1. [4] Suppose the statement $p \leftrightarrow q$ is false. Find all combinations of truth values for $p$, $q$, and $r$ such that the statement $(\neg r \leftrightarrow p) \vee (r \wedge \neg q)$ is true.

$p \Leftrightarrow q$ false $\Leftrightarrow$ $p$ T & $q$ F ①
or $p$ F & $q$ T ②

① $\neg r$ T ($r$ F) so $\neg r \Leftrightarrow p$ T
or $r$ T so $r \wedge \neg q$ T

② $\neg q$ F $\therefore$ $r \wedge \neg q$ never true
Need $\neg r \Leftrightarrow p$ true $\therefore$ $\neg r$ F (so $r$ T)

Sol'n

| p | q | r |
|---|---|---|
| T | F | T |
| T | F | F |
| F | T | T |

2. [3] Use known logical equivalences to show that $q \rightarrow (p \rightarrow q)$ is a tautology.

$$
\begin{aligned}
q \rightarrow (p \rightarrow q) &\Leftrightarrow \neg q \vee (p \rightarrow q) && \text{Known LE.}\\
&\Leftrightarrow \neg q \vee (\neg p \vee q) && \text{" "}\\
&\Leftrightarrow \underbrace{(\neg q \vee q)}_{T} \vee \neg p && \text{Assoc \& Comm.}\\
&\Leftrightarrow 1 \vee \neg p && \text{Known tautology}\\
&\Leftrightarrow \underline{1} && \text{Dominance}
\end{aligned}
$$

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

$\neg(p \rightarrow q) \Leftrightarrow \neg(\neg p \vee q)$
$\Leftrightarrow p \wedge \neg q$

T The negation of the statement $p \rightarrow q$ is $p \wedge \neg q$.
F The converse of the statement $p \rightarrow \neg q$ is $p \rightarrow q$.
T The statement $\exists x, (x^2 = 2) \rightarrow (2x < x)$ is true for the universe $\mathbb{Q}$.
T The statement "*Every picture tells a story*" has a hidden existential ~~existential~~ quantifier.

4. [4] Use known logical equivalences and rules of inference to show that the following argument is valid.

$$\begin{array}{rc} & \neg(p \wedge q) \\ & r \vee q \\ & \neg p \to \neg r \\ \hline \therefore & p \leftrightarrow r \end{array}$$

| | | |
|---|---|---|
| 1. | $\neg(p \wedge q)$ | Premise |
| 2. | $r \vee q$ | " |
| 3. | $\neg p \to \neg r$ | " |
| 4. | $\neg p \vee \neg q$ | 1, De Morgan |
| 5. | $p \to \neg q$ | 4, Known LE. |
| 6. | $\neg q \to r$ | 2, " " |
| 7. | $p \to r$ | 5, 6 Chain Rule |
| 8. | $r \to p$ | 3, Contrapos. |
| 9. | $(p \to r) \wedge (r \to p)$ | 7, 8, Conjunction |
| 10 | $p \Leftrightarrow r$ | 9, Known LE |

5. [3] Let $A, B$ and $C$ be sets. Give a counterexample to show that the statement "*If* $A \cup C = B \cup C$ *then* $A = B$." is false.

Let $A = \phi$, $B = \{1\}$, $C = \{1,2\}$

Then $A \cup C = \{1,2\}$

& $B \cup C = \{1,2\}$

$\therefore A \cup C = B \cup C$ but $A \neq B$

6. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters $A$ and $B$ represent sets.

T If $x \in A \cap B$, then $(x, x) \in A \times B$.

F $\mathcal{P}(\emptyset) = \emptyset$.

T If $A \neq B$, then $A \cap B \subsetneq A \cup B$.

F If the symmetric difference of $A$ and $B$ is not empty, then $A \not\subseteq B$.

7. Let $A, B$ and $C$ be sets. Suppose that $A \subseteq B$ and $B \subseteq C$.

(a) [3] Prove that $A \subseteq C$.

Take any $x \in A$.
Since $A \subseteq B$, $x \in B$
Since $B \subseteq C$, $x \in C$.
$\therefore A \subseteq C$

(b) [1] For the sets $A, B$ and $C$ as above, suppose that $(B \setminus A) \neq \emptyset$. Is it true that $A \subsetneq C$? Explain.

Yes. Since $B \setminus A \neq \emptyset$, there is an element $y \in B$ s.t. $y \notin A$.
Since $B \subseteq C$, $y \in C$
$\therefore \exists\, y \in C$ s.t. $y \notin A$.

