UNIVERSITY OF VICTORIA
EXAMINATIONS DECEMBER 2024

MATH 122: Logic and Foundations

CRN: 12079 (A01), 12080 (A02), 12081 (A03), 12082 (A04)

Instructors: K. Krishnan (A01), S. Dore-Hall (A02) & J. Noel (A03 & A04)

Duration: 3 Hours.

Answers are to be written on the exam paper.
No calculator is necessary, but a Sharp EL-510R (plus some letters) calculator is allowed.

This exam consists of 20 questions for a total of 80 marks. In order to receive full or partial credit you must show your work and justify your answers, unless otherwise instructed. Question 1 consists of 28 true/false questions labelled TF 1 to TF 28 to be answered on the bubble sheet at the end of the booklet. True is A and False is B on the bubble sheet.

There are 12 numbered pages, not including covers, and one bubble sheet.

Students must count the number of pages before beginning and report any discrepancy immediately to the invigilator.

Do your rough work here. Nothing written here will be marked.

1. [14] Use the **bubble sheet** provided on the last page of the test booklet to indicate whether each statement is **True (A)** or **False (B)**. No justification is necessary.

[TF 1] The contrapositive of $p \wedge \neg r$ is $r \wedge \neg p$.

[TF 2] If the statement $p \to q$ is false, then q must be false.

[TF 3] $p \wedge \neg p \Rightarrow p \vee \neg p$

[TF 4] The negation of $\exists x, \forall y, p(x) \leftrightarrow q(x, y)$ is logically equivalent to $\forall x, \exists y, (p(x) \wedge \neg q(x, y)) \vee (\neg p(x) \wedge q(x, y))$.

[TF 5] If $x \in \mathbb{R} \setminus \{1\}$ and $n \in \mathbb{N}$, then $1 + x + x^2 + \cdots + x^n = \frac{x^{n+1}-1}{x-1}$.

[TF 6] For any positive integers n and k, the remainder when n is divided by k is $n - \lfloor n/k \rfloor k$.

[TF 7] The Fundamental Theorem of Arithmetic states that every integer $n \geq 2$ has a unique greatest common divisor and least common multiple.

[TF 8] $\mathbb{R}$ is the set of all rational numbers.

[TF 9] If a and b are positive integers, then a and b are relatively prime if and only if $lcm(a, b) = gcd(a, b)$.

[TF 10] $lcm(2^2 5^1 11^1, 2^3 3^9 5^1 11^{99}) = 2^3 3^9 5^1 11^{99}$.

[TF 11] For $a, b \in \mathbb{N}$, the fraction a/b is rational if and only if it is in lowest terms.

[TF 12] Let n be a positive integer. If the sum of the digits of the base 7 representation of n is divisible by 6, then n is divisible by 6.

In questions TF 13–TF 15, let $X = \{1, \{1\}, \{1, 2\}, \{3\}\}$

[TF 13] $\emptyset \in X$.

[TF 14] $(1 \in X) \wedge (\{1\} \in X) \wedge (\{1\} \subseteq X) \wedge (\{\{1\}\} \subseteq X)$.

[TF 15] $|X \cup \{1, 2, 3, 4\}| = 8$.

[TF 16] The set $A = \{\emptyset, \{\emptyset\}\}$ satisfies $A \subseteq \mathcal{P}(A)$.

[TF 17] $\emptyset$ is a relation on $\emptyset$.

[TF 18] The relation $\mathcal{R} = \{(1, 2), (2, 3), (3, 1)\}$ on $\{1, 2, 3\}$ is antisymmetric.

[TF 19] The relation $\sim$ on $\mathbb{Z}$ where $a \sim b$ if and only if a and b are relatively prime is an equivalence relation.

[TF 20] There does not exist an injective function from an infinite set to a finite set.

[TF 21] The function $f : \mathbb{Z} \to \mathbb{Q}$ defined by $f(x) = x/2$ for $x \in \mathbb{Z}$ has an inverse.

