1. [3] Use a truth table to determine whether $\neg p \vee (q \to (p \wedge q))$ is a tautology. (Here, and elsewhere, be sure to clearly state your answer to the question!)

p	q	$\neg p$	$p \wedge q$	$q \to (p \wedge q)$	$\neg p \vee (q \to (p \wedge q))$
F	F	T	F	T	T
F	T	T	F	F	T
T	F	F	F	T	T
T	T	F	T	T	T

It is a tautology

2. [4] Use known logical equivalences to show that $\neg(p \vee (\neg p \wedge q))$ is logically equivalent to $\neg p \wedge \neg q$.

$\neg(p \vee (\neg p \wedge q)) \Leftrightarrow \neg((p \vee \neg p) \wedge (p \vee q))$ Dist

$\Leftrightarrow \neg(T \wedge (p \vee q))$ Known tautology

$\Leftrightarrow \neg(p \vee q)$ Identity

$\Leftrightarrow \neg p \wedge \neg q$ DeMorgan.

3. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

F The converse of $p \to \neg q$ is $\neg p \to q$.

T If $p \to q$ is false, the truth value of $(\neg p \vee \neg q) \to (p \leftrightarrow q)$ is false.

T The statement $\exists x, \exists y, (2x = -3y) \wedge (xy < 0)$ is true if the universe of x, y is $\mathbb{Z}$.

T The fact that there are infinitely many prime numbers can be proved by showing that for every natural number n there exists a prime number p with $p > n$.

4. [4] Write the following argument in symbolic form, and then use known logical equivalences and inference rules to show that it is valid. Please clearly indicate which letters correspond to which statements. Justify each step.

I am not wearing a pink tie or I am wearing a red shirt.
If it is not Saturday, then I am wearing a pink tie.
I am not wearing a red shirt
$\therefore$ It is Saturday

Let p be "I am wearing a pink tie"
r " "I am wearing a red shirt"
s " "It is Saturday"

The argument is

$\neg p \vee r$
$\neg s \to p$
$\neg r$
$\therefore s$

1.	$\neg p \vee r$	Premise
2.	$\neg s \to p$	Premise
3.	$\neg r$	Premise
4.	$\neg r \to \neg p$	1, Known LE.
5.	$\neg p \to s$	2, contrapos.
6.	$\neg r \to s$	4, 5 Chain Rule
7.	$\therefore s$	3, 6 M.P.

5. [3] Explain in detail why $n = 9$ is a counterexample to the following argument.

1. If n is prime, then n is not a sum of two odd numbers.
2. If n is even, then it is a sum of two odd numbers.
3. n is odd or n is a sum of two odd numbers.
4. $\therefore$ If n is not prime, then n is even.

9 not prime $\therefore$ 1 T
9 not even $\therefore$ 2 T
9 is odd $\therefore$ 3 T
All premises T

9 not prime & 9 not even
$\therefore$ 4 F
conclusion F.

$\therefore$ We have a counterexample.

6. [3] Prove that for all sets A, B, and C, $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

$$\begin{aligned} A \backslash (B \cap C) &= A \cap (B \cap C)^c && \text{Known} \\ &= A \cap (B^c \cup C^c) && \text{DeMorgan} \\ &= (A \cap B^c) \cup (A \cap C^c) && \text{Dist.} \\ &= (A \backslash B) \cup (A \backslash C) && \text{Known} \end{aligned}$$

7. [4] Let A and B be sets such that $A \cup B = B$. Prove that $A \cap B = A$.

Want $A \cap B \underset{1}{\subseteq} A$ and $A \underset{2}{\subseteq} A \cap B$

1 T by def'n of intersection.

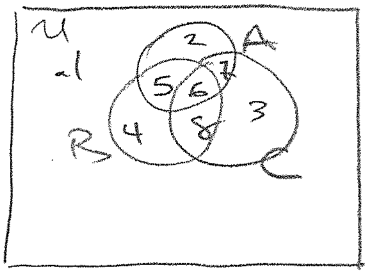
2 Take any $x \in A$.
Then $x \in A \cup B$ (def'n of union)
Since $A \cup B = B$, $x \in B$.
$\therefore$ $x \in A \cap B$
$\therefore$ $A \subseteq A \cap B$

By 1 & 2 $A \cap B = A$

8. [2] Let $A = \{1, \{2,3\}, \{1,2,3\}\}$ and $B = \{1, 2, \{1,2,3\}\}$. Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

F $A \subseteq B$.
T $\emptyset \subseteq A \setminus B$.
F $2 \in A \cap B$.
F $|A \cup B| = 6$.

