UNIVERSITY OF VICTORIA
EXAMINATIONS APRIL 2019
MATH 122, SECTIONS [A01], [A02], [A03], [A04], [A05]

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Lecture Section: ______________ Given (First) Name: ______________

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Points:	8	7	8	6	5	5	9	9	7	8	8	80
Score:												

THIS EXAM CONTAINS 13 PAGES (INCLUDING THIS COVER PAGE) AS WELL AS THE YELLOW INSTRUCTIONS PAGE AT THE END. COUNT THE NUMBER OF PAGES IN THIS EXAMINATION PAPER BEFORE BEGINNING TO WRITE, AND REPORT ANY DISCREPANCY IMMEDIATELY TO THE INVIGILATORS. THIS EXAM CONSISTS OF 26 QUESTIONS, TO BE ANSWERED ON THE EXAM PAPER. DURATION: 3 HOURS

Do not open this packet until instructed to do so.

1. Identifying information: Enter your Student ID, Name, and Lecture Section at the top of **this page** now. Fill out and sign the **yellow instruction page** on the back of this exam now.
2. Only the following materials are permitted:
   (a) Your UVic Student Identity Card (place it on your desk now).
   (b) Pens, pencils, erasers, and a ruler are permitted at your desk. If you have a pencil case it must be stored with your belongings in the front of the room.
   (c) A calculator is permitted. If you use a calculator, it must be a Sharp brand calculator with model number beginning EL510-R.
3. No notes, outside paper, or aid other than the ones listed above is permitted. You are responsible for ensuring that any unauthorized material is stored with your belongings at the front of the room.
4. Show all calculations on this paper for all problems, unless the question specifies otherwise. We may disallow any answer given without appropriate justification.
5. If you need to leave the room during the exam, raise your hand until an invigilator comes to you. Students may not leave during the first 30 minutes or the last 15 minutes of the exam.

Do not open this packet until instructed to do so.

(3 points) 1. Suppose the statement $(\neg p \vee r)$ is true. Find all combinations of truth values for p, r, s such that $(r \leftrightarrow s) \wedge (p \vee \neg s)$ is true.

$\neg p \vee r$ T $\Rightarrow$ p F or p, r both T.

Need $r \Leftrightarrow s$ & $p \vee \neg s$ both T

$$\begin{pmatrix} p & r & s & \\ F & F & F & \checkmark \\ \cancel{F} & \cancel{T} & \cancel{T} & \times \\ T & T & T & \checkmark \end{pmatrix}$$

(3 points) 2. Use known logical equivalences to prove that $(p \wedge s) \rightarrow r$ is logically equivalent to $p \rightarrow (s \rightarrow r)$.

$(p \wedge s) \rightarrow r$

$\Leftrightarrow \neg(p \wedge s) \vee r$ Known LE.

$\Leftrightarrow (\neg p \vee \neg s) \vee r$ DeMorgan

$\Leftrightarrow \neg p \vee (\neg s \vee r)$ Assoc.

$\Leftrightarrow \neg p \vee (s \rightarrow r)$ Known LE

$\Leftrightarrow p \rightarrow (s \rightarrow r)$ Known LE.

(2 points) 3. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F The negation of "There is an answer to every question" is "There is no answer to every question".

T The statement $\forall x, [(x < 0) \rightarrow (5x > 0)]$ is true for the universe $\mathbb{N}$.

F The statements $\forall x\ \exists y, x+y = 0$ and $\exists y\ \forall x, x+y = 0$ are logically equivalent for the universe $\mathbb{Q}$.

T The statements $\forall x\ \exists y, xy = 0$ and $\exists y\ \forall x, xy = 0$ are logically equivalent for the universe $\mathbb{Q}$.

(7 points) 4. Determine whether each of the following arguments is valid or invalid. If valid, use known logical equivalences and laws of inference to prove that it is valid. If invalid, provide a well-explained counterexample to demonstrate that it is invalid.

