UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2019

MATH 122: Logic and Foundations

CRN: 12155 (A01), 12156 (A02), 12158 (A04), 12159 (A05)

Instructors: G. MacGillivray (A01), W. Thompson (A02 & A05), K. Mynhardt (A04)

Section and Instructor : ______________________________

NAME: ______________________________

V00#: ______________________________

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R(plus some letters) calculator is allowed.

This exam consists of 26 questions, for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

There are 10 pages (numbered), not including covers.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

1. [3] Use any method to determine whether $(\neg p \to q) \wedge (q \leftrightarrow p)$ is a contradiction. Write a sentence that explains your conclusion.

p	q	$\neg p$	① $q \Leftrightarrow p$	② $\neg p \to q$	① $\wedge$ ②
F	F	T	T	F	F
F	T	T	F	T	F
T	F	F	F	T	F
T	T	F	T	T	T

Not a contradiction.
Stmt true if p & q are T

2. [4] Use known logical equivalences to show that $(p \wedge q) \to \neg r$ is logically equivalent to $q \to \neg(p \wedge r)$. $\Leftrightarrow \neg q \vee (\neg p \vee \neg r)$

$(p \wedge q) \to \neg r$

$\Leftrightarrow \neg(p \wedge q) \vee \neg r$ Known

$\Leftrightarrow (\neg p \vee \neg q) \vee \neg r$ DeMorgan

$\Leftrightarrow (\neg q \vee \neg p) \vee \neg r$ Comm.

$\Leftrightarrow \neg q \vee (\neg p \vee \neg r)$ Assoc.

$\Leftrightarrow q \to \neg(p \wedge r)$ Known, DeMorgan

3. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T q logically implies $p \to q$.

F The negation of "*if you practice regularly then you are good at tennis*" is "*If you don't practice regularly then you are not good at tennis*".

F The converse of "*some Kawasaki motorcycles are fast*" is "*All fast motorcycles are Kawasakis*".

T The contrapositive of "*all mathematicians like logic*" is "*anyone who does not like logic is not a mathematician*".

4. [2] Suppose the universe is $\mathcal{U} = \{1, 2, 3, 4\}$. Determine the truth value of the statement $\exists x, \forall y, (x^2 < 5) \to (y < 2)$. Explain your reasoning.

True. Take $x=3$. Then $x^2<5$ is false $\therefore$ $(x^2<5) \to (y<2)$ is true no matter the value of y.

5. [4] Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{l} \neg(q \wedge \neg p) \\ q \vee r \\ \neg r \\ \hline \therefore\ p \end{array}$$

1.	$\neg(q \wedge \neg p)$	Premise
2.	$q \vee r$	"
3.	$\neg r$	"
4.	$\neg q \vee \cancel{\neg\neg} p$	1. DeMorgan + Dbl Neg'n.
5.	$q \to p$	4. LE.
6.	$\neg r \to q$	2. LE.
7.	$\therefore q$	3, 6 M.P.
8.	$\therefore p$	5, 7 M.P.

6. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why you have shown that it is invalid.

$$\begin{array}{l} p \to \neg q \ \checkmark \\ q \vee r \ \checkmark \\ \neg r \ \checkmark \\ \hline \therefore\ p \ \times \end{array}$$

$\begin{pmatrix} p & q & r \\ F & T & F \end{pmatrix}$ The given truth assignment is s.t. all premises are T & the conclusion is F $\therefore$ The argument is invalid.

7. [2] Let $A = \{a, b, \{c\}, \{a, b\}\}$. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

F $c \in A$.

T $\emptyset \subseteq A$.

T $\{a, b\} \in A \cap \mathcal{P}(A)$.

F $|A| = 3$.

8. [4] Let A and B be sets such that $A \setminus B = \emptyset$. Prove that $A \subseteq B$, using an argument that starts with "*Take any* $x \in A$ ...". Then, use the universe $\mathcal{U} = \{1, 2\}$ to give an example that shows the sets A and B need not be equal.

Suppose $A \setminus B = \emptyset$.

Take any $x \in A$.

Since $A \setminus B = \emptyset$, $x \notin A \setminus B$.

$\therefore$ $x \in B$ by def'n of set difference.

$\therefore$ $A \subseteq B$

Let $A = \emptyset$ & $B = \{1\}$.

Then $A \setminus B = \emptyset$ and $A \neq B$.