8. [4] Let $A$ and $B$ be sets. Use any method except a Venn diagram to prove the identity $(A \cap B)^c = A^c \cup B^c$. Hint: there is a short argument that uses set-builder notation.

$$
\begin{aligned}
(A \cap B)^C &= \{x : x \notin A \cap B\} \\
&= \{x : \neg(x \in A \cap B)\} \\
&= \{x : \neg((x \in A) \wedge (x \in B))\} \\
&= \{x : \neg(x \in A) \vee \neg(x \in B)\} \\
&= \{x : x \notin A \vee x \notin B \\
&= \{x : x \in A^C \vee x \in B^C\} = A^C \cup B^C
\end{aligned}
$$

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T For any set $A$, there exists a reflexive relation on $A$.

F If $\mathcal{R}$ is a symmetric relation on a set $A$ with at least two elements, then there exist $a, b \in A$ such that $(a, b), (b, a) \in \mathcal{R}$.

F If $\mathcal{R}$ is a relation on $\{a, b, c\}$ such that $(a, b) \in \mathcal{R}$ and $(b, a) \notin \mathcal{R}$, then $\mathcal{R}$ is antisymmetric.

T If $\mathcal{R}$ is a transitive relation on $\{a, b, c\}$ such that $(a, b) \in \mathcal{R}$ and $(a, c) \notin \mathcal{R}$, then $(b, c) \notin \mathcal{R}$.

10. [4] Let $\sim$ be the relation on $\mathbb{N}$ defined by $a \sim b$ if and only if $\frac{a}{b+2} \geq \frac{b}{a+2}$. Prove that $\sim$ is reflexive and anti-symmetric.

reflexive. Let $n \in \mathbb{N}$. Want $n \sim n$.

We have $\frac{n}{n+2} \geq \frac{n}{n+2}$ $\therefore n \sim n$

$\therefore \sim$ is reflexive.

anti-symm. Suppose $a \sim b \wedge b \sim a$.

WANT $a = b$. Since $a \sim b$, $\frac{a}{b+2} \geq \frac{b}{a+2}$

Since $b \sim a$ $\frac{b}{a+2} \geq \frac{a}{b+2}$.

$\therefore \frac{a}{b+2} = \frac{b}{a+2}$ so $a(a+2) = b(b+2)$

$\therefore a^2+2a = b^2+2b$

$\therefore a^2+2a+1 = b^2+2b+1$

$\therefore (a+1)^2 = (b+1)^2$

Since $a, b \in \mathbb{N}$, $a+1 = b+1$ $\therefore a = b$

11. [2] Suppose that $\mathcal{R}$ is an equivalence relation on the set $A = \{a, b, c, d, e\}$ such that the partition of $A$ into equivalence classes determined by $\mathcal{R}$ is $\{\{a, b, c\}, \{d\}, \{e\}\}$. Write $\mathcal{R}$ as a set of ordered pairs.

$\mathcal{R} = \{(a,a), (a,b), (a,c), (b,a), (b,b), (b,c), (c,a), (c,b), (c,c), (d,d), (e,e)\}$

12. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

$7 = (-2)(-3) + 1$ $\quad -3 \overset{?}{=} \left\lceil \frac{7}{-2} \right\rceil$

T Suppose $a, b, q, r \in \mathbb{Z}$ and $b < 0$. If $a = bq + r$, $0 \leq r < |b|$, then $q = \lceil a/b \rceil$.

F If $|A| = 10$ and $f : A \to B$ is onto, then $|B| = 10$.

F A function $f : A \to B$ is 1-1 if and only if for every $a \in A$ there exists exactly one $b \in B$ such that $(a, b) \in f$.

T If the function $f : A \to B$ is invertible, then the inverse function, $f^{-1} : B \to A$ is 1-1 and onto.

13. [4] Let $f : \{0, 1, \ldots\} \to \{0, 1, \ldots\}$ be defined by $f(x) = x^2 - 5x + 6$. Show that $f$ is neither 1-1 nor onto.
$= (x-3)(x-2)$

Not 1-1 $\quad f(2) = 0 = f(3)$ but $2 \neq 3$.

Not onto $\quad$ Claim: There is no $x$ s.t. $f(x) = 1$

$f(x) = 1 \Leftrightarrow x^2 - 5x + 6 = 1 \Leftrightarrow x^2 - 5x + 5 = 0$

$\Leftrightarrow x = \dfrac{+5 \pm \sqrt{(-5)^2 - 4 \cdot 5}}{2} = \dfrac{5 \pm \sqrt{5}}{2}$

Neither $\dfrac{5+\sqrt{5}}{2}$ nor $\dfrac{5-\sqrt{5}}{2}$ is in the domain. $\therefore \nexists$ s.t. $f(x) = 1$.

