = 32$.

☐ T ☐ F $\emptyset \subseteq A$.

3. [2] Using basic known equivalences, show that $(\neg p \wedge q) \vee \neg(p \vee q)$ is logically equivalent to $\neg p$.

4. [4] Prove the following logical argument, giving a list of statements and reasons.

$$\begin{array}{ll} & p \vee q \\ & \neg p \vee r \\ & r \rightarrow s \\ \hline \therefore & q \vee s \end{array}$$

#	statement	reason
1.	$p \vee q$	premise
2.	$\neg p \vee r$	premise
3.	$r \rightarrow s$	premise
4. $\downarrow$		

5. Let A, B and C be sets.

   (a) [3] Prove that $A \setminus (B \cap C^c) \subseteq (A \cap B^c) \cup (A \cap C)$.

   (b) [2] Use a Venn diagram to investigate whether these sets may, in fact, be equal. Make a conjecture. Do not prove it.

6. [3] Let $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $b|c$ then $a|c$.

7. Let A and B be nonempty sets. Consider the statement: *if* $A \times B = B \times A$ *then* $A = B$.

   (a) [1] Write the contrapositive of the given statement.

   (b) [3] Prove the statement in (a).

   (c) [1] What does the result in part (b) tell you about the original statement?

   (d) [1] Does the truth value of the original statement change if $A = \emptyset$? Explain.

8. [4] Prove that the set of rational numbers is countable. Use a diagram to illustrate your proof.

9. [4] Consider the relation $\mathcal{R}$ defined on the set $\mathbb{Z}$ of integers by $(a, b) \in \mathcal{R}$ if and only if $a - b \leq 5$. Consider the statements below. If a statement is true, prove it. If it is false, give a counterexample.

   (a) $\mathcal{R}$ is reflexive.

   (b) $\mathcal{R}$ is symmetric.

   (c) $\mathcal{R}$ is antisymmetric.

   (d) $\mathcal{R}$ is transitive.

10. [3] Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is one-to-one then f is one-to-one.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 4 + |2x + 3|$.

(a) [1] Determine $\operatorname{rng} f$.

(b) [2] Give reasons why f is neither one-to-one nor onto.

(c) [1] Explain how to replace the target $\mathbb{R}$ of f with a set $B \subseteq \mathbb{R}$ so that the function $g : \mathbb{R} \to B$, defined by $g(x) = f(x)$ for all $x \in \mathbb{R}$, is onto.

(d) [1] Explain how to replace the domain $\mathbb{R}$ of g with a set $A \subseteq \mathbb{R}$ so that the function $h : A \to B$, defined by $h(x) = g(x)$ for all $x \in A$, is one-to-one and onto.

(e) [2] Find a formula for h^{-1}.

12. [3] Let a and b be integers and let p be a prime such that $\gcd(a, p^2) = p$ and $\gcd(b, p^3) = p^2$. Determine $\gcd(ab, p^4)$.

13. [2] Use the Fundamental Theorem of Arithmetic to prove that every integer $n \geq 2$ is divisible by a prime number.

14. [3] Determine the last digit of 33^{66}.

15. [3] Find the positive integer b if $(122)_b = (203)_7$.

16. [5] Let a_n be the sequence recursively defined by $a_0 = 1$, $a_1 = -3$, $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. Use strong induction to show that $a_n = (-3)^n$ for all integers $n \geq 0$.

17. (a) [2] Assume that $1+2+\cdots+k=\frac{(k+(1/2))^2}{2}$ for some $k \geq 1$. Use this hypothesis to prove that

$$1+2+\cdots+(k+1)=\frac{((k+1)+(1/2))^2}{2}.$$

(b) [2] Is the statement $1+2+\cdots+n=\frac{(n+(1/2))^2}{2}$ true for all integers $n \geq 1$? Explain.

18. [2] Let $a_1, a_2, a_3, \ldots$ be the sequence recursively defined by $a_1 = 1$ and, for $n > 1$, $a_n = 3a_{n-1}+1$. Find the first 4 terms of the sequence and conjecture a formula for a_n. Do not prove it.

19. Let $A = \{a, b, c\}$ and $B = \{u, x, y, z\}$. Answer the following questions. No reasons are necessary.

(a) [1] There are $\boxed{}$ functions from A to B.

(b) [1] There are $\boxed{}$ $1-1$ functions from A to B.

(c) [2] There are $\boxed{}$ functions f from A to B such that $f(a) = x$ or $f(a) = y$.

20. Let $S = \{1, 2, ..., 1000\}$.

(a) [2] Explain why the number of integers in S divisible by 11 is $\lfloor 1000/11 \rfloor = 90$. State a general result in which 11 is replaced by an arbitrary positive integer b.

(b) [3] How many integers in S are divisible by at least one of the numbers $3, 5, 11$?

(c) [2] How many integers in S relatively prime to 165?

/ END OF EXAMINATION