

MATH 122 – Logic and Foundations

Date: Dec. 13, 2023

Final Exam

Time Limit: 3 hours

Instructors:

A01 & A04 – CRNs 12137 & 12140 – M. Edwards

A02 – CRN 12138 – A. Sodhi

A03 – CRN 12139 – K. Krishnan

Before starting your test, enter your name and student ID (clearly) on this page.

This exam contains 22 pages (including this one, and scrap pages)
and 25 questions, worth a total of 80 marks.

You may not open the exam until instructed to do so. In the meantime, please read the following instructions:

- No textbooks or class notes are allowed on this exam. A calculator is not necessary, but a Sharp EL-510R, EL-510RN, or EL-510RNB calculator is allowed.
- Answer each question in the space immediately below that question. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed.

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- [3] 1. Find all possible truth values of p , q , and r such that $(p \rightarrow q) \rightarrow \neg r$ is false.

Need $\underbrace{p \rightarrow q}_{p \text{ FALSE}} \text{ TRUE} \ \& \ \underbrace{\neg r}_{r \text{ TRUE}} \text{ FALSE}$
 or p, q true

\therefore desired truth values $\begin{pmatrix} p & q & r \\ F & F & T \\ F & T & T \\ T & T & T \end{pmatrix}$

- [4] 2. Use known logical equivalences to show that $p \rightarrow (q \vee r)$ is logically equivalent to $(p \wedge \neg r) \rightarrow q$.

$(p \wedge \neg r) \rightarrow q \Leftrightarrow \neg(p \wedge \neg r) \vee q \Leftrightarrow (\neg p \vee r) \vee q$

$p \rightarrow (q \vee r)$
 $\Leftrightarrow \neg p \vee (q \vee r)$
 $\Leftrightarrow \neg p \vee (r \vee q)$
 $\Leftrightarrow (\neg p \vee r) \vee q$
 $\Leftrightarrow \neg(p \wedge \neg r) \vee q$
 $\Leftrightarrow (p \wedge \neg r) \rightarrow q$

Known LE.
 Comm.
 Assoc
 DeMorgan
 Known LE.

- [2] 3. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

True False

- (a) True False If $(\neg p) \Rightarrow q$ then $p \Rightarrow (\neg q)$.
- (b) True False The negation of $\forall x, \exists y, x \vee y$ is $\exists y, \forall x, (\neg x) \wedge (\neg y)$.
- (c) True False The statement $\forall x, x \leq 3x$ is true for the universe of positive real numbers.
- (d) True False If an argument is valid, then its premises are all true.

$\neg p \rightarrow q \Leftrightarrow \neg q \rightarrow p$

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- [4] 4. Use known logical equivalences and inference rules to show that the following argument is valid.

$$\begin{array}{l} (\neg q) \rightarrow p \\ (\neg r) \rightarrow s \\ (\neg q) \wedge u \\ \hline \neg s \\ \hline \therefore p \wedge r \end{array}$$

1. $\neg q \rightarrow p$
2. $\neg r \rightarrow s$
3. $\neg q \wedge u$
4. $\neg s$
5. $\neg q$
6. $\therefore p$
7. $\neg s \rightarrow r$
8. $\therefore r$
9. $p \wedge r$

Premise
 "
 "
 "
 3, Conj. Simpl.
 1, 5 M.P.
 2, Contrapos.
 4, 7 M.P.
 6, 8

- [3] 5. Give a counterexample to show that the following argument is invalid.

$$\begin{array}{l} p \rightarrow q \\ (\neg q) \rightarrow (\neg s) \\ r \rightarrow s \\ \hline \therefore p \end{array}$$

For this truth assignment, the premises are all T, but the conclusion is F. \therefore The argument is invalid.

$\left(\begin{array}{cccc} p & q & r & s \\ F & T & F & T \end{array} \right)$

- [2] 6. Let A , B , and C be sets. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

True False

- (a) If $A \subseteq B$, then $A^c \subseteq B^c$.
- (b) If $A \subseteq \mathbb{Z}$ and $B \subseteq \mathbb{Q}$, then $A \subseteq B$.
- (c) If $10 \in A$ and $3 \in B$, then $30 \in A \times B$.
- (d) If $C \in \mathcal{P}(A)$, then $A \cap C = C$.