9. [3] Give a counterexample to show the following statement is false: *For all sets* A, B *and* C, $(A \backslash B) \cup (B \backslash C) = A \backslash C$. You should clearly define your sets and explain why your example shows that the statement is false.



$\mathcal{U} = \{1, 2, \ldots, 8\}$

$A = \{2, 5, 6, 7\}$, $B = \{4, 5, 6, 8\}$

$C = \{3, 6, 7, 8\}$.

$(A \backslash B) \cup (B \backslash C)$

$= \{2, 7\} \cup \{4, 5\} = \{2, 4, 5, 7\}$

$(A \backslash C) = \{2, 5\} \neq (A \backslash B) \cup (B \backslash C)$

For the given sets A, B, C,

$(A \backslash B) \cup (B \backslash C) \neq A \backslash C$

$\therefore$ stmt false in general

10. [2] Let $\mathcal{R}$ be the relation on $\mathbb{Z}$ defined by $(x, y) \in \mathcal{R} \Leftrightarrow 3x + 4y = 1$. Prove that $\mathcal{R}$ is antisymmetric.

Suppose $(x, y), (y, x) \in \mathcal{R}$. Want $x = y$

Since $(x, y) \in R$, $3x + 4y = 1$

" $(y, x) \in R$, $3y + 4x = 1$

$\therefore 3x + 4y = 3y + 4x$

$\therefore x = y$

$\therefore \mathcal{R}$ is anti-symmetric.

11. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

F Any relation on $\{1, 2, 3\}$ contains at most 2^3 ordered pairs.

F If $\mathcal{R}$ is an antisymmetric relation on $\mathbb{Z}$ and $(1, 2) \notin \mathcal{R}$, then $(2, 1) \in \mathcal{R}$.

T The relation $\mathcal{R} = \{(1, 1), (2, 2), (3, 3)\}$ on $\{1, 2, 3\}$ is transitive.

T If $\sim$ is an equivalence relation and $x \sim y$, then the equivalence class of x equals the equivalence class of y.

12. (a) [3] Let $\sim$ be the relation on $\mathbb{Z}$ defined by $a \sim b \Leftrightarrow 6 \mid a^2 - b^2$. Prove that $\sim$ is an equivalence relation.

Refl: Take any $x \in \mathbb{Z}$. Then $6 \mid x^2 - x^2 = 0$
$\therefore x \sim x \quad \therefore \sim$ is reflexive.

Symm Suppose $x \sim y$. Then $6 \mid x^2 - y^2$
$\therefore \exists k \in \mathbb{Z}$ s.t. $6k = x^2 - y^2$
$\therefore 6(-k) = y^2 - x^2 \quad \therefore 6 \mid y^2 - x^2$
$\therefore y \sim x \quad \therefore \sim$ is symmetric.

Trans Suppose $x \sim y$ & $y \sim z$
$\therefore 6 \mid x^2 - y^2$ & $6 \mid y^2 - z^2$
$\therefore 6 \mid (x^2 - y^2) + (y^2 - z^2) = x^2 - z^2$
$\therefore x \sim z \quad \therefore \sim$ is transitive.

(b) [1] Which of $1, 2, \ldots, 7$ belong to the equivalence class $[11]$?

1 b/c $11^2 - 1^2 = 120 \wedge 6 \mid 120$
5 b/c $11^2 - 5^2 = 96 \wedge 6 \mid 96$.
7 b/c $11^2 - 7^2 = 72 \wedge 6 \mid 72$

13. [2] Show that the function $f : \mathbb{N} \to \{4, 5, \ldots\}$ defined by $f(x) = x^2 + 3$ is not onto.

If $f(x) = 5$, then $x^2 + 3 = 5$
$\therefore x^2 = 2$
$\therefore x = \pm\sqrt{2}$

But $\sqrt{2}, -\sqrt{2} \notin \mathbb{N}$. $\therefore 5$ is not a value of f $\therefore f$ is not onto.

14. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

T If $f : A \to B$ is onto then the range of f is B.
F Every function $g : \{a, b, c, d, e\} \to \{x, y, z\}$ is onto.
T There is a bijection $f : \{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$ if and only if $m = n$.
T If $f : A \to B$ is 1-1, then so is $\iota_B \circ f$.

15. [4] Suppose $f : A \to B$ and $g : B \to C$ are both 1-1 functions. Prove that $g \circ f : A \to C$ is 1-1.