(a)

$$\begin{array}{ll} \neg(p \wedge r) & F \\ \neg r \to s & T \\ p & T \\ \hline \therefore \; s & F \end{array}$$

~~$\begin{pmatrix} p & r & s \\ T & T & F \end{pmatrix}$~~ ∴ try to show valid

1. $\neg(p \wedge r)$ Premise
2. $\neg r \to s$ Premise
3. p Premise
4. $\neg p \vee \neg r$ 1, DeMorgan
5. $p \to \neg r$ 4, Known LE.
6. $p \to s$ 5,2 Chain Rule
7. ∴ s 6,3 M.P.

(b)

$$\begin{array}{ll} \neg(p \wedge r) & T \\ s \vee \neg r & T \\ s \to p & T \\ \hline \therefore \; p \wedge s & F \end{array}$$

$\begin{pmatrix} p & r & s \\ F & F & F \end{pmatrix}$

This truth assignment is s.t. all premises are T & the conclusion is F.

∴ It is a counterexample

∴ The argument is not valid.

(3 points) 5. Let A, B, and C be sets such that $A \subseteq B \cup C$. Use the definition of subset to prove that $A \setminus B \subseteq C$. Your proof must begin with "Take any $x \in A \setminus B$".

Take any $x \in A \setminus B$.

$\therefore$ $x \in A$ and $x \notin B$.

Since $A \subseteq B \cup C$, $x \in B \cup C$

$\therefore$ $x \in B$ or $x \in C$

Since we know $x \notin B$, $x \in C$

$\therefore$ $A \setminus B \subseteq C$

(3 points) 6. Provide a well-explained counterexample to show that the following statement is false: *For all sets A, B, and C,* $A \setminus (B \cap C) = (A \setminus B) \cap (A \setminus C)$.

Let $\mathcal{U} = \{1,2,3\}$, $A = \{1,2\}$, $B = \{1\}$, $C = \{2\}$.

Then $B \cap C = \phi$ $\therefore$ $A \setminus (B \cap C) = A = \{1,2\}$

Now $A \setminus B = \{1,2\} \setminus \{1\} = \{2\}$

$A \setminus C = \{1,2\} \setminus \{2\} = \{1\}$

$\therefore$ $(A \setminus B) \cap (A \setminus C) = \{2\} \cap \{1\} = \phi \neq \{1,2\}$

$\therefore$ $A \setminus (B \cap C) \neq (A \setminus B) \cap (A \setminus C)$ for all A, B, C.

(2 points) 7. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F For all sets A and B, if $A \cap B = \emptyset$, then $A = \emptyset$ or $B = \emptyset$.

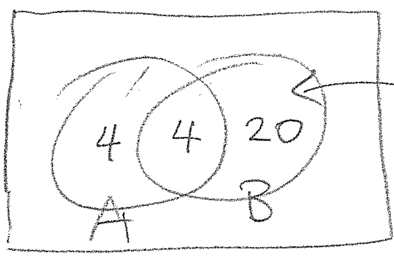
T $\{\emptyset\} \subseteq \mathcal{P}(A)$.

F For all sets A, B, and C, if $A \oplus B = C$, then $A \subseteq C$ and $B \subseteq C$.

T $|\{\emptyset, \{\emptyset\}, 8, \emptyset\}| = 3$.

(3 points) 8. Suppose that A and B are sets such that $|A \cup B| = 28$, $|A| = 2|A \cap B|$, and $|B| = 3|A|$. What is $|B \setminus A|$?

$$|A \cup B| = |A| + |B| - |A \cap B|$$
$$\therefore \quad 28 = \cancel{2}|A \cap B| + 3|A| - \cancel{|A \cap B|}$$
$$= |A \cap B| + 3 \cdot 2 \cdot |A \cap B|$$
$$= 7 \cdot |A \cap B| \qquad \therefore |A \cap B| = 4$$
$$|A| = 8$$
$$|B| = 24$$



$$\therefore |B \setminus A| = 20$$

(3 points) 9. Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 4$, and for all $n \geq 2$, $a_n = 7a_{n-1} + 4$.

(a) Calculate a_2, a_3, and a_4.

$$a_1 = 4$$
$$a_2 = 7 \cdot a_1 + 4 = 7 \cdot 4 + 4$$
$$a_3 = 7\, a_2 + 4 = 7(7 \cdot 4 + 4) + 4$$
$$= 7^2 \cdot 4 + 7 \cdot 4 + 4$$
$$a_4 = 7a_3 + 4 = 7(7^2 \cdot 4 + 7 \cdot 4 + 4) + 4$$
$$= 7^3 \cdot 4 + 7^2 \cdot 4 + 7 \cdot 4 + 4$$

(b) Use your work in (a) to guess a closed form formula for a_n. (That is, your formula should not involve recursion, nor should it express a_n as a sum using Σ notation or $\cdots$.) Do **not** prove that your formula is correct.