[TF 22] Every function f from a set A to a set B is a subset of $A \times B$.

[TF 23] Let A and B be finite sets. If $f : A \to B$ is surjective, then it must also be injective.

[TF 24] If $|A| = 3$, then $|\mathcal{P}(\mathcal{P}(A))| = 2^8$.

[TF 25] The set of prime numbers is uncountable.

[TF 26] If A and B are sets such that $A \subsetneq B$, then $|A| < |B|$.

[TF 27] Every uncountable set has a countable subset.

[TF 28] The set $\{x \in \mathbb{R} : x^2 = 2\}$ is uncountable.

2. [3] Use a truth table to determine whether $\neg(\neg p \to q) \vee (\neg p \leftrightarrow q)$ is a tautology. Write a sentence that explains how your work shows that your conclusion is correct.

p	q	$\neg p$	$\neg p \to q$	① $\neg(\neg p \to q)$	② $\neg p \leftrightarrow q$	① $\vee$ ②
F	F	T	F	T	F	T
F	T	T	T	F	T	T
T	F	F	T	F	T	T
T	T	F	T	F	F	F

when p is T & q is T, the stmt is F

∴ It is not a tautology.

3. [4] Use the Laws of Logic and known logical equivalences to show that $(p \to (p \wedge \neg q)) \wedge q$ is logically equivalent to $\neg p \wedge q$.

	$(p \to (p \wedge \neg q)) \wedge q$	
$\Leftrightarrow$	$(\neg p \vee (p \wedge \neg q)) \wedge q$	Known LE.
$\Leftrightarrow$	$(\overset{T}{(\neg p \vee p)} \wedge (\neg p \vee \neg q)) \wedge q$	Dist
$\Leftrightarrow$	$(T \wedge (\neg p \vee \neg q)) \wedge q$	Known tautology
$\Leftrightarrow$	$(\neg p \vee \neg q) \wedge q$	Identity
$\Leftrightarrow$	$(\neg p \wedge q) \vee \overset{F}{(\neg q \wedge q)}$	Dist.
$\Leftrightarrow$	$(\neg p \wedge q) \vee F$	Known contradiction
$\Leftrightarrow$	$\neg p \wedge q$	Identity

4. [4] Use the Laws of Logic, known logical equivalences and inference rules to show that the following argument is valid. Jusitfy each step.

$$\begin{array}{ll} & p \to (q \to \neg r) \\ & \neg s \to r \\ & p \\ \hline \therefore & q \to s \end{array}$$

1. $p \to (q \to \neg r)$ Premise
2. $\neg s \to r$ Premise
3. p Premise
4. $\therefore q \to \neg r$ 1,3 M.P.
5. $\neg r \to s$ 2, Contrapos.
6. $\therefore q \to s$ 4,5 Chain Rule

5. [3] Give a counterexample to show that the following argument is invalid. Be sure to explain why your counterexample shows that it is invalid.

$$\begin{array}{ll} & p \leftrightarrow r \ \checkmark \\ & (\neg r \vee s) \to q \ \checkmark \\ \hline \therefore & q \ \times \end{array}$$

$\left(\begin{array}{cccc} p & q & r & s \\ T & F & T & F \end{array}\right)$

The given truth assignment is such that the premises are all T & the conclusion is F

$\therefore$ The argument is invalid.

6. [4] Let $a, b, c \in \mathbb{Z}$. Prove that if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$.

Suppose $a \mid b$ and $a \mid c$

$\therefore$ $\exists\ k \in \mathbb{Z}$ s.t. $ak = b$

& $\exists\ l \in \mathbb{Z}$ s.t. $al = c$

Now

$$\begin{aligned} b + c &= ak + al \\ &= a(k + l) \end{aligned}$$

Since $k, l \in \mathbb{Z}$, $k + l \in \mathbb{Z}$.

