UNIVERSITY OF VICTORIA
DECEMBER EXAMINATIONS 2018

MATH 122: Logic and Foundations

CRN: 12202 (A01), 12203 (A02), and 12205 (A04)
Please circle your section number

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NAME: Review session

V00#: April 2019

Duration: 3 Hours.

Answers are to be written on the exam paper.

No calculator is necessary, but a Sharp EL-510R, Sharp EL-510RNB, or Sharp EL-510RTB calculator is allowed.

This exam consists of 29 questions, for a total of 80 marks. Unless otherwise specified, in order to receive full or partial credit, you must show your work and justify your answers.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Suppose the statement $p \leftrightarrow \neg q$ is true. Find all combinations of truth values for p, q, and r such that the statement $(r \leftrightarrow p) \wedge \neg(r \vee \neg q)$ is true.

$p \leftrightarrow \neg q$

p	q	r	$r \leftrightarrow p$	$(r \vee \neg q)$	$\neg(r \vee \neg q)$
1	0	0	0	1	0
1	0	1	1	1	0
0	1	0	1	0	1
0	1	1	0	1	0

So $p:0$, $q:1$, $r:0$

2. [3] Use known logical equivalences to show that $(q \to p) \to \neg q$ is logically equivalent to $\neg(p \wedge q)$.

$(q \to p) \to \neg q \Leftrightarrow \neg(q \to p) \vee \neg q$ Known L.E.

$\Leftrightarrow \neg(\neg q \vee p) \vee \neg q$ Known L.E.

$\Leftrightarrow (q \wedge \neg p) \vee \neg q$ DeMorgan's Law & double neg.

$\Leftrightarrow (q \vee \neg q) \wedge (\neg p \vee \neg q)$ Distrib.

$\Leftrightarrow 1 \wedge (\neg p \vee \neg q)$ $[q \vee \neg q \Leftrightarrow 1]$

$\Leftrightarrow (\neg p \vee \neg q)$ $[1 \wedge S \Leftrightarrow S]$

$\Leftrightarrow \neg(p \wedge q)$ DeMorgan's Law

3. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T The contrapositive of the statement "*if this question has the answer false, then the next question has the answer true*" is "*if the next question has the answer false, then this question has the answer true*".

F The converse of the statement $p \to \neg q$ is $\neg p \to q$.

T The statement $\forall x, (x^2 = 3) \to (x > 10)$ is true for the universe $\mathbb{Q}$.

F The negation of the statement $\exists x, \exists y, xy = \pi$ is $\forall x, \exists y, xy \neq \pi$.

4. [3] Use rules of inference and known logical equivalences to show that the following argument is valid.

$$\begin{array}{rc} & q \vee r \\ & \neg(q \wedge \neg p) \\ & \neg r \\ \hline \therefore & p \end{array}$$

1. $q \vee r$ Premise
2. $\neg(q \wedge \neg p)$ Premise
3. $\neg r$ Premise
4. $\neg q \rightarrow r$ L.E. to #1
5. $\neg q \vee p$ DeMorgan & double neg w/ #2
6. $q \rightarrow p$ L.E. to #5
7. $\neg r \rightarrow q$ L.E. w/ #4 (contrapositive)
8. $\neg r \rightarrow p$ C.R. w/ #7 & #6
9. $\therefore p$ M.P. w/ #8 & #3

5. [3] Give a counterexample to show that the statement "*For all sets* $A, B,$ *and* C, *if* $A \setminus C = B \setminus C$, *then* $A = B$." is false. Remember to explain why your example demonstrates that the statement is false.

$\mathcal{U} = \{1, 4, 5, 6, 7, 8\}$ $A \setminus C = \{6\}$ =
$A = \{5, 6, 8\}$ $B \setminus C = \{6\}$
$B = \{6, 7, 8\}$ $A \neq B$ $(5 \in A \cap B^c)$
$C = \{4, 5, 7, 8\}$

6. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters $p, q,$ and r represent statements.

<u>T</u> If $p \Rightarrow q$, $q \Rightarrow r$, and $r \Rightarrow p$, then p, q, and r are all logically equivalent.

