1. [5] Check the correct box (T = True, F = False) to the left of each statement. No reasons are necessary.

☐ T ☒ F If the universe is $\mathbb{R}$, the statements $\forall x,\ \exists y,\ xy \geq 0$ and $\exists y,\ \forall x,\ xy \geq 0$ are logically equivalent.

☐ T ☒ F The negation of the statement "*all dogs bite*" is "*all dogs don't bite.*"

☒ T ☐ F If A and B are sets and $A \neq B$, then $A \oplus B \neq \emptyset$.

☐ T ☒ F Let $\mathcal{R}$ be a binary relation on a set A and let $a \in A$. If $\mathcal{R}$ is not reflexive, then $(a, a) \notin \mathcal{R}$.

☐ T ☒ F For any $x \in \mathbb{R}$, $\lceil x \rceil = \lfloor x \rfloor + 1$.

☐ T ☒ F For functions f and g, $(g \circ f)(x) = g(x)f(x)$ whenever both sides are defined.

☒ T ☐ F Any two non-empty open intervals of real numbers have the same cardinality.

☒ T ☐ F If X is a countable set and $x \notin X$, then $X \cup \{x\}$ is countable.

☐ T ☒ F If a, b, c are nonzero integers such that $c|a$ and $c|b$, then $\gcd(a, b)|c$.

☐ T ☒ F If a, b, c, n are integers such that $ac \equiv bc \pmod{n}$, then either $a \equiv b \pmod{n}$ or $c \equiv 0 \pmod{n}$.

2. [3] Let $A = \{a, b, \{a, b, c\}, \{\emptyset, \{\emptyset\}\}, \{c\}\}$. Check the correct box to the left of each statement. No reasons are necessary.

☒ T ☐ F $\{a, b\} \subseteq A$

☐ T ☒ F $\{a, b\} \in A$

☐ T ☒ F $\{\emptyset\} \in A$

☐ T ☒ F $|A| = 8$

☒ T ☐ F $|\mathcal{P}(A)| = 32.$

☒ T ☐ F $\emptyset \subseteq A.$

3. [2] Using basic known equivalences, show that $(\neg p \wedge q) \vee \neg(p \vee q)$ is logically equivalent to $\neg p$.

$(\neg p \wedge q) \vee \neg(p \vee q)$	$\Leftrightarrow (\neg p \wedge q) \vee (\neg p \wedge \neg q)$	DeMorgan
	$\Leftrightarrow \neg p \wedge (q \vee \neg q)$	Dist've
	$\Leftrightarrow \neg p \wedge \underline{1}$	Known Tautology
	$\Leftrightarrow \neg p$	Identity

4. [4] Prove the following logical argument, giving a list of statements and reasons.

$$\begin{array}{ll} & p \vee q \\ & \neg p \vee r \\ & r \to s \\ \hline \therefore & q \vee s \end{array}$$

#	statement	reason
1.	$p \vee q$	premise
2.	$\neg p \vee r$	premise
3.	$r \to s$	premise
4.	$p \to r$	L.E to 2
5.	$p \to s$	4,3 Chain Rule
6.	$q \vee p$	1, Comm.
7.	$\neg q \to p$	L.E. to 6
8.	$\neg q \to s$	7,5 Chain Rule.
9.	$\cancel{\neg\neg} q \vee s$	L.E. to 8, Dbl Neg'n.

5. Let A, B and C be sets.

(a) [3] Prove that $A \setminus (B \cap C^c) \subseteq (A \cap B^c) \cup (A \cap C)$.

Take any $x \in A \setminus (B \cap C^c)$.

$\therefore$ $x \in A$ and $x \notin (B \cap C^c)$

Since $x \notin B \cap C^c$, $x \notin B$ or $x \notin C^c$

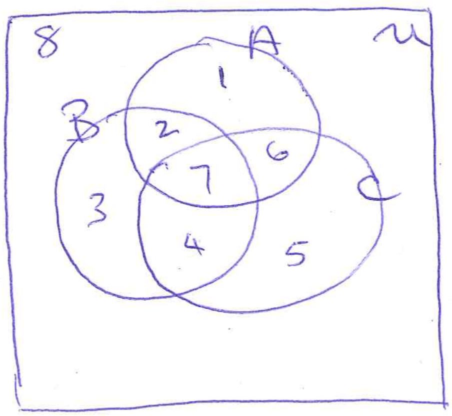
$\therefore$ $x \in B^c$ or $x \in C$

If $x \in B^c$, then $x \in A \cap B^c$, so $x \in RHS$

If $x \in C$, then $x \in A \cap C$, so $x \in RHS$

In either case, $x \in (A \cap B^c) \cup (A \cap C)$ $\square$

(b) [2] Use a Venn diagram to investigate whether these sets may, in fact, be equal. Make a conjecture. Do not prove it.