$\therefore$ $a \mid (b + c)$

7. [3] Find the base 16 (hexadecimal) representation of 2766. Use the symbols A, B, C, D, E, F to represent the digits $10, 11, 12, 13, 14, 15$, respectively.

$$\begin{aligned} 2766 &= 172 \times 16 + 14 & \quad E \\ 172 &= 10 \times 16 + 12 & \quad C \\ 10 &= 0 \times 16 + 10 & \quad A \end{aligned}$$

$\therefore$ $2766 = (ACE)_{16}$

8. [4] Use the Euclidean Algorithm to find $gcd(3774, 1512)$ and then use your work to find integers x and y such that $3774x + 1512y = gcd(3774, 1512)$.

$$3774 = 1512 \times 2 + 750$$
$$1512 = 750 \times 2 + 12$$
$$750 = 12 \times 62 + 6 \leftarrow \therefore gcd(3774, 1512) = 6$$
$$12 = 6 \times 2 + 0$$

$$\begin{aligned} 6 &= 750 - 12 \times 62 \\ &= 750 - (1512 - 750 \times 2) \times 62 \\ &= 750 \times 125 - 1512 \times 62 \\ &= (3774 - 1512 \times 2) \times 125 - 1512 \times 62 \\ &= 3774 \times 125 - 1512 \times 312 \\ &= 3774 \times \underbrace{125}_{x} + 1512 \times \underbrace{(-312)}_{y} \end{aligned}$$

9. [3] What is the remainder when $(-113)^{400}$ is divided by 13? Show your work.

$$-113 \equiv -113 + \underbrace{9 \times 13}_{117} \equiv 4 \pmod{13}$$

$$\therefore (-113)^{400} \equiv 4^{400} \equiv (4^2)^{200} \equiv 16^{200} \equiv 3^{200} \pmod{13}$$

$$3^{200} \equiv 3^2 \cdot 3^{198} \equiv 3^2 \cdot (3^3)^{66} \equiv 3^2 \cdot 1^{66} \equiv 3^2 \equiv 9 \pmod{13}$$

$$\uparrow$$
$$27 \equiv 1 \pmod{13}$$

$\therefore$ The remainder is 9

10. [3] Let p be a prime number and let a and b be integers. Use the Fundamental Theorem of Arithmetic to prove that, if $p \mid ab$ and $p \nmid a$, then $p \mid b$.

Since $p \nmid a$, the prime p does not appear in the prime factorization of a.

Since $p \mid ab$, the prime p appears in the prime factorization of ab

Since the prime factors of ab are the primes that divide a or divide b, p is a prime factor of b

$\therefore p \mid b$

11. [4] Use induction to prove that $1 + 3 + 5 + \cdots + (2n - 1) = n^2$ for every integer $n \geq 1$.

Basis: When $n=1$, LHS $= 1$ & RHS $= 1^2 = 1$
$\therefore$ Stmt true when $n=1$.

IH: Suppose there is an integer $k \geq 1$ s.t. $1+3+\cdots+(2k-1) = k^2$

IS Want $1+3+\cdots+(2(k+1)-1) = (k+1)^2$

$1+3+\cdots+(2(k+1)-1)$

$= \underbrace{1+3+\cdots+(2k-1)} + (2k+1)$ (meaning of $\cdots$)

$= k^2 + 2k+1$ by IH.

$= (k+1)^2$, as wanted

$\therefore$ By induction $1+3+\cdots+(2n-1) = n^2 \ \forall n \geq 1$

12. Let $b_0, b_1, \ldots$ be the sequence recursively defined $b_0 = 5$ and $b_n = 2b_{n-1} + 5$ for $n \geq 1$.

(a) [2] Find b_1, b_2, b_3 and b_4. Leave each of these as a sum rather than computing a numerical value.