<u>F</u> If the conclusion of an argument can never be true, then the argument is not valid.

<u>T</u> If q is a tautology and p is any statement, then $p \Rightarrow q$.

<u>T</u> If $s(x)$ is an open statement and the universe of x is not empty, then $[\forall x, s(x)] \Rightarrow [\exists x, s(x)]$.

7. [3] Let A and B be sets. Suppose that $A \subseteq B$. Prove that $A \cup B = B$.

Notice that $B \subseteq A \cup B$, because $A \cup B = \{x : x \in A \vee x \in B\}$.
Now, consider any $y \in A \cup B$.
Then there are two cases: $y \in A$ or $y \in B$.
If $y \in A$, then because $A \subseteq B$, $y \in B$.
If $y \in B$, then $y \in B$.
Either way, $y \in A \cup B \Rightarrow y \in B$.
$\therefore A \cup B \subseteq B$
so $B \subseteq A \cup B$ and $A \cup B \subseteq B$, therefor $A \cup B = B$. ∎

8. [3] Let A and B be sets. Show that $(A^c \cap B)^c \cap (A^c \setminus B)^c = A$. Hint: set-theoretic identities.

$(A^c \cap B)^c \cap (A^c \setminus B)^c = (A^c \cap B)^c \cap (A^c \cap B^c)^c$ $(X \setminus Y = X \cap Y^c)$
$= (A \cup B^c) \cap (A \cup B)$ D.M. & double ^{c}.
$= A \cup (B^c \cap B)$ Dist.
$= A \cup (\emptyset)$ $(X^c \cap X = \emptyset)$
$= A$ $(Y \cup \emptyset = Y)$

9. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. The letters A, B, and C represent sets.

F If the symmetric difference $A \oplus B \neq \emptyset$, then A is not a subset of B.
F $\mathcal{P}(\{1,2\}) = \{\{1\},\{2\},\{1,2\}\}$.
F If $A \subsetneq B$, then $A^c \subsetneq B^c$.
F $(A \setminus B) \cup C = (A \setminus C) \cup (B \setminus C)$.

statement

10. [4] Use induction to prove that any integer $n \geq 12$ can be written as a sum of 3s and 7s. (Note: a sum that involves only threes is acceptable, and the same for sevens.)

Base cases: $12 = 4 \cdot 3 + 0 \cdot 7$

$13 = 2 \cdot 3 + 1 \cdot 7$

$14 = 0 \cdot 3 + 2 \cdot 7$

IH: Suppose $\exists k \in \mathbb{Z}$ s.t. $k \geq 14$ and $\forall n \in \{12, 13, \ldots, k\}$ the statement is true.

IS: Consider $k+1$

$(k+1) - 3 = k - 2$, and $k \geq 14$ so $k - 2 \geq 12$

so by IH there are non-neg. integers a & b s.t. $k - 2 = a \cdot 3 + b \cdot 7$

Then $k+1 = (a+1) \cdot 3 + b \cdot 7$, so statement is true for $n = k+1$.

$\therefore$ By PMI statement is true $\forall n \geq 12$. ∎

11. [3] Let $a_0, a_1, \ldots$ be the sequence defined by $a_0 = 9$, and $a_n = 6a_{n-1} + 9$ for $n \geq 1$. Find a_1, a_2, a_3 and a_4 – there is no need to simplify – then use your work to guess a formula for a_n. It is not necessary to prove that your formula is correct, but note that it is not sufficient to express a_n as a sum.

$a_1 = 6 \cdot a_0 + 9 = 6 \cdot 9 + 9 = 63$

$a_2 = 6 \cdot a_1 + 9 = 6^2 \cdot 9 + 6 \cdot 9 + 9$

$a_3 = 6 \cdot a_2 + 9 = 6^3 \cdot 9 + 6^2 \cdot 9 + 6 \cdot 9 + 9$

$a_4 = 6 \cdot a_3 + 9 = 6^4 \cdot 9 + 6^3 \cdot 9 + 6^2 \cdot 9 + 6 \cdot 9 + 9$

$a_n = 9(1 + 6 + 6^2 + \cdots + 6^n) = 9\left(\frac{6^{n+1} - 1}{6 - 1}\right)$

check: $n=1$ then $9\left(\frac{6^2 - 1}{6 - 1}\right) = 9\left(\frac{35}{5}\right) = 63$ ✓

12. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. Unless otherwise specified, lower case letters represent integers.

