1. [3] Find all truth values for p, q and r for which $(\neg q) \wedge (r \to (\neg p))$ is true.

Need $\neg q$ T (q F) and $r \to \neg p$ T

For $r \to \neg p$ to be T,

either r is F or r & $\neg p$ both T

$\therefore$ p F

$\therefore$ possibilities

p	q	r
F	F	F
T	F	F
F	F	T

2. [4] Use known logical equivalences to show that $p \to (q \vee r)$ is logically equivalent to $(p \wedge (\neg r)) \to q$.

$p \to (q \vee r) \Leftrightarrow \neg p \vee (q \vee r)$ Known LE

$\Leftrightarrow \neg p \vee (r \vee q)$ Comm.

$\Leftrightarrow (\neg p \vee r) \vee q$ Assoc

$\Leftrightarrow \neg(\neg p \vee r) \to q$ Known LE

$\Leftrightarrow (p \wedge \neg r) \to q$ DeMorgan

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $(\neg p) \Leftrightarrow q$ then $p \Leftrightarrow (\neg q)$.

F The negation of $\forall x, \exists y, x \vee y$ is $\exists y, \forall x, (\neg x) \wedge (\neg y)$.

F The statement $\forall x,\ x \leq 5x$ is true for the universe $\mathbb{R}$.

F If an argument is valid, then its conclusion is true.

4. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{l} (\neg p) \to q \\ (\neg r) \to (\neg q) \\ \neg r \\ \hline \therefore\ p \end{array}$$

1. ¬p → q   Premise
2. ¬r → ¬q   "
3. ¬r   "
4. ¬q → p   1, Contrapos.
5. ¬r → p   2, 4 Chain Rule
6. ∴ p   3, 5 M.P.

5. [3] Give a counterexample to show that the following argument is invalid.

$$\begin{array}{l} (\neg p) \to q \\ (\neg q) \to (\neg s) \\ r \to s \\ p \leftrightarrow (\neg r) \\ \hline \therefore\ p \end{array}$$

Want: Prem's T & Concl. F.

p	q	r	s
F	T	T	T

For the given T.A., all prem's are T & the concl. is F

∴ The arg is not valid.

6. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is True or False. No justification is necessary.

F If $A \not\subseteq B$, then $B \subseteq A$.

F If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \subseteq B$.

F If $10 \in A$ and $3 \in B$ then $30 \in A \times B$.

T If $C \in \mathcal{P}(A)$ then $A \setminus C \in \mathcal{P}(A)$.

7. [4] Suppose that $A \subseteq B$. Prove that $A \subseteq A \cap B$, using an argument that starts with "*Take any* $x \in A$ ...". Are the sets A and $A \cap B$ actually equal? Explain.

Suppose $A \subseteq B$. Take any $x \in A$.
Since $A \subseteq B$, $x \in B$. $\therefore x \in A \cap B$.
$\therefore A \subseteq A \cap B$.

Yes, $A = A \cap B$. It is always true that $A \cap B \subseteq A$ (by def'n.)
$\therefore A = A \cap B$.

8. [4] Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$. Hint: set-theoretic identities.

$(A \setminus B^c) \cup (A^c \cup B)^c$

$= (A \cap B^{cc}) \cup (A^{cc} \cap B^c)$ { Known = DeMorgan Dbl Comp

$= A \cap (B \cup B^c)$ Dist.

$= A \cap \mathcal{U}$ Known =

$= A$ Identity

9. [2] Let $A = \{1, 2, 3, 4, 5, 6, 7\}$. Fill in each blank. No justification is necessary.

(a) The number of subsets of A that contain neither 1 nor 2 is 2^5.