9. [4] Let A, B and C be sets. Show that $(A \cap B) \cup (A \cup B^c)^c = B$. Hint: there is a short argument that uses set-theoretic identities.

$$
\begin{aligned}
&(A \cap B) \cup (A \cup B^c)^c \\
&= (A \cap B) \cup (A^c \cap B^{\cancel{cc}}) && \text{DeMorgan, Dbl Comp.} \\
&= (A \cup A^c) \cap B && \text{Dist, Known} \\
&= B && \text{Identity}
\end{aligned}
$$

10. [2] Let A, B and C be sets. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T $A \cap B = A \cup B$ if and only if $A = B$.

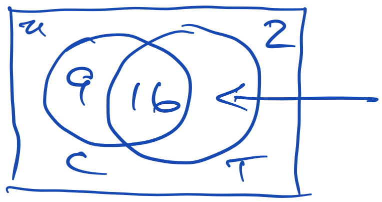
T If $A \oplus B = A$, then $B = \emptyset$.

F If $A \cap B = A \cap C$, then $B = C$.

F If $|A| = 3$ and $|B| = 4$, then $|A \times B| = 3^4$.

11. [2] In a group of 32 mathematicians, 25 like coffee and 16 like both coffee and tea. Two mathematicians like neither coffee nor tea. How many mathematicians like tea but not coffee?

Let C be the set who like coffee
& T " " " " " tea



$32 - 9 - 16 - 2 = 5$ like tea but not coffee

12. [3] Let $a, b, c, d \in \mathbb{Z}$. Prove that if $a|b$ and $c|d$, then $ac|bd$.

Suppose $a|b$ and $c|d$.

$\therefore$ There are integers k & l s.t $ak = b$ & $cl = d$

$\therefore$ $bd = (ak)(cl)$
$= (ac)(kl)$

Since $k, l \in \mathbb{Z}$, $kl \in \mathbb{Z}$.

$\therefore$ $ac \mid bd$.

13. [2] All lower case letters in this question represent integers. Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T If p is prime and $p \mid ab$, then $p \mid a$ or $p \mid b$.

F $\text{lcm}(2^3 7^2, 2^1 5^2 7^1) = 2^1 5^2 7^1$.

T $\gcd(a, b) \mid \text{lcm}(a, b)$.

T If $\gcd(a, b) = 2$, then $\gcd(2a, 2b) = 4$.

14. [4] Use the Euclidean Algorithm to find $d = gcd(1250, 560)$ and then use your work to find integers x and y such that $1250x + 560y = d$.

$1250 = 560 \times 2 + 130$

$560 = 130 \times 4 + 40$

$130 = 40 \times 3 + 10$

$40 = 10 \times 4 + 0$

$\therefore gcd(1250, 560) = 10$

$$\begin{aligned} 10 &= 130 - 40 \times 3 \\ &= 130 - (560 - 130 \times 4) \times 3 \\ &= 130 \times 13 - 560 \times 3 \\ &= (1250 - 560 \times 2) \times 13 - 560 \times 3 \\ &= 1250 \times 13 - 560 \times 29 \\ &= 1250 \times \underbrace{13}_{x} + 560 \underbrace{(-29)}_{y} \end{aligned}$$

15. [3] Find the base 6 representation of 2019.

$2019 = 6 \times 336 + 3$

$336 = 6 \times 56 + 0$

$56 = 6 \times 9 + 2$

$9 = 6 \times 1 + 3$

$1 = 6 \times 0 + 1$

$\therefore 2019 = (13203)_6$

16. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T The last digit of 99^{99} is 9.

T If $a \equiv 0 \pmod{12}$ then $a \equiv 0 \pmod{4}$.

F If $ak = b$ then every prime divisor of b is a divisor of a.

T $(1101010)_2 = (6A)_{16}$.

17. Let k be an integer such that $k \equiv -1 \pmod 4$.

(a) [2] What is the remainder when $7k^3 - 9$ is divided by 4?

$7k^3 - 9 \equiv 7(-1)^3 - 9$
$\equiv -7 - 9 \equiv -16 \equiv 0 \pmod 4$

$\therefore$ The remainder is 0

(b) [2] What is the last digit of the base 4 representation of k? Why?

We know $k \equiv -1 \equiv 3 \pmod 4$

$\therefore$ The last digit is 3.

18. [4] Use induction to prove that $0 \cdot 0! + 1 \cdot 1! + 2 \cdot 2! + \cdots + n \cdot n! = (n + 1)! - 1$ for all integers $n \geq 0$.