Guess $$a_n = 7^{n-1} \cdot 4 + 7^{n-2} \cdot 4 + \cdots + 7 \cdot 4 + 4$$
$$= 4(1 + 7 + \cdots + 7^{n-1})$$
$$= \overset{2}{\cancel{4}} \cdot \frac{7^n - 1}{\underset{3}{\cancel{7-1}}} = \frac{2(7^n - 1)}{3}$$

(5 points) 10. Use induction to prove that, for all integers $n \geq 0$,

$$0 \cdot 2^0 + 1 \cdot 2^1 + 2 \cdot 2^2 + \cdots + n \cdot 2^n = 2 + (n-1)2^{n+1}.$$

Basis : when $n = 0$,

$LHS = 0 \cdot 2^0 = 0$

$RHS = 2 + (0-1) \cdot 2^{0+1} = 2 - 2 = 0$

$\therefore$ The stmt is True when $n = 0$.

IH. Suppose there is an integer $k \geq 0$ s.t. $0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + n \cdot 2^n = 2 + (n-1)2^{n+1}$ is true for $n = 0, 1, \ldots, k$.

IS Want $0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + (k+1) 2^{k+1}$

$$= 2 + ((k+1)-1) 2^{(k+1)+1}$$
$$= 2 + k \cdot 2^{k+2}$$

Look at

$$0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + (k+1) 2^{k+1}$$
$$= \underbrace{0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + k \cdot 2^k} + (k+1) 2^{k+1}$$
$$= 2 + (k-1) 2^{k+1} + (k+1) 2^{k+1} \quad \text{by IH.}$$
$$= 2 + 2^{k+1} (k - 1 + k + 1)$$
$$= 2 + 2^{k+1} \cdot 2k$$
$$= 2 + k \cdot 2^{k+2} \quad \text{as wanted}$$

$\therefore$ By PMI,

$$0 \cdot 2^0 + 1 \cdot 2^1 + \cdots + n \cdot 2^n = 2 + (n-1) 2^{n+1}$$

$\forall n \geq 0$.

(5 points) 11. Let $s_1, s_2, \ldots$ be the sequence recursively defined by $s_1 = 1$, $s_2 = 1$, $s_3 = 1$, and for all $n \geq 4$, $s_n = s_{n-3} + s_{n-2} + s_{n-1}$.
Use induction to prove that, for all integers $n \geq 1$, $s_n < 2^n$.

Basis $s_1 = 1 < 2^1$, $s_2 = 1 < 2^2$, $s_3 = 1 < 2^3$

$\therefore$ The stmt is true when $n = 1, 2$ or 3.

IH Suppose there is an integer $k \geq 3$ s.t.

$s_n < 2^n$

for $n = 1, 2, \ldots, k$.

IS Want $s_{k+1} < 2^{k+1}$

Look at s_{k+1}. Since $k+1 \geq 3+1 = 4$ we have

$$\begin{aligned} s_{k+1} &= s_{k-2} + s_{k-1} + s_k \\ &< 2^{k-2} + 2^{k-1} + 2^k \\ &= 2^{k-2} + 2 \cdot 2^{k-2} + 2^2 \cdot 2^{k-2} \\ &= 7 \cdot 2^{k-2} < \underbrace{8}_{=2^3} \cdot 2^{k-2} = 2^{k+1}, \text{ as wanted} \end{aligned}$$

$\therefore$ By PMI $s_n < 2^n$ $\forall n \geq 1$

(2 points) 12. Let a and b be positive integers. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F If $\gcd(a, b) = 15$ then $\gcd(4a, b) = 15$.

F If $6 \mid ab$, then $6 \mid a$ or $6 \mid b$.

T For any prime number p, if $p \mid a$ and $p \mid b^2$, then $p \mid (6a - b)$.