Suppose $g \circ f(a_1) = g \circ f(a_2)$

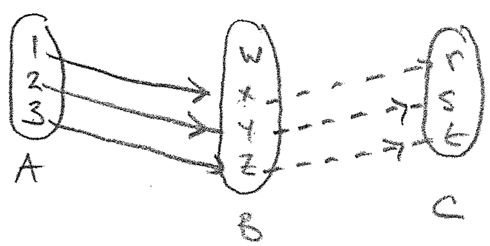
$\therefore$ $g(f(a_1)) = g(f(a_2))$

Since g is 1-1, $f(a_1) = f(a_2)$

Since f is 1-1, $a_1 = a_2$

$\therefore$ $g \circ f$ is 1-1

16. [3] Let $A = \{1, 2, 3\}, B = \{w, x, y, z\}, C = \{r, s, t\}$. Let $f : A \to B$ be given by $f = \{(1, x), (3, z), (2, y)\}$. Find a function $g : B \to C$ such that $g \circ f : A \to C$ is $g \circ f = \{(1, r), (2, s), (3, t)\}$. Explain why $g \circ f$ is invertible and find $(g \circ f)^{-1}$.



$g = \{(w, r), (x, r), (y, s), (z, t)\}$

Look: $(g \circ f)^{-1} = \{(r, 1), (s, 2), (t, 3)\}$
(pairs are reversed)

Since the inverse exists, $g \circ f$ is invertible

17. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

T If $a \in \mathbb{Z}$ and $5|a^3$ then $5|a^2$.

T If $a, b \in \mathbb{N}$ then $gcd(a, b) \mid lcm(a, b)$.

T If p is prime and $a \in \mathbb{Z}$ then $gcd(a, p) \in \{1, p\}$.

T $5 \mid (2345)_{15}$.

18. [4] Prove that divisibility is a transitive relation on $\mathbb{Z}$. That is, prove that for all $a, b, c \in \mathbb{Z}$, if $a|b$ and $b|c$, then $a|c$.

Suppose $a|b$ & $b|c$.

$\therefore \exists\ k, l \in \mathbb{Z}$ s.t. $ak = b$ & $bl = c$

$\therefore\ c = bl = (ak)l = a(kl)$

Since $k, l \in \mathbb{Z}$, $kl \in \mathbb{Z}$

$\therefore\ a|c$

19. [3] Find all ordered pairs of integers (c, d) such that $(cd)_5 = (dc)_9$.

$(cd)_5 = 5c + d$ & $(dc)_9 = 9d + c$

$\therefore\ 5c + d = 9d + c$ $\quad \therefore\ 4c = 8d$

$\therefore\ c = 2d$

c, d base 5 digits $\therefore\ c, d \in \{0, 1, 2, 3, 4\}$

$d = 0 \Rightarrow c = 0$ $\quad \therefore\ (0,0)$ is a sol'n

$d = 1 \Rightarrow c = 2$ $\quad \therefore\ (2,1)$ " " "

$d = 2 \Rightarrow c = 4$ $\quad \therefore\ (4,2)$ " " "

$d = 3 \Rightarrow c = 6$ X not a base 5 digit

$d = 4 \Rightarrow c = 8$ X " " " " "

$\therefore$ 3 sol'ns : $(0,0), (2,1), (4,2)$

20. [2] Use the blank to indicate whether each statement is True (T) or False (F). All variables are integers. No justification is necessary.

T If p is an odd prime number, then $p \equiv 1 \pmod 4$ or $p \equiv 3 \pmod 4$.

T $(d_2d_1d_0)_7 \equiv d_2 + d_1 + d_0 \pmod 3$.

T If $a \equiv 5 \pmod 6$, then $a^3 \equiv 2 \pmod 3$.

T The integer n is odd if and only if $\lfloor n/2 \rfloor < \lceil n/2 \rceil$.

21. [4] Use the Euclidean Algorithm to show that $gcd(2017, 122) = 1$ and then use your work to find integers x and y such that $2017x + 122y = 1$.