Basis: When $n=0$, LHS $= 0 \cdot 0! = 0$
& RHS $= (0+1)! - 1 = 0$.

$\therefore$ The stmt is true when $n=0$.

IH: Suppose there is an integer $k \geq 0$
s.t. $0 \cdot 0! + 1 \cdot 1! + \cdots + n \cdot n! = (n+1)! - 1$
for $n = 0, 1, \ldots, k$.

IS: Want $0 \cdot 0! + 1 \cdot 1! + \cdots + (k+1)(k+1)!$
$= ((k+1)+1)! - 1$
$= (k+2)! - 1$

Look at

$0 \cdot 0! + 1 \cdot 1! + \cdots + (k+1)(k+1)!$

$= \underbrace{0 \cdot 0! + 1 \cdot 1! + \cdots + k \cdot k!} + (k+1)(k+1)!$

$= (k+1)! - 1 + (k+1)(k+1)!$ by IH.

$= ((k+1)+1)(k+1)! - 1$, as wanted

$\therefore$ By induction $0 \cdot 0! + 1 \cdot 1! + \cdots + n \cdot n!$
$= (n+1)! - 1$ $\forall n \geq 0$

19. [3] Let $a_1, a_2, \ldots$ be the sequence recursively defined by $a_1 = 0$, and $a_n = 3a_{n-1} + 7$ for $n \geq 2$. Express each of a_2, a_3, a_4, and a_5 as a sum of terms that involve 3, 7 and exponents. Then, use your work to guess a (correct) formula for a_n. It is not necessary prove that your formula is correct. (Suggestion: first use your work to express a_n as a sum of n terms, as above, and then use that sum to obtain a formula.)

$a_1 = 0$

$a_2 = 3 \cdot a_1 + 7 = 3 \cdot 0 + 7$

$a_3 = 3 \cdot a_2 + 7 = 3(3 \cdot 0 + 7) + 7 = 3^2 \cdot 0 + 3 \cdot 7 + 7$

$a_4 = 3 \cdot a_3 + 7 = 3(3^2 \cdot 0 + 3 \cdot 7 + 7) + 7$

$= 3^3 \cdot 0 + 3^2 \cdot 7 + 3 \cdot 7 + 7$

$a_5 = 3a_4 + 7 = 3(3^3 \cdot 0 + 3^2 \cdot 7 + 3 \cdot 7 + 7) + 7$

$= 3^4 \cdot 0 + 3^3 \cdot 7 + 3^2 \cdot 7 + 3 \cdot 7 + 7$

Guess: $a_n = 7(3^{n-2} + 3^{n-1} + \cdots + 1)$

$= 7 \cdot \dfrac{3^{n-1} - 1}{3 - 1} = 7 \cdot \dfrac{3^{n-1} - 1}{2}$

20. [4] Let $b_0, b_1, \ldots$ be the sequence defined by $b_0 = 5$, $b_1 = 10$ and $b_n = 4b_{n-1} - 4b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 5 \cdot 2^n$ for all $n \geq 0$.

Basis: When $n = 0$, $b_0 = 5 = 5 \cdot 2^0$ ✓

& when $n = 1$, $b_1 = 10 = 5 \cdot 2^1$ ✓

$\therefore$ The stmt is true when $n=0$ & when $n=1$.

IH. Suppose there is an integer $k \geq 1$ s.t. $b_n = 5 \cdot 2^n$ for $n = 0, 1, \ldots, k$.

IS Want: $b_{k+1} = 5 \cdot 2^{k+1}$

Look at b_{k+1}. Since $k+1 \geq 2$ we can use the recursion.

$\therefore b_{k+1} = 4b_k - 4b_{k-1}$

$= 4(5 \cdot 2^k) - 4(5 \cdot 2^{k-1})$

$= 4 \cdot 5 \cdot 2^k - 2 \cdot 5 \cdot 2^k$

$= 2 \cdot 5 \cdot 2^k = 5 \cdot 2^{k+1}$, as wanted

$\therefore$ By induction $b_n = 5 \cdot 2^n$ $\forall n \geq 0$.

21. [4] Let $f : \mathbb{R} \to [0, \infty)$ be defined by $f(x) = 2x^2$. Show that f is onto but not 1-1. (Note: $[0, \infty) = \{x : x \geq 0\}$.)