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- [4] 7. Let A and B be sets. Show that $(A \setminus B^c) \cup (A^c \cup B)^c = A$.

Hint: Use set-theoretic identities.

$$\begin{aligned} & (A \setminus B^c) \cup (A^c \cup B)^c \\ &= (A \cap B) \cup (A \cap B^c) \\ &= A \cap (B \cup B^c) \\ &= A \end{aligned}$$

Def'n, Dbl Comp x2
DeMorgan
Dist.

Known equality
Identity

8. Let $A \subseteq B$.

- [3] (a) Prove that $A \setminus B = \emptyset$.

Prove the contrapositive: If $A \setminus B \neq \emptyset$, then $A \not\subseteq B$.

Suppose $A \setminus B \neq \emptyset$.

\therefore There exists x s.t. $x \in A$ & $x \notin B$

$\therefore A \not\subseteq B$

- [1] (b) Prove that $A = A \cap B$.

It is always true that $A \cap B \subseteq A$.

\therefore Need to show $A \subseteq A \cap B$ (if $A \subseteq B$)

Take any $x \in A$. Since $A \subseteq B$, $x \in B$

$\therefore x \in A \cap B$ $\therefore A \subseteq A \cap B$

$\therefore A = A \cap B$

- [2] 9. Let $A = \{1, 2, 3\}$. Fill in each blank. No justification is necessary.

(a) The number of subsets of A that contain 1 but not 2 is 2.

(b) The number of non-empty proper subsets of A is $6 = 2^3 - 2$.

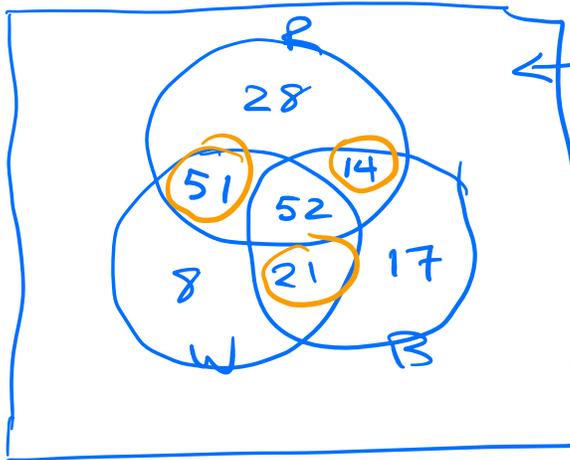
(c) The number of onto (surjective) functions $f: A \rightarrow \{w, x, y, z\}$ is 0.

(d) The number of relations on A is $2^{3 \times 3} = 2^9$.

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- [2] 10. The three most common colours in the 193 of the member nations of the United Nations are red, white, and blue. There are 52 flags that contain all three colours, 103 that contain both red and white, 66 that contain both red and blue, 73 that contain both white and blue, 145 that contain red, 132 that contain white, and 104 that contain blue. How many flags contain exactly two of the three colours?

let R, W, B be the set of member nations that have colour Red, White, Blue, resp



$$\begin{aligned} & \leftarrow 193 - |R \cup W \cup B| \\ & = 193 - 191 \\ & = 2 \end{aligned}$$

$$\begin{aligned} & \# \text{ with exactly 2 of} \\ & \text{The 3 colours} \\ & = 51 + 14 + 21 \\ & = 86 \end{aligned}$$

- [2] 11. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning. All variables are integers.

True False

- (a) If $a \mid b$ and $b \mid a$, then $a = b$.
- (b) If $d \mid (a + b)$, then $d \mid a$ and $d \mid b$.
- (c) The last digit of 3^{51} is 3.
- (d) If $a \equiv 0 \pmod{3}$, then $a \not\equiv 0 \pmod{2}$.

$$\begin{aligned} 3^{51} &= 3 \cdot 3^{50} \\ &= 3 \cdot (3^2)^{25} \\ &= 3 \cdot (-1)^{25} \\ &= -3 \\ &\equiv 7 \pmod{10} \end{aligned}$$

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- [4] 15. Let $n = (d_3d_2d_1d_0)_7$. Prove that $6 \mid n$ if and only if $6 \mid (d_3 + d_2 + d_1 + d_0)$.