F If $s(n)$ is an open statement such that $s(k)$ logically implies $s(k+1)$ for all $k \geq 0$, then $s(n)$ is true for all $n \geq 0$.

T There is an integer x such that $(1x)_{10} = (x1)_4$. $x = 3$

F Whether a number is even or odd depends on the base in which it is represented.

T If $p_1, p_2, \ldots, p_n$ are odd prime numbers, then the number $N = p_1 p_2 \cdots p_n + 2$ is not divisible by any of $p_1, p_2, \ldots, p_n$.

13. [4] Let $b_0, b_1, \ldots$ be defined by $b_0 = 0$, $b_1 = 2$, and $b_n = 4b_{n-1} - 4b_{n-2}$, $n \geq 2$. Use induction to prove that $b_n = n2^n$ for all $n \geq 0$.

Base: $n=0$ $b_0 = 0$ and $0 \cdot 2^0 = 0 \cdot 1 = 0$, so $b_0 = 0 \cdot 2^0$.
$n=1$ $b_1 = 2$ and $1 \cdot 2^1 = 1 \cdot 2 = 2$, so $b_1 = 1 \cdot 2^1$.

I.H.: Suppose $\exists k \in \mathbb{Z}$ s.t. $k \geq 1$ and $\forall n \in \{0, 1, \ldots, k\}$ $b_n = n2^n$ is true.

I.S. Consider b_{k+1}. Because $k \geq 1$, so $k+1 \geq 2$, so:

$$\begin{aligned} b_{k+1} &= 4 \cdot b_k - 4b_{k-1} \quad \text{(by def. of the seq.)} \\ &= 4 \cdot k \cdot 2^k - 4(k-1)2^{k-1} \quad \text{(by IH w/ } k, k-1\text{)} \\ &= k \cdot 2^{k+2} - (k-1)2^{k+1} \\ &= 2^{k+1}[2k - (k-1)] \\ &= 2^{k+1}[k+1], \text{ as desired} \end{aligned}$$

$\therefore$ By PMI $b_n = n2^n$ for all $n \geq 0$. ∎

14. [3] Let a, b, and c be integers such that $3a \mid b$ and $9b \mid c$. Prove that $27a \mid c$.

$3a \mid b$, so $\exists k \in \mathbb{Z}$ s.t. $b = 3ak$
$9b \mid c$, so $\exists \ell \in \mathbb{Z}$ s.t. $c = 9b\ell$

Then $c = 9b\ell = 9(3ak)\ell$
$= (27a)k\ell$

$k\ell \in \mathbb{Z}$, so $27a \mid c$. ∎

15. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All lower case letters denote integers.

T If $m > 1$ and $a = km + b$, then $a \equiv b \pmod{m}$.

T $(2233)_4 = (AF)_{16}$

T If $lcm(a, b) < ab$, then a and b are not relatively prime.

T If $b > 1$ is even, then a number written in base b is even if and only if its last digit is even.

16. [3] Use the Euclidean Algorithm to find $d = gcd(2200, 572)$, and then use your work to find integers x and y such that $2200x + 572y = d$.

$2200 = 3 \cdot 572 + 484 \longrightarrow 484 = 2200 - 3 \cdot 572$

$572 = 1 \cdot 484 + 88 \longrightarrow 88 = 572 - 1 \cdot 484$

$484 = 5 \cdot 88 + 44 \longrightarrow 44 = 484 - 5 \cdot 88$

$88 = 2 \cdot 44 + 0 \leftarrow$ stop!

$gcd(2200, 572) = 44$

$44 = 484 - 5 \cdot 88 = 484 - 5(572 - 484)$

$= 6 \cdot 484 - 5 \cdot 572 = 6(2200 - 3 \cdot 572) - 5 \cdot 572$

$= 6 \cdot 2200 - 23 \cdot 572$

17. Let $a, b, c, p \in \mathbb{Z}$.

(a) [3] Prove that if p is prime and $p|ab$, then $p|a$ or $p|b$.