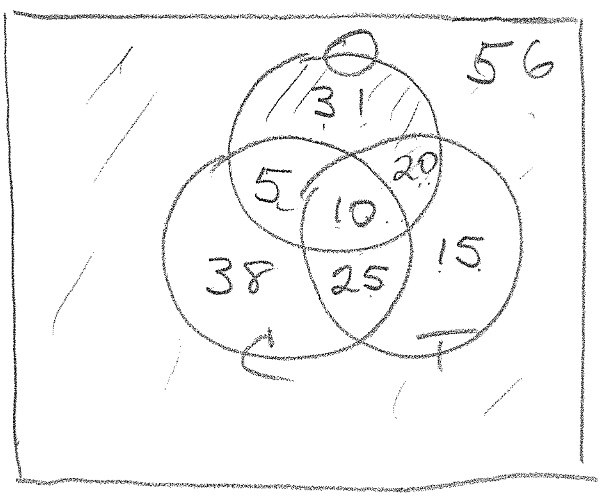
(b) The number of functions $f : A \to \{w, x, y, z\}$ where $f(1) = x$, $f(2) = w$ and $f(5) \neq x$, is $1 \cdot 1 \cdot 4 \cdot 4 \cdot 3 \cdot 4 \cdot 4$.

(c) $|A \times A| =$ $7 \times 7 = 49$.

(d) The number of relations on A is 2^{49}.

10. [2] An airline surveyed 200 passengers and recorded the following information about the type of non-alcoholic beverages they liked: 78 passengers liked coffee, 70 liked tea, and 66 liked orange juice; 35 liked both coffee and tea, 30 liked both tea and orange juice, 15 liked both coffee and orange juice, and 10 liked all three types of beverages. How many passengers liked only orange juice or none of the three beverages?

Let C: set who like coffee
T: set who like tea
O: " " " OJ



31 like only OJ
56 " none

∴ answer = 87

11. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

T If $d \mid a$ and $d \mid b$, then $d \mid (b - a)$.

F If $a \mid bc$, then $a \mid b$ or $a \mid c$.

F $\text{lcm}(2^4 7^6, 2^1 3^2 7^2) = 2^5 3^2 7^8$.

T If $\gcd(a, b) = 5$ then there exist integers x and y such that $ax + by = 20$.

12. [4] Use the Euclidean Algorithm to find $d = gcd(2500, 1120)$ and then use your work to find integers x and y such that $2500x + 1120y = d$.

$2500 = 1120 \times 2 + 260$
$1120 = 260 \times 4 + 80$
$260 = 80 \times 3 + 20 \leftarrow d = gcd(2500, 1120)$
$80 = 20 \times 4 + 0$

$20 = 260 - 3 \times 80$
$= 260 - 3 \times (1120 - 4 \times 260)$
$= 13 \times 260 - 3 \times 1120$
$= 13 \times (2500 - 2 \times 1120) - 3 \times 1120$
$= 13 \times 2500 - 29 \times 1120$
$= 2500\underbrace{(13)}_{x} + 1120\underbrace{(-29)}_{y}$

13. [4] Find the base 9 representation of 2018.

$2018 = 224 \times 9 + 2$
$224 = 24 \times 9 + 8$
$24 = 2 \times 9 + 6$
$2 = 0 \times 9 + 2$

$\uparrow$ $\therefore 2018 = (2682)_9$

14. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All variables are integers.

T The integers 1000012^{66} and 122^{66} have the same last digit.
F If $a \equiv 0 \pmod 3$ then $a \not\equiv 0 \pmod 2$.
T If $ak = b$ then every prime divisor of a is a divisor of b.
F $(110100)_2 = (C0)_{16}$.

15. [4] Let $n = (d_3d_2d_1d_0)_{10}$. Prove that $3 \mid n$ if and only if $3 \mid (d_3 + d_2 + d_1 + d_0)$.

$3 \mid n \iff n \equiv 0 \pmod 3$

Now $n = (d_3 d_2 d_1 d_0)_{10}$

$= d_3 10^3 + d_2 10^2 + d_1 \cdot 10 + d_0$

$10 \equiv 1 \pmod 3 \rightarrow \quad \equiv d_3 \cdot 1^3 + d_2 1^2 + d_1 \cdot 1 + d_0$

$\equiv d_3 + d_2 + d_1 + d_0 \pmod 3$

$\therefore 3 \mid n \iff n \equiv 0 \pmod 3$

$\iff d_3 + d_2 + d_1 + d_0 \equiv 0 \pmod 3$

$\iff 3 \mid d_3 + d_2 + d_1 + d_0$

16. [4] Use induction to prove that $1(2) + 2(3) + \cdots + n(n+1) = n(n+1)(n+2)/3$ for all integers $n \geq 1$

Basis When $n=1$, LHS $= 1(2) = 2$

& RHS $= 1 \cdot 2 \cdot 3/3 = 2$

$\therefore$ Stmt true when $n=1$.