Take any $y \in [0, \infty)$

Then $f(x) = y \iff 2x^2 = y$

$\iff x^2 = y/2 \iff x = \pm\sqrt{\frac{y}{2}}$

Since $y \in [0, \infty)$, $\sqrt{\frac{y}{2}}$ is defined.

$\therefore \exists x \in \mathbb{R}$ s.t. $f(x) = y$

$\therefore f$ is onto.

Since $f(1) = 2 \cdot 1^2 = 2$

& $f(-1) = 2(-1)^2 = 2$

the function f is not 1-1.

22. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T $\lfloor n/2 \rfloor + \lceil n/2 \rceil = n$ for all integers n.

F If $|A| = 122$ and there exists a 1-1 function $f : A \to B$, then $|B| = 122$.

F A function $f : A \to B$ is onto if for every $a \in A$ there exists $b \in B$ such that $(a, b) \in f$.

T For any set A, the identity function on A is invertible.

23. [4] For sets A, B and C, let $f : A \to B$ and $g : B \to C$ be 1-1 functions. Prove that $g \circ f : A \to C$ is 1-1.

Suppose $g \circ f(x_1) = g \circ f(x_2)$

Then $g(f(x_1)) = g(f(x_2))$

Since g is 1-1, $f(x_1) = f(x_2)$

Since f is 1-1, $x_1 = x_2$

$\therefore$ $g \circ f$ is 1-1.

24. Let $A = \{1, 2, \ldots, 50\}$. Fill in the blanks. No justification is necessary.

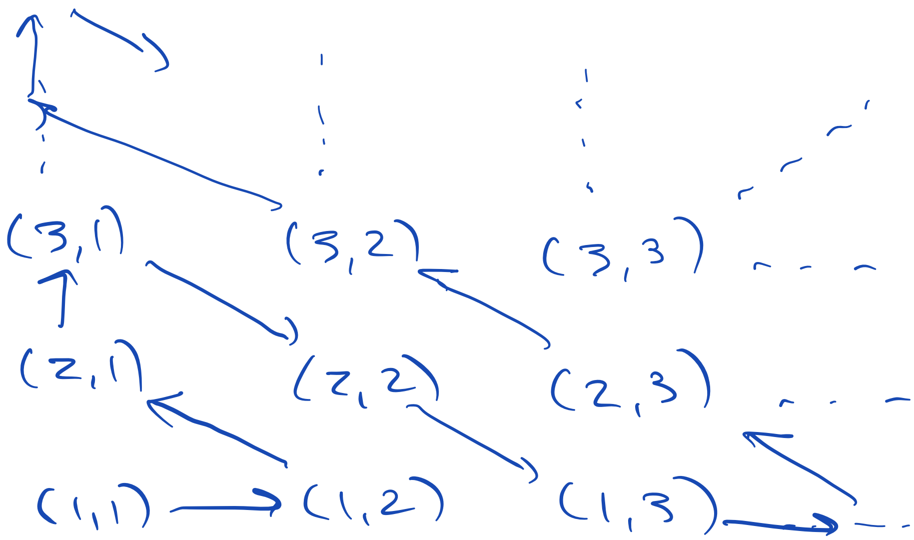
(a) [1] The number of functions $f : A \to \{1, 2, 3\}$ is 3^{50}.

(b) [1] The number of subsets of $A \times A$ is $2^{50 \times 50}$.

(c) [1] The number of 1-1, onto functions $f : A \to A$ is $50 \cdot 49 \cdot \ldots 1 = 50!$

(d) [1] The number of proper subsets of A that contain 50 and do not contain 49 is $1 \times 1 \times 2^{48}$

25. [3] Use a diagonal sweeping argument to prove that $\mathbb{N} \times \mathbb{N}$ is countable.



The array contains every element of $\mathbb{N} \times \mathbb{N}$. The sequence indicated by the arrows contains every element of the array & $\therefore$ of $\mathbb{N} \times \mathbb{N}$.

$\therefore$ $\mathbb{N} \times \mathbb{N}$ is countable.

26. [2] Use the blank to indicate whether each statement is **True (T)** or **False (F)**. No justification is necessary.

T Any subset of $\{2, 4, 6, 8, \ldots\}$ is countable.

T If $a < b$ then the closed interval $[a, b]$ of real numbers is uncountable.

F The intersection of any two uncountable sets is uncountable.

T For any set A there exists a set B such that $|B| > |A|$.

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