$A = \{1,2,6,7\}$ $\quad$ $U = \{1,2,\ldots,8\}$

$B = \{2,3,4,7\}$ $\quad$ $C = \{4,5,6,7\}$

$C^c = \{1,2,3,8\}$ $\quad$ $B \cap C^c = \{2,3\}$

$LHS = A \setminus (B \cap C^c) = \{1,6,7\}$.

$A \cap B^c = A \cap \{1,5,6,8\} = \{1,6\}$.

$A \cap C = \{6,7\}$

$RHS = (A \cap B^c) \cup (A \cap C)$

$= \{1,6,7\}$.

$\therefore$ We think they are equal

6. [3] Let $a, b, c \in \mathbb{Z}$. Prove that if $a|b$ and $b|c$ then $a|c$.

Since $a|b$, $\exists k \in \mathbb{Z}$ s.t. $ak = b$

Since $b|c$, $\exists \ell \in \mathbb{Z}$ s.t. $b\ell = c$

$\therefore$ $c = b\ell = (ak)\ell = a(k\ell)$

Since $(k\ell) \in \mathbb{Z}$, $a|c$

7. Let A and B be nonempty sets. Consider the statement: *if* $A \times B = B \times A$ *then* $A = B$.

(a) [1] Write the contrapositive of the given statement.

If $A \neq B$, then $A \times B \neq B \times A$.

(b) [3] Prove the statement in (a).

Suppose $A \neq B$. Then one of these sets has an element not in the other. Suppose there is an element $x \in A$ s.t. $x \notin B$. Since $B \neq \emptyset$ it has an element, b. Then $(x, b) \in A \times B$, and $(x, b) \notin B \times A$ b/c $x \notin B$. $\therefore$ $A \times B \neq B \times A$. The case where $\exists\, y \in B$ s.t. $y \notin A$ is similar.

(c) [1] What does the result in part (b) tell you about the original statement?

That it is true $\because$ a stmt and its contrapositive are logically equivalent

(d) [1] Does the truth value of the original statement change if $A = \emptyset$? Explain.

Yes. If $A = \emptyset$, then $A \times B = \emptyset = B \times A$ no matter what set B is.

8. [4] Prove that the set of rational numbers is countable. Use a diagram to illustrate your proof.

Consider the array:

$$
\begin{array}{cccccc}
\frac{0}{1} \rightarrow & \frac{1}{1} & \frac{-1}{1} \rightarrow & \frac{2}{1} & \frac{-2}{1} & \cdots \\
\frac{0}{-1} & \frac{1}{-1} & \frac{-1}{-1} & \frac{2}{-1} & \frac{-2}{-1} & \cdots \\
\downarrow & & & & & \\
\frac{0}{2} & \frac{1}{2} & \frac{-1}{2} & \frac{2}{2} & \frac{-2}{2} & \cdots
\end{array}
$$

It contains every rational #.

The sequence indicated by the arrows contains every element of the array, and $\therefore$ every element of $\mathbb{Q}$

$\therefore$ $\mathbb{Q}$ is countable

9. [4] Consider the relation $\mathcal{R}$ defined on the set $\mathbb{Z}$ of integers by $(a,b) \in \mathcal{R}$ if and only if $a - b \leq 5$. Consider the statements below. If a statement is true, prove it. If it is false, give a counterexample.

(a) $\mathcal{R}$ is reflexive. True.

Take any $x \in \mathbb{Z}$. Then $x - x = 0 \leq 5$.
$\therefore (x,x) \in R$ $\therefore R$ reflexive.

(b) $\mathcal{R}$ is symmetric. False

$(0,10) \in R$ b/c $0 - 10 = -10 \leq 5$.
But $(10,0) \notin R$ b/c $10 - 0 = 10 \nleq 5$.

(c) $\mathcal{R}$ is antisymmetric. False.

$(1,0) \in R$ b/c $1 - 0 = 1 \leq 5$
$(0,1) \in R$ b/c $0 - 1 = -1 \leq 5$
But $1 \neq 0$.

(d) $\mathcal{R}$ is transitive. False

$(10,5) \in R$ b/c $10 - 5 = 5 \leq 5$
$(5,0) \in R$ b/c $5 - 0 = 5 \leq 5$
But $(10,0) \notin R$ b/c $10 - 0 = 10 \nleq 5$.

10. [3] Let $f : A \to B$ and $g : B \to C$ be functions. Prove that if $g \circ f$ is one-to-one then f is one-to-one.