IH. Assume there is an integer $k \geq 1$ s.t. $1(2) + 2(3) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$ for $n = 1, 2, \ldots, k$.

IS. Want $1(2) + 2(3) + \cdots + (k+1)(k+2) = (k+1)(k+2)(k+3)/3$

Look at LHS

$1(2) + 2(3) + \cdots + (k+1)(k+2)$

$= \underbrace{1(2) + 2(3) + \cdots + k(k+1)} + (k+1)(k+2)$

$= \frac{k(k+1)(k+2)}{3} + \frac{3(k+1)(k+2)}{3}$ by IH

$= \frac{(k+1)(k+2)}{3}\left[k+3\right]$ as wanted.

$\therefore$ By PMI, $1(2) + (2)(3) + \cdots + n(n+1) = \frac{n(n+1)(n+2)}{3}$

17. [4] Let $a_1, a_2, \ldots$ be the sequence defined by $a_1 = 6$, and $a_n = a_{n-1} + 6n$ for $n \geq 2$. Find a_2, a_3, a_4 and a_5, then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary prove that your formula is correct.

$a_2 = a_1 + 6 \cdot 2 = 6 + 6 \cdot 2$

$a_3 = a_2 + 6 \cdot 3 = (6 + 6 \cdot 2) + 6 \cdot 3$

$a_4 = a_3 + 6 \cdot 4 = (6 + 6 \cdot 2 + 6 \cdot 3) + 6 \cdot 4$

$a_5 = a_4 + 6 \cdot 5 = (6 + 6 \cdot 2 + 6 \cdot 3 + 6 \cdot 4) + 6 \cdot 5$

Guess $a_n = 6 + 6 \cdot 2 + 6 \cdot 3 + \cdots + 6 \cdot n$

$= 6(1 + 2 + \cdots + n)$

$= 6\, n(n+1)/2 = 3n(n+1)$

18. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 2$, $b_1 = 5$ and $b_n = 5b_{n-1} - 6b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 2^n + 3^n$ for all $n \geq 0$.

Basis When $n=0$, $b_0 = 2 = 2^0 + 3^0$ ✓

" $n=1$, $b_1 = 5 = 2^1 + 3^1$ ✓

$\therefore$ Stmt true when $n=0$ and when $n=1$.

IH. Assume there is an integer $k \geq 1$ s.t. $b_n = 2^n + 3^n$ for $n = 0, 1, \ldots, k$.

IS Want: $b_{k+1} = 2^{k+1} + 3^{k+1}$.

Look at b_{k+1}. Since $k+1 \geq 2$,

$b_{k+1} = 5b_k - 6b_{k-1}$

$= 5(2^k + 3^k) - 6(2^{k-1} + 3^{k-1})$ by IH

$= 5 \cdot 2^k + 5 \cdot 3^k - 6 \cdot 2^{k-1} - 6 \cdot 3^{k-1}$

$= 5 \cdot 2^k + 5 \cdot 3^k - 3 \cdot 2^k - 2 \cdot 3^k$

$= 2 \cdot 2^k + 3 \cdot 3^k$

$= 2^{k+1} + 3^{k+1}$, as wanted

$\therefore$ By PMI $b_n = 2^n + 3^n$ $\forall n \geq 0$

19. [4] Let $f : \mathbb{Z} \to \mathbb{Z}$ be defined by $f(x) = 3x$. Prove that f is 1-1 but not onto.

1-1: Suppose $f(x_1) = f(x_2)$

Then $3x_1 = 3x_2$

$\therefore x_1 = x_2$

$\therefore f$ is 1-1

Not onto Way 1: Notice $f(x)$ is always a multiple of 3

$\therefore \nexists x$ s.t. $f(x) = 1$

$\therefore f$ is not onto.

Way 2: Take any $y \in \mathbb{Z}$.