$2017 = 122 \cdot 16 + 65$
$122 = 65 \cdot 1 + 57$
$65 = 57 \cdot 1 + 8$
$57 = 8 \cdot 7 + 1 \Leftarrow$ gcd
$8 = \underline{1} \cdot 8 + 0$

$1 = 57 - 8 \cdot 7$
$= 57 - 7(65 - 57) = 8 \cdot 57 - 7 \cdot 65.$
$= 8(122 - 65) - 7 \cdot 65 = 8 \cdot 122 - 15 \cdot 65$
$= 8 \cdot 122 - 15(2017 - 16 \cdot 122)$
$= 248 \cdot 122 - 15 \cdot 2017$
$= 2017(\underbrace{-15}_{x}) + 122(\underbrace{248}_{y})$

22. [3] Suppose $k \equiv 3 \pmod 5$. Find the remainder when $11k^3 + 12$ is divided by 5.

Want r s.t. $0 \le r \le 4$ and $11k^3 + 12 \equiv r \pmod 5$

$11k^3 + 12 \equiv 1 \cdot k^3 + 2$
$\equiv 3^3 + 2 \equiv 4 \pmod 5$

$\therefore$ The remainder is 4

23. [3] Let p be a prime number. Prove that if $p|ab$, then $p|a$ or $p|b$. Give an example to show that the statement can be false if p is not prime.

Let p be prime. If $p|a$ then we're done.
Otherwise $gcd(a, p) = 1$
$\therefore \exists x, y \in \mathbb{Z}$ s.t. $ax + py = 1$
$\therefore abx + pby = b$

Since $p|ab$, $p|abx$
Also $p|pby$ $\therefore p|abx + pby = b$.

24. [4] Use induction to prove that $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ for all $n \geq 1$.

Basis When $n=1$, LHS $=1$ & RHS $= \frac{1(1+1)}{2} = 1$

$\therefore$ stmt true when $n=1$.

I.H. Assume the stmt is true for $n = 1, 2, \ldots, k$ for some $k \geq 1$

i.e. $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ if $1 \leq n \leq k$.

IS. Want $1 + 2 + \cdots + k+1 = \frac{(k+1)((k+1)+1)}{2}$

$$\begin{aligned} LHS &= 1 + 2 + \cdots + (k+1) \\ &= \underbrace{1 + 2 + \cdots + k} + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1)\frac{2}{2} \\ &= \frac{k+1}{2}\left[k+2\right] \checkmark \end{aligned}$$

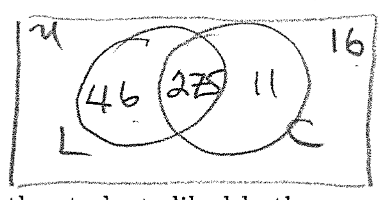
as wanted.

$\therefore$ By induction $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ $\forall n \geq 1$.

25. In a group of 348 students, 321 liked Math 122, and 286 liked Math 101. Sixteen students liked neither of these courses.

Let L be the set who liked 122

C " " " " " 101



$|L| + |C| - |L \cap C| = 348 - 16$

$\therefore |L \cap C| = 321 + 286 - 332 = 275$

(a) [1] How many of the students liked both courses?

275

(b) [1] How many of the 348 students did not like Math 101?

$16 + 46 = 62$

26. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2,\ b_1 = 5$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + 3^n$ for all $n \geq 0$.

Basis: When $n = 0$, $b_0 = 2 = 2^0 + 3^0$ ✓
" $n = 1$, $b_1 = 5 = 2^1 + 3^1$ ✓

∴ Stmt true when $n=0$ & when $n=1$.

IH: Assume $b_0 = 2^0 + 3^0$,
$b_1 = 2^1 + 3^1$,
⋮
$b_k = 2^k + 3^k$ for some $k \geq 1$.

IS: Want $b_{k+1} = 2^{k+1} + 3^{k+1}$

Look at $b_{k+1} = 5b_k - 6b_{k-1}$ (b/c $k+1 \geq 2$)
$= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$
$= 5 \cdot 2^k + 5 \cdot 3^k - 3 \cdot 2^k - 2 \cdot 3^k$
$= 2 \cdot 2^k + 3 \cdot 3^k$
$= 2^{k+1} + 3^{k+1}$ ✓ as wanted

∴ By induction $b_n = 2^n + 3^n$ $\forall n \geq 0$

27. [2] Use the blank to indicate whether each statement is True (T) or False (F). No justification is necessary.

T The number of proper subsets of a finite set A equals the number elements of the power set of A minus 1.

T There is no set A for which there are exactly 2^{10} relations on A.

F If $|A| = 7$ and $|B| = 10$, then $|A \times B| = 17$.

F The number of 1-1 functions $f : \{1, 2, \ldots, 5\} \to \{1, 2, \ldots, 100\}$ is 100!.

/END