14. Let $X = \{a, b, c, d, e, f\}$, and let $f : X \to X$ be given in the table below

| $x =$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $f(x) =$ | $b$ | $c$ | $a$ | $f$ | $d$ | $e$ |

(a) [1] Find $g = f \circ f$.

| $x =$ | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $f(x) =$ | $b$ | $c$ | $a$ | $f$ | $d$ | $e$ |
| $g(x) =$ | c | a | b | e | f | d |

(b) [2] Use any method to show that $g = f^{-1}$.

$f = \{(a,b), (b,c), (c,a), (d,f), (e,d), (f,e)\}$

$g = \{(b,a), (c,b), (a,c), (f,d), (d,e), (e,f)\}$.

$g$ is a fn and is obtained by reversing the ordered pairs in $f$ $\therefore g = f^{-1}$

15. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F When $-17$ is divided by 4, the remainder is $-1$.

T If $p$ and $q$ are different prime numbers, then $lcm(p, q) = pq$.

F $(1101101)_2 = (A5)_{16}$.

T If $3k = 15\ell$, then $5 \mid k$.

16. Consider the integers $m = 900$ and $n = 189 = 3^3 7$. Fill in the blanks:

(a) [1] The prime factorization of $m$ is $2^2 3^2 5^2$.        $900 = 3^2 \times 100$
$= 3^2 \times 2^2 \times 5^2$

(b) [1] $\gcd(m, n) = 2^0 3^2 5^0 7^0$ and $lcm(m, n) = 2^2 3^3 5^2 7^1$

(c) [2] The smallest multiple $k$ of 189 such that $\gcd(m, k) = 45$ is $5 \times 189$

17. (a) [2] Prove that if $k \equiv 3 \pmod 8$, then $7 \cdot k^{2016} + 1$ is divisible by 8.

Want $7 \cancel{k}^{2016} + 1 \equiv 0 \pmod 8$

$7k^{2016} + 1 \equiv 7 \cdot 3^{2016} + 1 \equiv 7 \cdot (3^2)^{1008} + 1$

$\equiv 7 \cdot \cancel{1^{1008}} + 1 \equiv 0 \pmod 8$

(b) [2] Is there an integer $n > 1$ such that the last digit of $122^n$ is 2? Are the infinitely many such integers $n$?

i.e. is there $n > 1$ s.t. $122^n \equiv 2 \pmod{10}$

$122^n \equiv 2^n \pmod{10}$ and $2^5 \equiv 32 \equiv 2 \pmod{10}$

$\therefore$ yes.

There are infinitely many. The powers of 2 (mod 10) are

2, 4, 8, 6, 2, 4, 8, 6, 2, ....

18. [2] Use the blank to indicate whether each statement is True or False. All variables are integers. No justification is necessary.

T $4 \mid (d_k d_{k-1} \ldots d_1 d_0)_{10}$ if and only if $4 \mid (d_1 d_0)_{10}$.

T If $a \equiv 1 \pmod 6$ then $a \equiv 1 \pmod 3$.

F If $p_1, p_2, \ldots, p_n$ are the first $n$ prime numbers, then the integer $N = p_1 p_2 \cdots p_n + 1$ is prime.

T It follows from $2016 = 7 \times 250 + 266$ that $\gcd(2016, 250) = \gcd(250, 266)$.

19. [4] Let $a, b$ and $c$ be integers such that $a + b + c = 0$. Let $d$ be an integer such that $d \mid a$ and $d \mid b$. Prove that $d \mid c$.

Suppose $d|a$ and $d|b$.
Then $\exists\ k \in \mathbb{Z}$ s.t. $dk = a$
& $\exists\ l \in \mathbb{Z}$ s.t. $dl = b$

Now $c = -a-b = -dk - dl = d(-k-l)$
Since $(-k-l) \in \mathbb{Z}$, $d|c$.

20. [4] Use induction to prove that $1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$, for all integers $n \geq 1$.

$0! = 1$
$n! = n(n-1)!,\ n \geq 1$

Basis. When $n=1$, LHS $= 1 \cdot 1! = 1$
& RHS $= (1+1)! - 1 = 2! - 1 = 1$
Since LHS = RHS, the stmt is true when n=1.

IH. Assume
$1 \cdot 1! + 2 \cdot 2! + \cdots + k \cdot k! = (k+1)! - 1$
for some $k \geq 1$.