We know $6 \mid n \Leftrightarrow n \equiv 0 \pmod{6}$

$$\Leftrightarrow d_3 7^3 + d_2 7^2 + d_1 7 + d_0 \equiv 0 \pmod{6}$$

$$\Leftrightarrow d_3 1^3 + d_2 1^2 + d_1 \cdot 1 + d_0 \equiv 0 \pmod{6}$$

b/c $7 \equiv 1 \pmod{6}$

$$\Leftrightarrow d_3 + d_2 + d_1 + d_0 \equiv 0 \pmod{6}$$

$$\Leftrightarrow 6 \mid d_3 + d_2 + d_1 + d_0$$

- [4] 16. Use induction to prove that $1(1!) + 2(2!) + \dots + n(n!) = (n+1)! - 1$ for all integers $n \geq 1$, where $n! = n(n-1) \cdots 2 \cdot 1$.

Basis When $n=1$, LHS = $1 \cdot 1! = 1$
 \neq RHS = $2! - 1 = 1$

\therefore The stmt is true when $n=1$

IH Suppose there is an integer $k > 1$

s.t. $1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k! = (k+1)! - 1$

for $n = 1, 2, \dots, k$

IS want to show:

$$1 \cdot 1! + 2 \cdot 2! + \dots + (k+1)(k+1)! = ((k+1)+1)! - 1$$

Look at LHS

$$\begin{aligned} & 1 \cdot 1! + 2 \cdot 2! + \dots + (k+1)(k+1)! \\ &= \underbrace{1 \cdot 1! + 2 \cdot 2! + \dots + k \cdot k!}_{(k+1)! - 1} + (k+1)(k+1)! \\ &= (k+1)! - 1 + (k+1)(k+1)! \quad \text{by IH.} \\ &= (1 + (k+1))(k+1)! - 1 = (k+2)(k+1)! - 1 \\ &= (k+2)! - 1 \quad \checkmark \end{aligned}$$

\therefore By induction $1 \cdot 1! + 2 \cdot 2! + \dots + n \cdot n! = (n+1)! - 1 \quad \forall n \geq 1$

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- [4] 17. Let a_0, a_1, a_2, \dots be the sequence defined by $a_0 = 4$, and $a_n = a_{n-1} + 4n$ for $n \geq 1$. Find a_1, a_2, a_3 , and a_4 , then use your work to obtain a formula for a_n (note: a formula, not just a summation). It is not necessary to prove that your formula is correct.

$$a_0 = 4$$

$$a_1 = a_0 + 4 \cdot 1 = 4 + 4 \cdot 1$$

$$a_2 = a_1 + 4 \cdot 2 = 4 + 4 \cdot 1 + 4 \cdot 2$$

$$a_3 = a_2 + 4 \cdot 3 = 4 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3$$

$$a_4 = a_3 + 4 \cdot 4 = 4 + 4 \cdot 1 + 4 \cdot 2 + 4 \cdot 3 + 4 \cdot 4$$

$$\begin{aligned} \text{Guess } a_n &= 4 + 4 \cdot 1 + 4 \cdot 2 + \dots + 4 \cdot n \\ &= 4 + 4(1 + 2 + \dots + n) \\ &= 4 + 4 \cdot \frac{n(n+1)}{2} \\ &= 4 + 2n(n+1) \end{aligned}$$

- [4] 18. Let b_0, b_1, \dots be the sequence defined by $b_0 = 6$, $b_1 = 23$ and $b_n = 7b_{n-1} - 12b_{n-2}$ for $n \geq 2$. Use induction to prove that $b_n = 3^n + 5 \cdot 4^n$ for all $n \geq 0$.