$b_0 = 5$

$b_1 = 2b_0 + 5 = 2 \cdot 5 + 5$

$b_2 = 2b_1 + 5 = 2(2 \cdot 5 + 5) + 5$
$= 2^2 \cdot 5 + 2 \cdot 5 + 5$

$b_3 = 2b_2 + 5 = 2(2^2 \cdot 5 + 2 \cdot 5 + 5) + 5$
$= 2^3 \cdot 5 + 2^2 \cdot 5 + 2 \cdot 5 + 5$

$b_4 = 2b_3 + 5 = 2(2^3 \cdot 5 + 2^2 \cdot 5 + 2 \cdot 5 + 5) + 5$
$= 2^4 \cdot 5 + 2^3 \cdot 5 + 2^2 \cdot 5 + 2 \cdot 5 + 5$

(b) [2] Based on your work from part (a), guess a formula for b_n for all integers $n \geq 0$. Give your final answer as a closed-form formula that is not a summation. You do not need to prove that your formula is correct.

Guess $b_n = 2^n \cdot 5 + 2^{n-1} \cdot 5 + \cdots + 2 \cdot 5 + 5$
$= 5(2^n + 2^{n-1} + \cdots + 2 + 1)$
$= 5(1 + 2 + \cdots + 2^n)$
$= 5 \cdot \dfrac{2^{n+1} - 1}{2 - 1}$

$= 5(2^{n+1} - 1)$

13. [4] Let $a_0, a_1, \ldots$ be the sequence recursively defined by $a_0 = 0$, $a_1 = 2$ and $a_n = 4a_{n-1} - 4a_{n-2}$ for all $n \geq 2$. Use induction to prove that $a_n = n2^n$ for all $n \geq 0$.

Basis : When $n=0$, $a_0 = 0 = 0 \cdot 2^0$ ✓
when $n=1$, $a_1 = 2 = 1 \cdot 2^1$ ✓
$\therefore$ the stmt is true when $n=0$ & when $n=1$.

IH : Suppose there is an integer $k \geq 1$
s.t. $a_n = n \cdot 2^n$ for $n = 0, 1, \ldots, k$

IS : Want $a_{k+1} = (k+1) 2^{k+1}$

Look at a_{k+1}. Since $k \geq 1$, we have $k+1 \geq 2$

$$\begin{aligned}
\therefore a_{k+1} &= 4a_k - 4a_{k-1} \\
&= 4 \cdot k \cdot 2^k - 4(k-1)2^{k-1} \quad \text{by IH} \\
&= 2 \cdot k \cdot 2^{k+1} - (k-1)2^{k+1} \\
&= (2k - (k-1))2^{k+1} \\
&= (k+1)2^{k+1}, \text{ as wanted}
\end{aligned}$$

$\therefore$ By induction, $a_n = n \cdot 2^n \ \forall n \geq 0$

14. [3] Let A, B and C be finite sets. What does the Principle of Inclusion-Exclusion say about $|A \cup B \cup C|$? Your answer should be an equation with $|A \cup B \cup C|$ on one side.

$$\begin{aligned}
|A \cup B \cup C| = &|A| + |B| + |C| \\
&- |A \cap B| - |A \cap C| - |B \cap C| \\
&+ |A \cap B \cap C|
\end{aligned}$$

15. [4] Let A and B be sets such that $A \subseteq A \cap B$. Prove that $A \cup B \subseteq B$. Your proof must begin with "Take any $x \in A \cup B \ldots$"

Suppose $A \subseteq A \cap B$.

Take any $x \in A \cup B$.

$\therefore x \in A$ or $x \in B$

If $x \in B$, we're done.

Suppose $x \in A$.