Suppose $f(a_1) = f(a_2)$ (Want $a_1 = a_2$)
$\therefore g(f(a_1)) = g(f(a_2))$
$\therefore g \circ f(a_1) = g \circ f(a_2)$
Since $g \circ f$ is 1-1, $a_1 = a_2$
$\therefore f$ is 1-1

11. Let $f : \mathbb{R} \to \mathbb{R}$ be the function defined by $f(x) = 4 + |2x + 3|$.

(a) [1] Determine rng f.

$|2x+3| \geq 0$, and all real #'s in $[0,\infty)$ can arise

$\therefore$ rng $f = [4,\infty)$

(b) [2] Give reasons why f is neither one-to-one nor onto.

$f(2) = 4 + |2\cdot 2 + 3| = 11$

$f(-5) = 4 + |2\cdot(-5) + 3| = 11$

Since $2 \neq -5$, f is not 1-1.

Since rng $f = [4,\infty)$, $\nexists$ x s.t. $f(x) = 0$

$\therefore$ f is not onto

(c) [1] Explain how to replace the target $\mathbb{R}$ of f with a set $B \subseteq \mathbb{R}$ so that the function $g : \mathbb{R} \to B$, defined by $g(x) = f(x)$ for all $x \in \mathbb{R}$, is onto.

Let $B =$ rng $f = [4,\infty)$.

Then, by definition of range, every value in B arises as a value of f.

(d) [1] Explain how to replace the domain $\mathbb{R}$ of g with a set $A \subseteq \mathbb{R}$ so that the function $h : A \to B$, defined by $h(x) = g(x)$ for all $x \in A$, is one-to-one and onto.

Take $A = [-\frac{3}{2},\infty)$. For $x \in A$,

$f(x) = 4 + |2x+3| = 4 + 2x + 3$

b/c $2x + 3 \geq 0$. Linear functions are 1-1, and every element of B arises as a value of the fn. h.

(e) [2] Find a formula for h^{-1}.

From (d),

$h(x) = 7 + 2x$

$y = h(x) \iff y = 7 + 2x$

$\iff \frac{y-7}{2} = x$

If $y \in [4,\infty)$ then $\frac{y-7}{2} \in [-\frac{3}{2},\infty)$

$\therefore h^{-1}(y) = \frac{y-7}{2}$

12. [3] Let a and b be integers and let p be a prime such that $\gcd(a, p^2) = p$ and $\gcd(b, p^3) = p^2$. Determine $\gcd(ab, p^4)$.

$\gcd(a,p^2) = p \Rightarrow$ exponent of p in prime fac'n of a is 1

$\gcd(b,p^3) = p^2 \Rightarrow$ exponent of p in prime fac'n of b is 2

$\therefore$ exp. of p in prime fac'n of ab is $1+2$

$\therefore \gcd(ab, p^4) = p^3$

13. [2] Use the Fundamental Theorem of Arithmetic to prove that every integer $n \geq 2$ is divisible by a prime number.

F.T.A. says every $n \geq 2$ can be uniquely written as

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$$

where each p_i is prime, each $\alpha_i \geq 1$, and $p_1 < p_2 < \cdots < p_k$

$\therefore n = p_1 (p_1^{\alpha_1 - 1} p_2^{\alpha_2} \cdots p_k^{\alpha_k})$ so $p_1 \mid n$

14. [3] Determine the last digit of 33^{66}.

Want $d \in \{0, 1, \ldots, 9\}$ s.t. $33^{66} \equiv d \pmod{10}$

$33^{66} \equiv 3^{66} \equiv (3^2)^{33} \equiv (-1)^{33} \equiv -1 \equiv 9 \pmod{10}$

$\therefore$ The last digit is 9

15. [3] Find the positive integer b if $(122)_b = (203)_7$.

$(122)_b = 1 \cdot b^2 + 2 \cdot b + 2$

$(203)_7 = 2 \cdot 7^2 + 0 \cdot 7 + 3 = 101$

$\therefore b^2 + 2b + 2 = 101$

$b^2 + 2b - 99 = 0$

$(b+11)(b-9) = 0$

$\therefore b = -11$ or 9

But -11 isn't a base, so $b = 9$

16. [5] Let a_n be the sequence recursively defined by $a_0 = 1$, $a_1 = -3$, $a_n = -6a_{n-1} - 9a_{n-2}$ for $n \geq 2$. Use strong induction to show that $a_n = (-3)^n$ for all integers $n \geq 0$.

Basis When $n=0$, $a_0 = 1 = (-3)^0$.
When $n=1$, $a_1 = -3 = (-3)^1$.
$\therefore$ stmt true when $n=0$ or $n=1$.

IH. Assume $a_0 = (-3)^0$,
$a_1 = (-3)^1$,
$\vdots$
$a_k = (-3)^k$ for some $k \geq 1$.

IS. Want $a_{k+1} = (-3)^{k+1}$.