Basis: When $n=0$ we have $b_0 = 6 = 3^0 + 5 \cdot 4^0$ ✓
 When $n=1$ we have $b_1 = 23 = 3^1 + 5 \cdot 4^1$ ✓
 \therefore The stmt is true when $n=0$ & when $n=1$.

IH: Suppose there is an integer $k \geq 1$ s.t.
 $b_n = 3^n + 5 \cdot 4^n$ for $n = 0, 1, \dots, k$

IS: want to show $b_{k+1} = 3^{k+1} + 5 \cdot 4^{k+1}$

Since $k+1 \geq 2$ we can use the recursion

$$\begin{aligned} \therefore b_{k+1} &= 7b_k - 12b_{k-1} \\ &= 7(3^k + 5 \cdot 4^k) - 12(3^{k-1} + 5 \cdot 4^{k-1}) \quad \text{by IH} \\ &= 7 \cdot 3^k - 12 \cdot 3^{k-1} + 7 \cdot 5 \cdot 4^k - 12 \cdot 5 \cdot 4^{k-1} \\ &= 7 \cdot 3^k - 4 \cdot 3^k + 7 \cdot 5 \cdot 4^k - 3 \cdot 5 \cdot 4^k \\ &= 3 \cdot 3^k + 20 \cdot 4^k \\ &= 3^{k+1} + 5 \cdot 4^{k+1} \quad \checkmark \end{aligned}$$

\therefore By induction, $b_n = 3^n + 5 \cdot 4^n \quad \forall n \geq 0$

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- [4] 19. Let $f : \mathbb{R} \rightarrow (-2, \infty)$ be defined by $f(x) = x^2 - 2$. Prove that f is onto (surjective) but not one-to-one (injective).

Take any $y \in (-2, \infty)$.

$$\begin{aligned} \text{Then } f(x) = y &\Leftrightarrow x^2 - 2 = y \\ &\Leftrightarrow x^2 = y + 2 \\ &\Leftrightarrow x = \pm \sqrt{y + 2} \end{aligned}$$

Since $y \in (-2, \infty)$, $y + 2 \geq 0 \therefore \sqrt{y + 2} \in \mathbb{R}$

$$\begin{aligned} \text{If } x = \sqrt{y + 2}, \text{ then } f(x) &= (\sqrt{y + 2})^2 - 2 \\ &= |y + 2| - 2 \\ &= y + 2 - 2 = y \end{aligned}$$

$\therefore f$ is onto.

b/c $y + 2 \geq 0$

$$\begin{aligned} \text{Since } f(-1) &= (-1)^2 - 2 = -1 \\ &\neq f(1) = 1^2 - 2 = -1 \end{aligned}$$

The function f is not 1-1.

- [2] 20. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

True **False**

- (a) For any integer x , $\lfloor \frac{2x+1}{2} \rfloor = \lceil \frac{2x+1}{2} \rceil - 1$.
- (b) There exists an onto (surjective) function from $\{1, 2, 3\}$ to $\{1, 2, 3, 4, 5\}$.
- (c) A function $f : A \rightarrow B$ is one-to-one (injective) if for every $a \in A$ there exists a unique $b \in B$ such that $(a, b) \in f$.
- (d) A function $f : A \rightarrow B$ has an inverse if and only if the relation g from B to A defined by $g = \{(b, a) : (a, b) \in f\}$ is a function.

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- [4] 21. For sets A , B , and C let $f : A \rightarrow B$ and $g : B \rightarrow C$ be one-to-one (injective) functions. Prove that $g \circ f : A \rightarrow C$ is one-to-one (injective).

Suppose $g \circ f(x_1) = g \circ f(x_2)$
 $\therefore g(f(x_1)) = g(f(x_2))$
 Since g is 1-1, $f(x_1) = f(x_2)$
 Since f is 1-1, $x_1 = x_2$
 $\therefore g \circ f$ is 1-1.