IS. Want $1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1)(k+1)!$
$= ((k+1)+1)! - 1$

Look at LHS:

$$1 \cdot 1! + 2 \cdot 2! + \cdots + (k+1)(k+1)!$$
$$= \underbrace{1 \cdot 1! + 2 \cdot 2! + \cdots + k\,k!}_{} + (k+1)(k+1)!$$
$$= (k+1)! - 1 + (k+1)(k+1)!$$
$$= (k+1)!\left[1 + (k+1)\right] - 1$$
$$= (k+2)(k+1)! - 1 = (k+2)! - 1 \checkmark$$

$\therefore$ By induction
$1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n+1)! - 1 \quad \forall n \geq 1$

21. [4] Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 3$, and $a_n = 5a_{n-1} + 3$ for $n \geq 1$. Find $a_1, a_2, a_3$ and $a_4$, then use your work to conjecture (i.e. guess) a formula for $a_n$. It is not necessary prove that your formula is correct, but note that it is not sufficient to express $a_n$ as a sum.

$$a_1 = 5a_0 + 3 = 5\cdot 3 + 3 = 18$$

$$a_2 = 5a_1 + 3 = 5(5\cdot 3 + 3) + 3$$
$$= 5^2\cdot 3 + 5\cdot 3 + 3 = 93$$

$$a_3 = 5a_2 + 3 = 5(5^2\cdot 3 + 5\cdot 3 + 3) + 3$$
$$= 5^3\cdot 3 + 5^2\cdot 3 + 5\cdot 3 + 3 = 468$$

$$a_4 = 5a_3 + 3 = 5(\quad \downarrow \quad) + 3$$
$$= 5^4\cdot 3 + 5^3\cdot 3 + 5^2\cdot 3 + 5\cdot 3 + 3 = 2343$$

Guess $a_n = 3(5^n + 5^{n-1} + 5^{n-2} + \cdots + 1) = 3\,\dfrac{5^{n+1}-1}{5-1}$

22. [4] Use the strong form of induction to prove that any integer $n \geq 8$ can be written as a sum of 3s and 5s. Note: the basis step should cover several values of $n$.

BASIS: Since $8 = 3+5$, $9 = 3+3+3$, $10 = 5+5$, each of 8, 9, 10 can be written as a sum of 3's & 5's.

IH Assume each of $8, 9, \ldots, k$ can be written as a sum of 3's & 5's, for some $k \geq 10$.

IS Consider $k+1$. Since $k \geq 10$, $k+1 \geq 11$ and $\therefore$ $k+1-3 \geq 8$.
By IH, $k+1-3$ can be written as a sum of 3's & 5's. Adding a 3 gives a representation of $k+1$ as a sum of 3's & 5's.

$\therefore$ By induction, any $n \geq 8$ can be written as a sum of 3's & 5's.

23. [3] Use a diagonal sweeping argument to prove that the set $\mathbb{N} \times \mathbb{N}$ is countable.

Consider the array

$$\begin{array}{ccccc}
(1,1) & (1,2) & (1,3) & (1,4) & \cdots \\
(2,1) & (2,2) & (2,3) & (2,4) & \cdots \\
(3,1) & (3,2) & (3,3) & (3,4) & \cdots \\
(4,1) & (4,2) & (4,3) & (4,4) & \cdots \\
\vdots & \vdots & \vdots & \vdots &
\end{array}$$

The array contains every element of $\mathbb{N} \times \mathbb{N}$. The sequence indicated by the arrows contains every element of the array, and $\therefore$ of $\mathbb{N} \times \mathbb{N}$.

$\therefore \mathbb{N} \times \mathbb{N}$ is countable.

24. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $A$ and $B$ are infinite sets such that $|A| = |B|$, then $|\mathcal{P}(A)| = |\mathcal{P}(B)|$.

F If $A$ and $B$ are infinite sets such that $|A| = |B|$ and $a \in A$, then $|A \setminus \{a\}| < |B|$

T The union of any two countable sets is countable.

F $|\mathbb{Q}| = |(0,1)|$.

25. (a) [1] Complete the statement of the Principle of Inclusion-Exclusion for two sets:

$$|A \cup B| = |A| + |B| - |A \cap B|$$

(b) [2] How many integers in the set $S = \{1, 2, 3, ..., 3000\}$ are divisible by 2 or 3?

A : set divisible by 2
B : set " " 3

Want $|A \cup B|$

$|A| = \left\lfloor \frac{3000}{2} \right\rfloor = 1500$ $|B| = \left\lfloor \frac{3000}{3} \right\rfloor = 1000$

$|A \cap B| = \left\lfloor \frac{3000}{2 \cdot 3} \right\rfloor = 500$ $\therefore$ $1500 + 1000 - 500$

26. [2] Let $A = \{1, 2, 3, 4, 5, 6\}$. Fill in each blank. No justification is necessary.

(a) $|A \times A| =$ $6 \times 6$.

(b) The number of subsets $X \subseteq A$ such that $\{1, 2, 3\} \subseteq X$ is $8$ = # subsets $\{4, 5, 6\}$.

(c) The number of functions $f : A \to \{w, x, y, z\}$ is $4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 \cdot 4 = 4^6$

(d) The number of 1-1 functions $g : \{a, b, c\} \to A$ is $6 \cdot 5 \cdot 4$.

END