T If $a \equiv b \pmod 8$ then $\overset{11}{\cancel{3}}a^2 \overset{+3}{\cancel{-5}} \equiv 11ab + 3 \pmod 8$.

(3 points) 13. Let a, b, and c be integers. Prove that if $a \mid b$ and $ab \mid c$, then $a^3 \mid bc$.

Suppose $a \mid b$ and $ab \mid c$.

$\therefore$ There exist $k_1, k_2 \in \mathbb{Z}$ s.t.

$ak_1 = b$ and $abk_2 = c$

$\therefore bc = (ak_1)(abk_2)$

$= ak_1 \cdot a \cdot (ak_1) \cdot k_2 = k_1^2 \cdot k_2 \cdot a^3$

Since $k_1^2 \cdot k_2 \in \mathbb{Z}$, $a^3 \mid bc$

(2 points) 14. Use the Euclidean Algorithm to determine $\gcd(7912, 3870)$.

$7912 = 3870 \cdot 2 + 172$

$3870 = 172 \cdot 22 + 86 \leftarrow \gcd$

$172 = 86 \cdot 2 + 0$

$\therefore \gcd(7912, 3870) = 86$

(2 points) 15. Explain why there are no integers x and y such that $30x + 12y = 10$.

$\gcd(30, 12) = 6$. Any integer c for which there exist $x, y \in \mathbb{Z}$ s.t. $30x + 12y = c$ must be a multiple of 6. Since 10 isn't a multiple of 6 the required integers x & y don't exist.

(2 points) 16. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F $\text{lcm}(2^2 \cdot 3^6,\ \ 2^3 \cdot 3^4) = 2^2 \cdot 3^4$

F $\gcd(20!,\ 25!) = 5!$

T If p is a prime number, and a is an integer such that a and p are not relatively prime, then $p \mid a$.

T If a is an integer, then a and $a+1$ are relatively prime.

(2 points) 17. Write 2773 in base 11, using A to represent 10.

$$2773 = 252 \cdot 11 + 1$$
$$252 = 22 \cdot 11 + 10$$
$$22 = 2 \cdot 11 + 0$$
$$2 = \underline{\underline{0}} \cdot 11 + 2$$

$\therefore\ 2773 = (20A1)_{11}$

(5 points) 18. (a) Suppose that $6(3x-5) \equiv 12 \pmod{17}$. What is the remainder when x is divided by 17?

$12 \equiv 6(3x-5) \equiv \cancel{18}^{1}x - 30 \pmod{17}$

$\therefore\ x \equiv 12 + 30 \equiv 42 \equiv 25 \equiv 8 \pmod{17}$

$\therefore$ The remainder is 8.

(b) What is the remainder when $a = 5 \cdot 2^{84} - 16$ is divided by 17?

Want r s.t. $0 \le r \le 16$ & $a \equiv r \pmod{17}$

$$a = 5 \cdot 2^{84} - 16 \equiv 5 \cdot (2^4)^{21} - 16$$
$$\equiv 5(-1)^{21} - 16$$
$$\equiv 5(-1) - 16$$
$$\equiv -21 \equiv -4 \equiv 13 \pmod{17}$$

$\therefore$ The remainder is 13

(2 points) 19. Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F $\emptyset$ is an equivalence relation on A.

T If $\mathcal{R}$ is an anti-symmetric relation on A and $(1,3) \in \mathcal{R}$, then $\mathcal{R}$ is not symmetric.

T If $\mathcal{R}$ is a relation on A, and $|\mathcal{R}| < 7$, then $\mathcal{R}$ is not reflexive.

F If $\mathcal{R}$ is an equivalence relation on A, and $(2,6) \in \mathcal{R}$, then $[2] \cap [6] = \emptyset$.

(5 points) 20. Let $\sim$ be the relation on $\mathbb{Z}$ defined by $a \sim b$ if and only if $5 \mid (3a - 3b)$. Prove that $\sim$ is an equivalence relation.

reflexive: Let $a \in \mathbb{Z}$. Then $5 \mid 3a - 3a = 0$

$\therefore \sim$ is reflexive

symmetric. Suppose $a \sim b$.