- [5] 22. Let $A = \{10, 11, 12, 13, \dots, 99\}$, and let \mathcal{R} be the relation on A defined by $(x, y) \in \mathcal{R}$ if and only if the sum of the digits of x equals the sum of the digits of y . For example $15 \mathcal{R} 24$ (i.e. $(15, 24) \in \mathcal{R}$) since $1 + 5 = 2 + 4$.

Prove that \mathcal{R} is an equivalence relation on A . How many distinct equivalence classes are there determined by \mathcal{R} ? Explain.

refl. let $x \in A$. Since the sum of x 's digits = the sum of x 's digits, $(x, x) \in \mathcal{R}$
 $\therefore \mathcal{R}$ is reflexive

Symm. Suppose $(x, y) \in \mathcal{R}$.
 \therefore the sum of x 's digits = the sum of y 's digits
 \therefore " " " y 's " = " " " x 's "
 $\therefore (y, x) \in \mathcal{R}$ $\therefore \mathcal{R}$ is symmetric

trans. Suppose $(x, y), (y, z) \in \mathcal{R}$
 \therefore the sum of x 's digits = the sum of y 's digits
 $\&$ " " " y 's " = " " " z 's "
 \therefore " " " x 's " = " " " z 's "
 $\therefore (x, z) \in \mathcal{R}$ $\&$ \mathcal{R} is transitive

equiv. classes = # possible sums of digits
 = 18

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[2] 23. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

True False

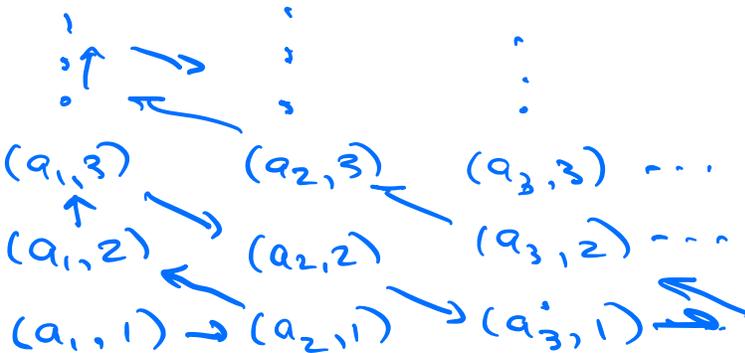
- (a) Suppose A is a set with $|A| = n$. There exists a reflexive relation \mathcal{R} on A that contains fewer than n ordered pairs.
- (b) If \mathcal{R} is a symmetric relation on $\{1, 2, 3, 4\}$ and $(4, 1) \in \mathcal{R}$, then \mathcal{R} is antisymmetric.
- (c) If \mathcal{R} is a reflexive relation on $\{1, 2, 3\}$ with $(1, 2) \in \mathcal{R}$, then \mathcal{R} is not a function.
- (d) If \mathcal{R} is an antisymmetric and transitive relation on $\{1, 2, 3, 4\}$ where $(3, 1) \in \mathcal{R}$ and $(4, 1) \in \mathcal{R}$, then $(3, 4) \in \mathcal{R}$.

$= \{1, 2, \dots, n\}$, $(1,1), (2,2), \dots, (n,n) \in A$

$\mathcal{R} = \{(3,1), (4,1)\}$

[3] 24. Let A be a countably infinite set. Using a diagonal sweeping argument, prove that $A \times \mathbb{N}$ is countable.

Let $A = \{a_1, a_2, \dots\}$



Every element of $A \times \mathbb{N}$ is in the array. The seq. indicated by the arrows contains every element of the array $\therefore A \times \mathbb{N}$ is countable.

[2] 25. Check the appropriate circle to indicate whether each statement is true or false. You do not need to provide any reasoning.

True False

- (a) If A is an uncountable set, and $B \subseteq A$, then B is uncountable.
- (b) For any real number b such that $b > 1$, the open interval $(1, b)$ is uncountable.
- (c) If A is a countable set, then $A \cup \{-1, 1\}$ is countable.
- (d) The set $\{\dots, -16, -9, -4, -1, 0, 1, 4, 9, 16, \dots\}$ is uncountable.

$0, 1, -1, 4, -4, 9, -9, \dots$

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