Suppose $p|ab$. If $p|a$, we are happy. Suppose therefore $p \nmid a$. Then because p is prime $gcd(p, a) = 1$.

$\therefore \exists x, y \in \mathbb{Z}$ s.t. $1 = px + ay$.

So $b = b(px + ay) = \underline{p}bx + \underline{ab}y$

But $p|pbx$ and $p|aby$ (because $p|ab$). $\therefore p|b$. ■

So $p|a$ or $p|b$.

(b) [1] Give an example to show that the statement *if $a|bc$, then $a|b$ or $a|c$* is not true for all integers a, b and c.

$a = 6,\ b = 2,\ c = 3$. Then $6 | 2 \cdot 3$, but $6 \nmid 2$ and $6 \nmid 3$.

18. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary. All lower case integers denote integers.

T $gcd(2^{50}5^{30}7^{10}, 60 \times 77) = 2^2 5^1 7^1$.

T If $a|b$ then $lcm(a, b) = |b|$.

F $125 \cdot 7^4 - 2018 \equiv -11 \pmod 7$.

F If $a \equiv 9 \pmod{12}$, then $a \equiv 3 \pmod 4$. ex.// $a = 9$

19. (a) [2] Find the last digit of 33^{123}.

$33 \equiv 3 \pmod{10}$

$(33)^4 \equiv 81 \equiv 1 \pmod{10}$

$(33)^{123} = (33^4)^{30} \cdot 33^3 \equiv 1^{30} \cdot (3)^3 \pmod{10}$

$1^{30} \cdot 3^3 = 27 \equiv 7 \pmod{10}$

7 is the last digit of 33^{123}

(b) [1] Find the last digit of the base 3 representation of 33^{123}.

$33^{123} \equiv 0 \pmod{3}$ because $33 = 3 \times 11$.

so last digit is 0 in base 3.

20. [3] Let A, B, C, D be sets such that $A \subseteq C$ and $B \subseteq D$. Prove that $A \times B \subseteq C \times D$.

Choose any $(x, y) \in A \times B$

Then $x \in A$ and $y \in B$.

But $A \subseteq C$, so $x \in C$.

And $B \subseteq D$, so $y \in D$.

So $(x, y) \in C \times D$.

$\therefore (x,y) \in A \times B \Rightarrow (x,y) \in C \times D$

so $A \times B \subseteq C \times D$. ∎

21. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T For any set A, the relation $\mathcal{R} = \emptyset$ is a symmetric, transitive, and antisymmetric relation on A.

T If $|A| = n$, and $\mathcal{R}$ is a relation on A that contains fewer than n ordered pairs, then $\mathcal{R}$ is not reflexive.

T If $|A| = n$, then there are 2^{n^2} relations on A.

F If $\mathcal{R}$ a symmetric and transitive relation on the set A, then $\mathcal{R}$ is reflexive.

22. [3] Let $\mathcal{R}$ be the relation on $\mathbb{N} = \{1, 2, \ldots\}$ where $(a, b) \in \mathcal{R}$ if and only if $\frac{a}{b} \le \frac{b}{a}$. Prove that $\mathcal{R}$ is reflexive and antisymmetric.

Reflexive: $\forall n \in \mathbb{N}$, $\frac{n}{n} \le \frac{n}{n}$, so $(n,n) \in R$.

Antisymm.: Suppose $(x,y) \in R$ and $(y,x) \in R$.

Then $\frac{x}{y} \le \frac{y}{x}$ and $\frac{y}{x} \le \frac{x}{y}$

so $\frac{x}{y} = \frac{y}{x}$

so $x^2 = y^2$

but $x > 0$ and $y > 0$, so $x = y$.

23. [2] Let $\mathcal{R}$ be an equivalence relation on $A = \{1, 2, 3, 4, 5, 6\}$. Suppose $\mathcal{R}$ has exactly three different equivalence classes, and $(1, 3), (1, 4), (6, 2) \in \mathcal{R}$. Find the partition of A into equivalence classes.