Then $f(x) = y \iff 3x = y$

$\iff x = y/3$

But if we take $y = 1$, then we must have $x = 1/3$ and $1/3 \notin \mathbb{Z}$

$\therefore \nexists x \in \mathbb{Z}$ s.t. $f(x) = 1$

$\therefore f$ is not onto

20. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F For any $x \in \mathbb{R}$, $\lfloor -x \rfloor = -\lfloor x \rfloor$.

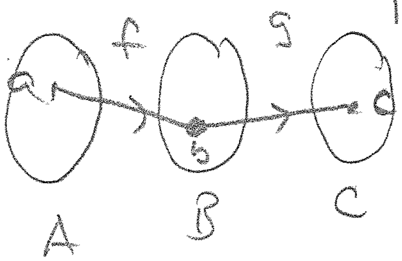
F There exists a one-to-one function from $\{1, 2, 3, 4\}$ to $\{1, 2, 3\}$.

F A function $f : A \to B$ is onto if for every element $a \in A$ there exists an element $b \in B$ such that $(a, b) \in f$.

T A function $f : A \to B$ has an inverse if and only if it is 1-1 and onto.

21. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be onto functions. Prove that $g \circ f : A \to C$ is onto.

Take any $c \in C$. Need $a \in A$ s.t. $g \circ f(a) = c$.



Since g is onto, $\exists\ b \in B$ s.t. $g(b) = c$

Since f is onto, $\exists\ a \in A$ s.t. $f(a) = b$.

$\therefore\ g \circ f(a) = g(f(a)) = g(b) = c$.

$\therefore$ g is onto

22. [5] Let $A = \{1, 2, \ldots, 11\}$, and let $\sim$ be the relation on A defined by $a \sim b \Leftrightarrow 2 \mid (a+b)$. Prove that $\sim$ is an equivalence relation on A and find the partition of A it determines.

reflexive : Take $x \in A$. Then $2 \mid x + x = 2x$,
$\therefore\ x \sim x$ & $\sim$ is reflexive.

Symmetric : Suppose $x \sim y$. Then $2 \mid x+y$.
$\therefore\ 2 \mid y + x$ $\therefore\ y \sim x$
$\therefore$ $\sim$ is symmetric.

transitive : Suppose $x \sim y$ & $y \sim z$.
Then $2 \mid x + y$ and $2 \mid y + z$.
$\therefore\ 2 \mid (x+y) - (y+z) = x - z$
$\therefore\ 2 \mid x - z + 2z = x + z$
$\therefore\ x \sim z$ $\therefore$ $\sim$ is transitive.

$\therefore$ $\sim$ is an equivalence rel'n.

By def'n $x \sim y \Leftrightarrow x+y$ is even.
$\therefore$ The partition of A determined by $\sim$ is $\{\{2,4,6,8,10\}, \{1,3,5,7,9,11\}\}$

23. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F The relation $\{(1,1)\}$ on $\{1,2,3\}$ is reflexive.

T If $\mathcal{R}$ is an antisymmetric relation on $\{1,2,3,4\}$ and $(1,2) \in \mathcal{R}$ then $\mathcal{R}$ is not symmetric.

F The relation $\sim$ on $\{2,4,6,\dots\}$ defined by $x \sim y$ if and only if $x - y = 2$ is transitive.

T If $\sim$ is an equivalence relation on $\mathbb{Z}$ such that $1 \not\sim 3$, then $1 \not\sim 2$ or $2 \not\sim 3$.

24. [3] Let A and B be countably infinite sets. Prove that $A \cup B$ is countable.

Suppose A & B are countably infinite.
∴ ∃ seq $a_1, a_2, \dots$ that contains all elements of A & ∃ seq $b_1, b_2, \dots$ that contains all elements of B.

∴ The seq
$a_1, b_1, a_2, b_2, a_3, b_3, \dots$
contains all elements of $A \cup B$.
Since $A \cup B$ is infinite (b/c A, B are) it is countably infinite.

25. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If A is a countable set and B is an uncountable set, then $A \cup B$ is uncountable.

T For any real numbers a, b such that $a < b$, the closed interval $[a, b]$ is uncountable.

F If $A \subsetneq B$ then $|A| < |B|$.

T The set $\{1, 2, -3, -4, 5, 6, -7, -8 \dots\}$ is countable.

/END