Look at a_{k+1}. Since $k+1 \geq 2$, we have

$$
\begin{aligned}
a_{k+1} &= -6a_k - 9a_{k-1} \\
&= -6(-3)^k - 9(-3)^{k-1} \\
&= 2(-3)^{k+1} - \cancel{(-3)^2(-3)^{k-1}} \to (-3)^{k+1} \\
&= (-3)^{k+1}(2-1) = (-3)^{k+1}, \text{ as wanted}
\end{aligned}
$$

$\therefore$ By strong induction, $a_n = (-3)^n \; \forall n \geq 0$

17. (a) [2] Assume that $1+2+\cdots+k=\frac{(k+(1/2))^2}{2}$ for some $k\geq 1$. Use this hypothesis to prove that

$$1+2+\cdots+(k+1)=\frac{((k+1)+(1/2))^2}{2}.$$

$$\begin{aligned}
LHS &= 1+2+\cdots+(k+1)\\
&= \underbrace{1+2+\cdots+k}+(k+1)\\
&= (k+(1/2))^2/2 + (k+1)\frac{2}{2}\\
&= \frac{1}{2}\left[k^2+k+\frac{1}{4}+2(k+1)\right]\\
&= \frac{1}{2}\left[k^2+3k+\frac{9}{4}\right] = \frac{((k+1)+\frac{1}{2})^2}{2} \quad \square
\end{aligned}$$

(b) [2] Is the statement $1+2+\cdots+n=\frac{(n+(1/2))^2}{2}$ true for all integers $n\geq 1$? Explain.

No. When $n=1$, $LHS=1$

$RHS = \frac{(1+1/2)^2}{2} = \frac{(\frac{3}{2})^2}{2} = \frac{9}{8}$

Part (a) is an induction step, but the base case isn't true.

18. [2] Let $a_1, a_2, a_3, \ldots$ be the sequence recursively defined by $a_1=1$ and, for $n>1$, $a_n=3a_{n-1}+1$. Find the first 4 terms of the sequence and conjecture a formula for a_n. Do not prove it.

$a_1 = 1$

$a_2 = 3a_1+1 = 3\cdot 1+1$

$a_3 = 3a_2+1 = 3(3\cdot 1+1)+1$

$= 3^2\cdot 1+3\cdot 1+1$

$a_4 = 3a_3+1 = 3(3^2\cdot 1+3\cdot 1+1)+1$

$= 3^3\cdot 1+3^2\cdot 1+3\cdot 1+1$

Conj: $a_n = 3^{n-1}+3^{n-2}+\cdots+1$

$= (3^n-1)/(3-1) \quad \forall n\geq 1$

19. Let $A = \{a, b, c\}$ and $B = \{u, x, y, z\}$. Answer the following questions. No reasons are necessary.

(a) [1] There are $\boxed{4^3}$ functions from A to B.

(b) [1] There are $\boxed{4!}$ $1-1$ functions from A to B. $4 \cdot 3 \cdot 2 = 4!$

(c) [2] There are $\boxed{32}$ functions f from A to B such that $f(a) = x$ or $f(a) = y$.

$1 \cdot 4^2 + 1 \cdot 4^2$

20. Let $S = \{1, 2, ..., 1000\}$.

(a) [2] Explain why the number of integers in S divisible by 11 is $\lfloor 1000/11 \rfloor = 90$. State a general result in which 11 is replaced by an arbitrary positive integer b.

By the division algorithm, if $a = bq + r$, $0 \leq r < b$ and $b \geq 1$, then $q = \lfloor a/b \rfloor$. $1 \cdot b, 2 \cdot b, \ldots, \lfloor a/b \rfloor \cdot b$ are all less than or equal to a, but $(\lfloor a/b \rfloor + 1) b > a$

$1000 = \left\lfloor \frac{1000}{11} \right\rfloor \cdot 11 + 10$

90

(b) [3] How many integers in S are divisible by at least one of the numbers 3, 5, 11?

A = set div. by 3
B = set " " 5
C = " " " 11.

Want $|A \cup B \cup C|$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

$$= \left\lfloor \frac{1000}{3} \right\rfloor + \left\lfloor \frac{1000}{5} \right\rfloor + \left\lfloor \frac{1000}{11} \right\rfloor - \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{33} \right\rfloor - \left\lfloor \frac{1000}{55} \right\rfloor + \left\lfloor \frac{1000}{165} \right\rfloor$$

$$= 515$$

(c) [2] How many integers in S relatively prime to 165?

$165 = 3 \times 5 \times 11$. Every int. in (b) ~~S~~ is divisible by one of these. And this is all of them

$\therefore$ ~~[illegible]~~ $1000 - 515$

/ END OF EXAMINATION