$[1] = \{1, 3, 4, \cancel{5}\}$ Classes $[1], [2], [5]$

$[2] = \{2, 6, \cancel{5}\}$

$[5] = \{5\}$ because if $5 \in [1]$ or $5 \in [2]$ there would be only two classes.

Partition: $\{\{1,3,4\}, \{2,6\}, \{5\}\}$

24. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T For any set A, the identity function ι_A is a reflexive relation on A.

F If A and B are sets with $|A| = 5$ and $|B| = 7$, then a function $f : A \to B$ contains exactly 7 ordered pairs.

F If $f : A \to B$ is 1-1, then $|A| \ge |B|$.

T There are $2^4 - 2$ onto functions $f : \{1, 2, 3, 4\} \to \{0, 1\}$.

25. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^2 + 5$.

(a) [2] Show that f is 1-1.

Suppose $\exists\, a, b \in (0, \infty)$ such that $f(a) = f(b)$.

Then $a^2 + 5 = b^2 + 5$

$a^2 = b^2$

But $a > 0$ and $b > 0$, so $a = b$.

$\therefore f(a) = f(b) \Rightarrow a = b$. ■

(b) [1] Show that $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = x^2 + 5$, is not 1-1.

$f(-1) = 6 = f(1)$ but $-1 \neq 1$.

26. Suppose $f : A \to B$ and $g : B \to C$ are both onto functions.

(a) [3] Prove that $g \circ f : A \to C$ is an onto function.

Suppose $z \in C$

Then because g is onto $\exists\, y \in B$ s.t. $g(y) = z$.

And because f is onto $\exists\, x \in A$ s.t. $f(x) = y$.

Then $(g \circ f)(x) = g(f(x)) = g(y) = z$.

So $\forall z \in C \; \exists\, x \in A$ s.t. $(g \circ f)(x) = z$.

■

(b) [1] Suppose f and g are also both 1-1. Is $g \circ f$ invertible? Explain.

Yes, then $g \circ f$ is bijective and so invertible.

27. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

F Every subset of an uncountable set is uncountable.

T If A and B are countable sets, then $A \cup B$ is countable.

T The relation $\mathcal{R}$ on the set of all subsets of the universe defined by $(X, Y) \in \mathcal{R}$ if and only if there is a 1-1 correspondence between X and Y is an equivalence relation.

T Any interval of real numbers with positive length has the same cardinality as $\mathbb{R}$.

28. [3] Use a diagonal sweeping argument to prove that the set $\mathbb{Z} \times \mathbb{Z}$ is countable.

The following sequence has every element of $\mathbb{Z}$ appearing in it:

$0, -1, 1, 2, -2, 3, -3, 4, -4, \ldots$

We will use this sequence to build a grid that we can sweep diagonally:

2:	(2,0)	⋮	⋮	⋮	
-2:	(-2,0)	(-2,-1)	(-2,1)	(-2,-2	...
1:	(1,0)	(1,-1)	(1,1)	(1,-2)	...
-1:	(-1,0)	(-1,-1)	(-1,1)	(-1,-2)	...
0:	(0,0) →	(0,-1)	(0,1) →	(0,-2)	(0,2) → etc.
	0	-1	1	-2	2

Note: to find (a,b), go to the row corresponding to a in the $\mathbb{Z}$ sequence and the column for b in the $\mathbb{Z}$ sequence.

Sweeping in the usual way gives us a sequence that has every element of the grid, and therefor every element of $\mathbb{Z} \times \mathbb{Z}$.

29. [2] Use the blank to indicate whether each statement is True or False. No justification is necessary.

T If $|A| = n$, then the number of non-empty subsets of A is $2^n - 1$.

T The number of 1-1, onto functions $f : \{1, 2, 3\} \to \{1, 2, 3\}$ is the same as the number of 1-1 functions $g : \{1, 2, 3\} \to \{1, 2, 3\}$.

T The number of subsets of $\{a, b, c, d, e\}$ that contain a or b is $2^5 - 2^3$.

T $|\mathcal{P}(X)| > |X|$ for any set X.

/ END