Since $A \subseteq A \cap B$, $x \in A \cap B$

$\therefore x \in A$ and $x \in B$

$\therefore x \in B$

$\therefore$ If $x \in A \cup B$, then $x \in B$

i.e. $A \cup B \subseteq B$

16. [4] Give a counterexample to show that the following statement is false: "*For any sets A, B and C, if $A \setminus B = A \setminus C$, then $B = C$.*"

Let $\mathcal{U} = \{1, 2\}$, $A = \emptyset$, $B = \{1\}$, $C = \{2\}$

Then $A \setminus B = \emptyset$

$B \setminus C = \emptyset$

but $B \neq C$

17. Let $\mathcal{R}$ be the relation on $\mathbb{Q}$ defined by $(x, y) \in \mathcal{R} \Leftrightarrow x - y \in \mathbb{Z}$.

(a) [3] Prove that $\mathcal{R}$ is an equivalence relation.

reflexive : Let $x \in \mathbb{Q}$. Then $x - x = 0 \in \mathbb{Z}$
$\therefore$ $(x, x) \in R$ and R is reflexive.

symmetric: Suppose $(x, y) \in R$. Then $x - y \in \mathbb{Z}$
Since $y - x = -(x - y)$, we have $y - x \in \mathbb{Z}$
$\therefore$ $(y, x) \in R$ and R is symmetric.

transitive: Suppose $(x, y), (y, z) \in R$
Then $x - y \in \mathbb{Z}$ and $y - z \in \mathbb{Z}$
$\therefore$ $\underbrace{(x - y) + (y \in z)}_{= x - z} \in \mathbb{Z}$
$\therefore$ $(x, z) \in R$, and R is transitive
$\therefore$ R is an equivalence relation

(b) [1] Find, or describe, the equivalence class [5].

$[5] = \{y : (5, y) \in R\} = \{y : 5 - y \in \mathbb{Z}$
Since $5 - y \in \mathbb{Z} \Leftrightarrow y \in \mathbb{Z}$, $[5] = \mathbb{Z}$.

18. [3] Let $f : \mathbb{N} \to \mathbb{R}$ be defined by $f(n) = \frac{1}{n+5}$. Prove that f is injective.

Suppose $f(n_1) = f(n_2)$
Then $\dfrac{1}{n_1 + 5} = \dfrac{1}{n_2 + 5}$

$\therefore$ $n_1 + 5 = n_2 + 5$
$\therefore$ $n_1 = n_2$ $\therefore$ f is injective.

19. [2] Let $f : (0,1) \to (1,4)$ be defined by $f(x) = (x+1)^2$. Find a function $g : (1,4) \to (0,1)$ such that $g \circ f$ is the identity function on $(0,1)$. Writing down a correct function is good enough; you do not need to prove that it has the property.

Take $g(x) = \sqrt{x} - 1$

Then $g \circ f(x)$

$= g(f(x))$

$= g((x+1)^2) = \sqrt{(x+1)^2} - 1$

$= x + 1 - 1$

$= x$ ✓

$y = (x+1)^2$

$\sqrt{y} = x+1$

$\therefore x = \sqrt{y} - 1$

can do this b/c $x, y \geq 0$.

20. [3] Prove that the set $\{2^a 3^b : a, b \in \mathbb{N}\}$ is countable.

$\{2^a3^b : a,b \in \mathbb{N}\} \subseteq \mathbb{Z}$. Any subset of a countable set is countable. Since $\mathbb{Z}$ is countable, so is $\{2^a3^b : a,b \in \mathbb{N}\}$.

$$\begin{array}{cccc}
2^3 3^0 & \cdots & & \\
2^2 3^0 & 2^2 3^1 & 2^2 3^2 & \cdots \\
2^1 3^0 & 2^1 3^1 & 2^1 3^2 & \cdots \\
2^0 3^0 & 2^0 3^1 & 2^0 3^2 & \cdots
\end{array}$$

Every element of $\{2^a3^b : a,b \in \mathbb{N}\}$ is in the array. The seq. indicated by the arrows contains every element of the array & $\therefore$ of $\{2^a3^b : a,b \in \mathbb{N}\}$.

$\therefore \{2^a3^b : a,b \in \mathbb{N}\}$ is countable

/END