Then $5 \mid 3a - 3b$

$\therefore 5 \mid -(3a - 3b) = 3b - 3a$

$\therefore b \sim a$

$\therefore \sim$ is symmetric

transitive. Suppose $a \sim b$ and $b \sim c$

Then $5 \mid 3a - 3b$ & $5 \mid 3b - 3c$

$\therefore 5 \mid (3a - 3b) + (3b - 3c) = 3a - 3c$

$\therefore a \sim c$

$\therefore \sim$ is transitive

$\therefore \sim$ is an equivalence relation.

(2 points) 21. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F The relation $\{(x^4, x) : x \in \mathbb{R}\}$ is a function from $\mathbb{R}$ to $\mathbb{R}$.

T Let $n \in \mathbb{N}$. The function $f : \{1, 2, ..., 2n\} \to \{0, 1, 2, 3, ..., n\}$ defined by $f(a) = \lfloor \frac{a}{2} \rfloor$ is surjective (i.e., onto).

T Let A and B be sets. Then a function $f : A \to B$ has an inverse if and only if it is both injective (i.e., one-to-one) and surjective (i.e., onto).

F Let A, B, and C be sets. If $f : A \to B$ and $g : B \to C$ are functions, then $g \circ f$ is bijective if and only if f and g are both bijective.

(4 points) 22. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = 2n + 1$.

(a) Prove that f is injective (i.e., one-to-one).

Suppose $f(n_1) = f(n_2)$

Then $\cancel{2}n_1 \cancel{+1} = \cancel{2}n_2 \cancel{+1}$

$\therefore n_1 = n_2$ $\therefore f$ is 1-1

(b) Prove that f is not surjective (i.e., onto).

Suppose $f(n) = y$.

Then $2n+1 = y$, so $n = \frac{y-1}{2}$

$\therefore$ If $y = 0$ then we need $n = -\frac{1}{2}$ in order to have $f(n) = y$.

But $-1/2 \notin \mathbb{Z}$ $\therefore \nexists n$ s.t. $f(n) = 0$

$\therefore f$ is not onto.

(2 points) 23. Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(n) = -n$.

(a) Show that $f \circ f$ is the identity function on $\mathbb{Z}$.

$f \circ f : \mathbb{Z} \to \mathbb{Z}$ and $f \circ f(n)$

$= f(f(n)) = f(-n) = --n = n$

$\therefore f \circ f = \iota_{\mathbb{Z}}$

(b) Is f invertible? Give a brief explanation of your answer.

Yes. Clearly $f(n) = -n$ is 1-1 & onto.

OR We need a fn $f^{-1} : \mathbb{Z} \to \mathbb{Z}$ s.t.

$f \circ f^{-1} = f^{-1} \circ f = \iota_{\mathbb{Z}}$.

By (a), $f^{-1} = f$ works. $\therefore f$ is invertible

(2 points) 24. Let $X = \{1, 2, 3\}$ and let $Y = \{1, 2, 3, 4, 5, 6, 7, 8\}$. Fill in each blank. No justification is necessary, and you do not need to simplify your answer.

- $|X \times X| =$ 3×3
- The number of subsets of Y that contain no elements of X is 2^5
- The number of injective (one-to-one) functions with domain X and codomain (target) Y is $8 \cdot 7 \cdot 6$
- The number of surjective (onto) functions with domain X and codomain (target) Y is 0

(3 points) 25. Let $A = \mathbb{N}$, and let $B = \left\{\frac{1}{n} : n \in \mathbb{N}\right\}$. Prove that $A \cup B$ is countable.

The sequence $1, \frac{1}{1}, 2, \frac{1}{2}, 3, \frac{1}{3}, \ldots$

contains every element of $A \cup B$.

$\therefore$ $A \cup B$ is countable.

(3 points) 26. Use the blank to indicate whether each statement is **True** (T) or **False** (F). No justification is necessary.

F If A is an infinite set, and $a \in A$, then $|A \setminus \{a\}| \neq |A|$.

F If A and B are infinite sets, then there is a bijection $f : A \to B$.

T If $B \subseteq \mathbb{Q}$, then B is countable.

T The open interval $(-1, 3)$ has the same cardinality as the open interval $(2, 9)$.

T There are sets A and B such that A is uncountable, B is uncountable, and $A \cap B$ is countable.

F There is a set B such that, for all sets A, $|A| \leq |B|$.

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[END]